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DOMINATIONS ON BIPARTITE STEINHAUS GRAPHS

Daekeun Lim

Abstract. In this paper, we give an upper bound for dominations of Steinhaus graphs, and the domination numbers of the bipartite Steinhaus graphs. Also, we give an upper bound for Nordhaus-Gaddum type result for the bipartite Steinhaus graphs.

1. Introduction

A *Steinhaus graph* G is a labeled graph G of order n whose adjacency matrix $A(G) = (a_{i,j})$ satisfy the Steinhaus property : $a_{i,j} = a_{i-1,j-1} + a_{i-1,j} \pmod 2$ for each $1 \le i < j \le n$. The first row in $A(G)$ is called the *generating string* of G. It is obvious that there are exactly 2^{n-1} Steinhaus graphs of order n. The vertices of a Steinhaus graph are usually labeled by their corresponding row numbers (see Figure 1).

	1	2	3	4	5	6	7	8
1	0	0	1	1	0	1	1	0
2	0	0	1	0	1	1	0	1
3	1	1	0	1	1	0	1	1
4	1	0	1	0	0	1	1	0
5	0	1	1	0	0	1	0	1
6	1	1	0	1	1	0	1	1
7	1	0	1	1	0	1	0	0
8	0	1	1	0	1	1	0	0

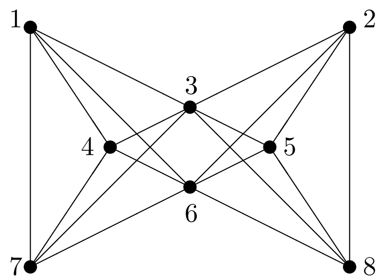

Figure 1 Steinhaus graph with the generating string 00110110

In this paper, $\lfloor x \rfloor$ is the floor of x and $\lceil x \rceil$ is the ceiling of x. We denote $log_2(x)$ by $lg(x)$. We now present Pascal's rectangle modulo two (see Figure 2). The rows of the rectangle are labelled $R_1^*, R_2^*, \cdots$, and so the kth element

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of R_n^* is 0 if $k > n$ and is $\binom{n-1}{k-1}$ (mod 2) if $1 \le k \le n$. We denote $R_{n,k}$ by the string formed by the first k elements of R_n^* and we set $R_n = R_{n,n}$.

$R_{1,8} \rightarrow$	1	0	0	0	0	0	0	0
$R_{2,8} \rightarrow$	1	1	0	0	0	0	0	0
$R_{3,8} \rightarrow$	1	0	1	0	0	0	0	0
$R_{4,8} \rightarrow$	1	1	1	1	0	0	0	0
$R_{5,8} \rightarrow$	1	0	0	0	1	0	0	0
$R_{6,8} \rightarrow$	1	1	0	0	1	1	0	0
$R_{7,8} \rightarrow$	1	0	1	0	1	0	1	0
$R_{8,8} \rightarrow$	1	1	1	1	1	1	1	1

Figure 2 Pascal's rectangle of length 8

Also, if k is a positive integer, then let $K = 2^{\lceil lg(k) \rceil}$ and $T = R_{K-k+1,K}$. If T is a string of zeros and ones, then T^k is the string T concatenated with itself $k-1$ times. For example, if $T = 10$, then $T^4 = 10101010$. In [3] , the generating strings for bipartite Steinhaus graphs are described as follows.

Theorem 1.1. ([3]) *A Steinhaus graph is bipartite if and only if its generating string is a prefix of either* $0^k T^{i2^m} 0^{K2^m}$ *or* $0^k T^{2^j} 0^m$ *for each positive integer* k*, positive odd integer* i *larger than* 1*, non-negative integers* j, m*.*

Moreover, the tight upper bound and a recurrence formula for the numbers of bipartite Steinhaus graphs are studied in [3].

A set $S \subseteq V(G)$ of a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a *γ-set.* It is studied domination numbers of Steinhaus graphs in [8].

Theorem 1.2. ([8]) *For any nontrivial Steinhaus graph* G *with* n *vertices,*

$$\gamma(G) \le \left\lceil \frac{n}{3} \right\rceil,$$

with equality holds if and only if G *is the path* P_n*.*

2. Domination numbers of bipartite Steinhaus graphs

First, we consider all bipartite Steinhaus graphs G whose generating string is a prefix of $0^k T^{i2^j}$ for some positive integer k, positive odd integer i and nonnegative integer j. Then if $k = 1$ or $n-1$, G is isomorphic to $K_{1,n-1}$. So, $\gamma(G) = 1$. Otherwise, it is clear that $\{k, k+1\}$ is a γ-set of G. So, $\gamma(G) = 2$. Therefore, we get the following theorem.

Theorem 2.1. *If a bipartite Steinhaus graph G has the generating string which is a prefix of $0^k T^{i2^j}$ for some positive integer k, positive odd integer i and nonnegative integers j, then*

$$\gamma(G) = \begin{cases} 1, & \text{if } k = 1, n-1; \\ 2, & \text{otherwise.} \end{cases}$$

From now on, assume that G with adjacent matrix, $(a_{i,j})$ has the generating string $a_{1,1}, a_{1,2} \cdots a_{1,n}$. Let us start from Lucas's Theorem (See [9]).

Theorem 2.2. *Let p be prime, and let $n = \sum a_i p^i$ and $m = \sum b_i p^i$ be the p-ary expansions of positive integers n and m. Then*

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots \pmod{p}.$$

Since the entry $a_{i,j}$ in the matrix $A(G)$ is given by $a_{i,j} \equiv \sum_{k=0}^{i-l} \binom{i-l}{k} a_{l,j-k}$ (mod 2) for all $1 \le l \le \frac{i}{2}$, we get the following lemma by applying $p = 2$ in Theorem 2.2.

Lemma 2.3. *If i, j, r are nonnegative integers, with $i, j \ge 1$, then*

$$a_{i+2^r, j+2^r} \equiv a_{i,j} + a_{i,j+2^r} \pmod{2}.$$

From now on, let l, m be positive integers, and denote that $L = 2^{\lceil lg(l) \rceil}$. Observe that in Pascal's rectangle modulo two, the number of ones in $R_{i,L}$ is at most $\frac{L}{4}$ for $1 \le i \le L$ except $i = \frac{L}{2}, L-1, L$. For convenience, we divide nontrivial bipartite Steinhaus graphs into three cases for $k = 1, 2$ and $k \ge 3$ respectively. Consider the case $k = 1$. Let G be a bipartite Steinhaus graph with generating string $01^l 0^m$. Then since $a_{1,j} = 0$ for $j \ge l+2$, we get the following facts by Lemma 2.3;

Fact 1. For $1 \le i < j$, $a_{i,j} = a_{i+L, j+L}$.
Fact 2. For $2 \le i \le L+1$, $(a_{i,j})_{i<j}$ is given by as follow;

$$(a_{i,j}) = \begin{cases} 0^{l-i} R_{i-1,L}, & \text{if } 2 \le i \le l+1; \\ \text{surfix of } R_{i-1,L} \text{ deleted by first } L-l \text{ entries}, & \text{if } l+2 \le i \le L+1. \end{cases}$$

Therefore, by applying the above observation and two Facts we get the following lemma.

Lemma 2.4. *If G has the generating string $01^l 0^m$, then an upper bound for maximal degree of G, $\Delta(G)$ is given by as follows;*
(1) *If $l = L-1$, $\Delta(G) = 2L-2 = 2l$.*
(2) *If $l = L$, $\Delta(G) = l+1$.*
(3) *Otherwise, $\Delta(G) \le \frac{3l}{2}$.*

The following theorem concerns the domination numbers of G for $k = 1$. Denote that for a vertex s of G, $N(s)$ is the set of vertices which are adjacent to s. In the following theorems, we assume that G has n-vertices.

Theorem 2.5. *If G has the generating string 01^l0^m, then*

$$\gamma(G) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & \text{if } l = 1; \\ \left\lceil \frac{n+1}{4} \right\rceil, & \text{if } l = 2; \\ \left\lceil \frac{n-1}{2l+1} \right\rceil + 1, & \text{if } l+1 \text{ is a power of two, } l \geq 3; \\ \left\lceil \frac{n-1}{l} \right\rceil + 1, & \text{if } l \text{ is a power of two, for } l \geq 4; \\ \left\lceil \frac{2(n-1)}{L} \right\rceil + 1, & \text{otherwise.} \end{cases}$$

Proof. First, for $l = 1$, G is the path P_n. So, $\gamma(G) = \left\lceil \frac{n}{3} \right\rceil$.
Next, for $l = 2$, observe that G is isomorphic to $P_2 \times P_{\lceil n/2 \rceil}$ or $P_2 \times P_{\lceil n/2 \rceil} - \{n+1\}$. Construct S as follow;

$$S = \begin{cases} \{1, 6, 9, 14, \ldots, n\}, & \text{if } n \equiv 0, 2 \pmod 4; \\ \{2, 4, 10, 13, \ldots, n\}, & \text{if } n \equiv 1, 3 \pmod 4). \end{cases}$$

Then it is clear that S dominate the vertex set of G. Since the degree of each vertex in $S - \{1, n\}$ is 3 which is maximal, for all $s \in S - \{1, n\}$, $N(s)$'s are pairwise disjoint. Therefore, S is a γ-set of G. So, $\gamma(G) = |S| = \left\lceil \frac{n+1}{4} \right\rceil$.

Next, if $l + 1$ is a power of two, then $S = \{1, l+2, 3l+4, \ldots\}$ dominates the vertex set of G. Since the degree of each vertex in $S - \{1\}$ is $2l + 1$ which is maximal by (1) in Lemma 2.4, for all $s \in S - \{1\}$, $N(s)$'s are pairwise disjoint. Therefore, S is a γ-set of G. So, $\gamma(G) = |S| = \left\lceil \frac{n-1}{2l} \right\rceil + 1$.

Next, if l is a power of two, then the set $S = \{1, l+1, 2l+1, \ldots\}$ dominates the vertex set of G. Since the degree of each vertex in $S - \{1\}$ is $l + 1$ which is maximal by (2) in Lemma 2.4, for all $s \in S - \{1\}$, $(N(s) - \{s-1\})$'s are pairwise disjoint. Therefore, S is a γ-set of G. So, $\gamma(G) = |S| = \left\lceil \frac{n-1}{l} \right\rceil + 1$.

Finally, if $l, l+1$ are not a power of two, then $S = \{1, \frac{L}{2}+1, L+1, \frac{3L}{2}+1, 2L+1, \ldots\}$ dominates the vertex set of G. It is straightforward to show that S is a γ-set of G. So, $\gamma(G) = |S| = \left\lceil \frac{2(n-1)}{L} \right\rceil + 1$.

□

Theorem 2.6. *If G has the generating string which is $00(10)^{j2^r}0^m$ for some nonnegative integer r and positive odd integer j, then*

$$\gamma(G) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil, & \text{if } j = 1, r = 0; \\ \left\lceil \frac{n-2}{2^r} \right\rceil, & \text{if } j = 1, r \geq 1; \\ \left\lceil \frac{n-2}{J2^{r-1}} \right\rceil + 1, & \text{otherwise,} \end{cases}$$

where $J = 2^{\lceil lg(j) \rceil}$.

Proof. It is clear that the induced subgraph $G - 1$ is a bipartite graph with generating string 01^l0^m which is one of $l = 2$ or l is a power of two or l is a multiple of two but not a power of two as in the Theorem 2.5. So, by Theorem 2.5, the proof is completed. □

For $k \geq 3$, $T = R_{K-k+1,K}$. So, the generating string of a bipartite Steinhaus graph is $0^k T^{j2^r} 0^m$ for some nonnegative integer r and positive odd integer j.

Theorem 2.7. *If $k \geq 3$ and G has the generating string $0^k T^{j2^r} 0^m$ for some nonnegative integer r and positive odd integer j, then*

$$\gamma(G) = \begin{cases} \left\lceil \frac{n-k}{K2^r} \right\rceil, & \text{if } j = 1 \text{ ;} \\ \left\lceil \frac{2(n-k)}{KJ2^r} \right\rceil + 1, & \text{otherwise} \end{cases}$$

Proof. It is clear that the induced subgraph $G - \{1, 2, \ldots, k\}$ is a bipartite graph with generating string $01^l 0^m$. So, by Theorem 2.5, the proof is completed. □

Since the domination number of any nontrivial bipartite graph is 2, we get the following Nordhaus-Gaddum type result.

Corollary 2.8. *For any nontrivial bipartite Steinhaus graph G with n vertices,*

$$\gamma(G) + \gamma(\overline{G}) \leq \left\lceil \frac{n+6}{3} \right\rceil$$

$$\gamma(G)\gamma(\overline{G}) \leq \left\lceil \frac{2n+3}{3} \right\rceil$$

with equality if and only if $G = P_n$.

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Daekeun Lim
Department of Mathematics
Keimyung University
Daegu 704-701, Korea
E-mail address: limd@kmu.ac.kr