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SCALARIZATION METHODS FOR MINTY-TYPE VECTOR VARIATIONAL INEQUALITIES*

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Abstract. Many kinds of Minty's lemmas show that Minty-type variational inequality problems are very closely related to Stampacchia-type variational inequality problems. Particularly, Minty-type vector variational inequality problems are deeply connected with vector optimization problems. Liu et al. [10] considered vector variational inequalities for set-valued mappings by using scalarization approaches considered by Konnov [8]. Lee et al. [9] considered two kinds of Stampacchia-type vector variational inequalities by using four kinds of Stampacchia-type scalar variational inequalities and obtain the relations of the solution sets between the six variational inequalities, which are more generalized results than those considered in [10]. In this paper, the author considers the Minty-type case corresponding to the Stampacchia-type case considered in [9].

1. Introduction and Preliminaries

Recently, there have been usually traditional concentrations on scalarization approaches [3, 4, 6, 8-11] which enable us to replace the vector problems under consideration with equivalent scalar problems in studying vector problems including vector optimization problems, vector variational inequality problems and vector equilibrium problems.

In particular, Slavov [11] discussed some scalarization techniques and one application of multi-objective optimization problems into a mathematical economics.

In 2009, Jimenez et al. [6] developed a scalarization method in order to obtain scalar versions of their results on the necessary and sufficient conditions for strict minimizers of a general vector optimization problem, through a variational approach.

Konnov [8] also considered a scalarization approach to connect vector variational inequalities into an equivalent scalar variational inequalities with a

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set-valued cost mapping. He gave an equivalence between weak and strong solutions of set-valued vector variational inequalities and suggested a new gap function for vector variational inequalities. He applied his results to vector optimization, vector network equilibrium and vector migration equilibrium problems.

In 2009, Guu et al. [4] extended the scalarization approaches of Giannessi et. al [3] to set-valued vector optimization problems and set-valued weak vector optimization problems. The scalar variational inequalities given by them is different from those given by Konnov [8].

In [10], Liu et al. introduced four kinds of scalar variational inequalities for studying vector variational inequalities for set-valued mappings by using the scalarinzation approach considered by Konnov [8].

In [9], Lee et al. obtained more generalized results than the corresponding results of Liu et al. [10].

In this paper, the author considers Minty-type cases corresponding to Stampacchia-type cases considered in [9].

An ordered Banach space (Y, P) is a real Banach space ordered by a nonempty closed, convex and pointed cone $P \subseteq Y$, that is, $\lambda P \subset P$ for $\lambda > 0$, $P + P = P$ and $P \cap \{-P\} = \{0\}$ with the apex at the origin in the following form of

$$x \geq y \Leftrightarrow x - y \in P \text{ for } x, y \in Y$$

and

$$x \not\geq y \Leftrightarrow x - y \notin P \text{ for } x, y \in Y.$$

If the interior $intP$ of P is nonempty, then a weak order in Y is defined by

$$y < x \Leftrightarrow x - y \in intP \text{ for } x, y \in Y$$

and

$$y \not< x \Leftrightarrow x - y \notin intP \text{ for } x, y \in Y.$$

Throughout this paper, X is a real Banach space with its topological dual X^* , $(Y, P) \equiv (\mathbb{R}^n, P)$, K a nonempty convex subset of X, N_0 be a nonempty subset of X^*, N a nonempty subset of the vector space $L(X, Y)$ of all the continuous linear mappings from X to Y and $\langle \ell, x\rangle$ the value of $\ell \in L(X, Y)$ at $x \in X$, where $P = \mathbb{R}^n_+ = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n;\ \lambda_i > 0,\ i = 1, 2, \cdots, n\}$. Let $\eta : K \times K \to K$ be a mapping, $T_i : K \to 2^{X^*}$ $(i = 1, 2, \cdots, n)$ be a set-valued mapping, $T_0 : K \to 2^{X^*}$ and $T : K \to 2^{L(X,Y)}$ be set-valued mappings defined by $T_0(x) = conv\Big(\bigcup_{i=1}^n T_i(x)\Big)$, the convex hull of $\bigcup_{i=1}^n T_i(x)$ and $T(x) = \prod_{i=1}^n T_i(x)$. Let $M_i : K \times N_0 \to 2^{X^*}$ $(i = 1, 2, \cdots, n)$ be a set-valued mapping, $M : K \times N \to 2^{L(X,Y)}$ a set-valued mapping defined by

$$M(x, s) = \prod_{i=1}^n M_i(x, s_i) \text{ for } s = (s_1, s_2, \cdots, s_n) \in N \subset L(X, Y)$$

and $M_0 : K \times N_0 \to 2^{X^*}$ be a set-valued mapping defined by

$$M_0(x, s_i) = conv\Big(\bigcup_{i=1}^{n} M_i(x, s_i)\Big) \text{ for } x \in K \text{ and } s = (s_1, s_2, \cdots, s_n) \in N.$$

Let f_i $(i = 1, 2, \cdots, n) : K \to \mathbb{R}$ be a mapping and $f(x) = (f_1(x), \cdots, f_n(x))$ for $x \in K$ and $B = \{(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} \lambda_i = 1\}$, which is a nonempty compact and convex subset of $\mathbb{R}^n$.

If $f(x) \equiv u$ for all $x \in K$, $\eta(x, y) = x - y$ for $x, y \in K$ and $M_0(x^*, s^*) = s^*$, then we obtain some corresponding results which have been investigated by many authors [1, 2, 5, 8, 9, 10].

By using a scalarization system, Konnov [8] and Liu et al. [10] converted vector variational inequalities into equivalent scalar variational inequalities. Inspired by works of Konnov [8] and Liu et al. [10], in this paper we introduce four kinds of Minty-type generalized scalar variational inequalities for studying MT-VVI-S and MT-VVI-W and then we study some relationships between those variational inequalities under suitable conditions.

The following well-known Kneser Minimax Theorem is essential in the proving of our main results.

Proposition 1.1. ([7]) *Let A be a nonempty convex set in a vector space and B a nonempty compact convex set in a Hausdorff topological vector space. Suppose that g is a real-valued function on $A \times B$ such that for each fixed $a \in A$, $g(a, \cdot)$ is lower semicontinuous and convex on B and for each fixed $b \in B$, $g(\cdot, b)$ is concave on A. Then*

$$\min_{b \in B} \sup_{a \in A} g(a, b) = \sup_{a \in A} \min_{b \in B} g(a, b).$$

2. Main results

We consider the following Minty-type vector(or scalar) variational inequalities:

(MT-VVI-S) Minty-type vector variational inequalities;
Find $x^* \in K$ such that there exists $s^* \in T(x^*)$ satisfying

$$\langle M(y, s^*), \eta(y, x^*)\rangle + f(y) - f(x^*) \not< 0, \text{ for } y \in K,$$

whose solution set is denoted by M-S_{VS}.

(MT-VVI-W) Minty-type vector variational inequalities;
Find $x^* \in K$ such that for each $y \in K$ there exists $s^* \in T(x^*)$ satisfying

$$\langle M(y, s^*), \eta(y, x^*)\rangle + f(y) - f(x^*) \not< 0,$$

whose solution set is denoted by M-S_{VW}.

(MT-SVI-S) Minty-type scalar variational inequalities;
Find $x^* \in K$ such that there exists $e^* \in T_0(x^*)$ satisfying

$$\langle M_0(y, e^*), \eta(y, x^*)\rangle + \langle \lambda, f(y) - f(x^*)\rangle \geq 0, \text{ for } y \in K, \ \lambda \in B,$$

whose solution set is denoted by M-S_{SS}.

(MT-SVI-1) Minty-type scalar variational inequalities;
Find $x^* \in K$ such that for each $y \in K$, there exists $e^* \in T_0(x^*)$ satisfying

$$\langle M_0(y, e^*), \eta(y, x^*)\rangle + \langle \lambda, f(y) - f(x^*)\rangle \geq 0, \text{ for } \lambda \in B,$$

whose solution set is denoted by M-S_{S1}.

(MT-SVI-2) Minty-type scalar variational inequalities;
Find $x^* \in K$ such that there exist $e^* \in T_0(x^*)$, $\lambda \in B$ satisfying

$$\langle M_0(y, e^*), \eta(y, x^*)\rangle + \langle \lambda, f(y) - f(x^*)\rangle \geq 0, \text{ for } y \in K,$$

whose solution set is denoted by M-S_{S2}.

(MT-SVI-W) Minty-type scalar variational inequalities;
Find $x^* \in K$ such that for each $y \in K$, there exist $e^* \in T_0(x^*)$, $\lambda \in B$ satisfying

$$\langle M_0(y, e^*), \eta(y, x^*)\rangle + \langle \lambda, f(y) - f(x^*)\rangle \geq 0,$$

whose solution set is denoted by M-S_{SW}.

Remark 2.1. By definitions, we obtain the following relations

$$\text{M-S}_{\text{VS}} \subset \text{M-S}_{\text{VW}},\ \text{M-S}_{\text{SS}} \subset \text{M-S}_{\text{S1}} \subset \text{M-S}_{\text{SW}} \text{ and } \text{M-S}_{\text{SS}} \subset \text{M-S}_{\text{S2}} \subset \text{M-S}_{\text{SW}}.$$

By using the scalarization approaches, we obtain the other relations between the solution sets under suitable conditions.

Theorem 2.1. *Let $M_0 : K \times N_0 \to X^*$ be a mapping such that $x \to M_0(\cdot, x)$ is continuous and $\eta : K \times K \to K$ be a mapping satisfying $\eta(x, y) + \eta(y, x) = 0$ for x, $y \in K$. Assume that for each $x \in K$, $T_i(x)$ is nonempty convex and weakly* compact and f_i is convex $(i = 1, 2, \cdots, n)$. Then M-S_{S2} = M-S_{SW} holds.*

Proof. Since $T_i(x)$ $(i = 1, 2, \cdots, n)$ is nonempty convex and weakly* compact and B is nonempty compact and convex, so are $T_0(x)$ and $T_0(x) \times B$. Let $x^* \in$ M-S_{SW}, then for each $y \in K$, there exists $e^* \in T_0(x^*)$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in B$ such that

$$\begin{aligned} &\langle M_0(y, e^*), \eta(y, x^*)\rangle + \sum_{i=1}^{n} \lambda_i (f_i(y) - f_i(x^*)) \\ &= \langle M_0(y, e^*), \eta(y, x^*)\rangle + \langle \lambda, f(y) - f(x^*)\rangle \geq 0. \end{aligned}$$

Define a function $g : K \times T_0(x^*) \times B \to \mathbb{R}$ by

$$\begin{aligned} g(a, b, \alpha) &= \langle M_0(y, b), \eta(x^*, a)\rangle + \langle \alpha, f(x^*) - f(a)\rangle \\ &= \langle M_0(y, b), \eta(x^*, a)\rangle + \sum_{i=1}^{n} \alpha_i (f_i(x^*) - f_i(a)) \end{aligned}$$

for $a \in K$, $b \in T_0(x^*)$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in B$. Then it follows that

$$\begin{aligned}
&\sup_{a \in K} \min_{(b,\alpha) \in T_0(x^*) \times B} g(a, b, \alpha) \\
&= \sup_{a \in K} \min_{(b,\alpha) \in T_0(x^*) \times B} \{\langle M_0(y, b), \eta(x^*, a)\rangle + \sum_{i=1}^{n} \alpha_i (f_i(x^*) - f_i(a)\} \\
&\leq 0.
\end{aligned}$$

On the other hand, for each fixed $a \in A$, $g(a, \cdot, \cdot)$ is continuous (with respect to the second argument in the weak* topology of X^* and with respect to the third one in the norm topology of $\mathbb{R}^n$) and convex on B and for each fixed $(b, \alpha) \in T_0(x^*) \times B$, $g(\cdot, b, \alpha)$ is concave on A from the convexity of f_i $(i = 1, 2, \cdots, n)$. By Proposition 1.1 we have

$$\begin{aligned}
&\min_{(b,\alpha) \in T_0(x^*) \times B} \sup_{a \in K} \{\langle M_0(y, b), \eta(x^*, a)\rangle + \langle \alpha, f(x^*) - f(a)\rangle\} \\
&= \min_{(b,\alpha) \in T_0(x^*) \times B} \sup_{a \in K} g(a, b, \alpha) \\
&= \sup_{a \in K} \min_{(b,\alpha) \in T_0(x^*) \times B} g(a, b, \alpha) \leq 0.
\end{aligned}$$

Thus there are $h^* \in T_0(x^*)$ and $\delta \in B$ such that

$$\langle M_0(y, h^*), \eta(x^*, y)\rangle + \langle \delta, f(x^*) - f(y)\rangle \leq 0 \text{ for all } y \in K,$$

which implies that $x^* \in$ M-S$_{\mathrm{S2}}$. □

Corollary 2.1. *Let $M_0 : K \times N_0 \to X^*$ be a mapping such that $x \to M_0(x, \cdot)$ is continuous and $\eta : K \times K \to K$ be a mapping satisfying $\eta(x, y) + \eta(y, x) = 0$ for x, $y \in K$. Assume that for each $x \in K$, $T_i(x)$ is nonempty convex and weakly* compact and f_i is convex $(i = 1, 2, \cdots, n)$. Then*

$$M\text{-S}_{\mathrm{SS}} \subseteq M\text{-S}_{\mathrm{S1}} \subseteq M\text{-S}_{\mathrm{S2}} = M\text{-S}_{\mathrm{SW}}.$$

Now we characterize the solution sets M-S$_{\mathrm{VS}}$ and M-S$_{\mathrm{VW}}$ via M-S$_{\mathrm{SS}}$, M-S$_{\mathrm{S1}}$, M-S$_{\mathrm{S2}}$ and M-S$_{\mathrm{VW}}$.

Theorem 2.2. *Let $M_0 : K \times N_0 \to X^*$ be a mapping such that $x \to M_0(x, \cdot)$ is continuous. Assume that for each $x \in K$, $T_i(x)$ is a nonempty convex and weakly* compact and f_i is convex $(i = 1, 2, \cdots, n)$, then the following assertions hold:*

(i) *M-S$_{\mathrm{VS}} \subseteq M$-S$_{\mathrm{VW}} \subseteq M$-S$_{\mathrm{S2}} = M$-S$_{\mathrm{SW}}$.*

(ii) *If $T_0(x) = \bigcup_{i=1}^{n} T_i(x)$ for $x \in k$, then*

$$M\text{-S}_{\mathrm{SS}} \subseteq M\text{-S}_{\mathrm{VS}} \quad \textit{and} \quad M\text{-S}_{\mathrm{S1}} \subseteq M\text{-S}_{\mathrm{VW}}.$$

Proof. (i) Let $x^* \in$ M-S$_{\mathrm{VW}}$, then for each $y \in K$,

$$\langle M(y, s^*), \eta(y, x^*)\rangle + f(y) - f(x^*) \not< 0 \text{ for some } s^* \in T(x^*).$$

Hence for some $j \in \{1, 2, 3, \cdots, n\}$, there is $s_j^* \in T_j(x^*)$ such that

$$\langle M_j(y, s_j^*), \eta(y, x^*)\rangle + f_j(y) - f_j(x^*) \geq 0.$$

Set $e^* = s_j^*$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ with $\gamma_j = 1$ and $\gamma_i = 0$ for $i \neq j$, then $e^* \in T_0(x^*)$, $\gamma \in B$ and consequently

$$\langle M_0(y, e^*), \eta(y, x^*)\rangle + \langle \gamma, f(y) - f(x^*)\rangle \geq 0,$$

which implies that $x^* \in$ M-S$_{\mathrm{SW}}$.

(ii) Assume that for each $x \in K$, $T_0(x) = \bigcup_{i=1}^{n} T_i(x)$. Let $x^* \in$ M-S$_{\mathrm{SS}}$, then there exists

$$e^* \in T_0(x^*) \text{ such that } \langle M_0(y, e^*), \eta(y, x^*)\rangle + \langle \lambda, f(y) - f(x^*)\rangle \geq 0$$

for $y \in K$, $\lambda \in B$.

Since for each $x \in K$, $T_0(x) = \bigcup_{i=1}^{n} T_i(x)$, there is $j \in \{1, 2, \cdots, n\}$ such that $e^* \in T_j(x^*)$, set $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ with $\lambda_j = 1$ and $\lambda_i = 0$ for $i \neq j$. From the above inequality, we have

$$\langle M_0(y, e^*), \eta(y, x^*)\rangle + f_j(y) - f_j(x^*) \geq 0, \text{ for } y \in K.$$

Choose arbitrary elements $s_i^* \in T_i(x^*)$ for $i \neq j$, and set $s_j^* = e^*$ and $s^* = (s_1^*, s_2^*, \cdots, s_n^*) \in T(x^*)$. It follows that

$$\langle M(y, s^*), \eta(y, x^*)\rangle + f(y) - f(x^*) \not< 0, \text{ for } y \in K,$$

which implies that $x^* \in$ M-S$_{\mathrm{VS}}$. Similarly, we can show that M-S$_{\mathrm{S1}} \subseteq$ M-S$_{\mathrm{VW}}$. $\square$

Remark 2.2. Our results generalize some results in [9, 10] as corollaries.

Theorem 2.3. *(MT-VVI-S) implies the following generalized form (G-VVI-S) Find $x^* \in K$ such that there exists $s^* \in T(x^*)$ satisfying*

$$\langle M(x^*, s^*), \eta(y, x^*)\rangle + f(y) - f(x^*) \not< 0, \text{ for } y \in K;$$

of (VVI-S) considered in [9].

Proof. Suppose that we can find $x^* \in K$ such that there exists $s^* \in T(x^*)$ satisfying

$$\langle M(y, s^*), \eta(y, x^*)\rangle + f(y) - f(x^*) \not< 0, \text{ for } y \in K.$$

Assume to the contrary that for some $y_0 \in K$, for all $x \in K$ and for all $s \in T(x)$, we have

$$\langle M(x, s), \eta(y, x)\rangle + f(y_0) - f(x) \geq 0.$$

Hence

$$\langle M(y_0, s^*), \eta(y_0, x^*)\rangle + f(y_0) - f(x^*) \geq 0,$$

which is a contradiction to our hypothesis. $\square$

Remark 2.3. By the same method, we obtain the implication of (MT-VVI-W) to the generalized form of (VVI-W) considered in [9].

Remark 2.4. By defining an appropriate monotone concept, we obtain the converse implications of Theorem 2.3 and Remark 2.3.

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