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SOME RESULTS ON REDUCIBILITY AND STRONG REDUCIBILITY

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Abstract. We investigate some properties of strongly reduced near-rings and left regular near-rings, also show that a strongly reduced near-ring is reduced. Next, we will characterize that reducibility, strong reducibility and left regularity in near-rings.

1. Introduction

Mason [2] introduced this notion and characterized left regular zero-symmetric unital near-rings. Also, several authors ([1], [3], [4], [6] etc.) studied them. In particular, Reddy and Murty [6] extended some results in [2] to the non-zero symmetric case. They observed that every left regular near-ring has some interesting property (*) in Reddy and Murty. In this paper we consider this property. Let R be a right near-ring and let R_c denote the constant part of R. We will define strong reducibility of rings. We show that strong reducibility is a general concept of the property (*). Left or right regular near-rings form one of the important class of strongly reduced near-rings. Using strong reducibility, we will characterize reducibility, strong reducibility in near-rings and left regular near-rings.

A near-ring R is said to be *left regular* if, for each $a \in R$, there exists $x \in R$ such that $a = xa^2$. Right regularity is defined in a symmetric way. Also, a near-ring R is said to be *left κ-regular* if, for each $a \in R$, there exists a positive integer n and an element $x \in R$ such that $a^n = xa^{n+1}$. Similarly, we can define right κ-regular.

For notation and basic results, we shall refer to Pilz [5].

2. Results

We say that a near-ring R has the *insertion of factors property* (briefly, IFP) provided that for all a, b, x in R with $ab = 0$ implies $axb = 0$, and R has the *strong IFP* if every homomorphic image of R has the IFP, equivalently, for

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any ideal I of R, for all a, b, x in R with $ab \in I$ implies $axb \in I$, which are introduced in [5].

Also, we say that R is *reduced* if R has no nonzero nilpotent elements, that is, for each a in R, $a^n = 0$, for some positive integer n implies $a = 0$. McCoy proved that R is reduced iff for each a in R, $a^2 = 0$ implies $a = 0$.

A near-ring R is called *reversible* if for any $a, b \in R$, $ab = 0$ implies $ba = 0$, and R is said to be *strongly reversible* if for any $a, b \in R$ and for each ideal I of R, $ab \in I$ implies $ba \in I$. On the other hand, we say that R has the *reversible IFP* in case R has the IFP and is reversible.

For a near-ring R, R_c denotes the constant part of R, that is, $R_c = \{a \in R \mid a0 = a\}$. A near-ring R is said to be *strongly reduced* if, for $a \in R$, $a^2 \in R_c$ implies $a \in R_c$. Obviously R is strongly reduced if and only if, for $a \in R$ and any positive integer n, $a^n \in R_c$ implies $a \in R_c$. We will show that a strongly reduced near-ring is reduced.

Lemma 2.1. *(1) Any subnear-ring of a strongly reduced near-ring is strongly reduced.*

(2) Every homomorphic image of a strongly reduced constant near-ring is strongly reduced.

(3) The direct product of strongly reduced near-rings is strongly reduced.

Lemma 2.2. *(1) All left or right regular near-rings are strongly reduced.*

(2) Every integral near-ring N is strongly reduced.

(3) The direct product of integral near-rings is strongly reduced.

Lemma 2.3. *Let R be a zero-symmetric and reduced near-ring. Then R has the reversible IFP.*

Proof. Suppose that a, b in R such that $ab = 0$. Then, since R is zero-symmetric, we have $(ba)^2 = baba = b0a = b0 = 0$ Reducibility implies that $ba = 0$. Next, assume that for all a, b, x in R with $ab = 0$. Then

$$(axb)^2 = axbaxb = ax0xb = ax0 = 0$$

This implies $axb = 0$, by reducibility. Hence R has the reversible IFP. □

We have the following statements from above lemmas.

Proposition 2.4. *Let R be a reduced near-ring with the condition: for any a in R, $a^nR = Ra^{n+2}$, for some positive integer n. Then R is a left κ-regular near-ring.*

Proposition 2.5. *Let R be a strongly reduced near-ring and let $a, b \in R$. Then we have the following.*

(1) R is reduced.

(2) If $ab^n \in R_c$ for some positive integer n, then $\{ab, ba\} \cup aRb \cup bRa \subseteq R_c$.

(3) If $ab^n = 0$ for some positive integer n, then $ab = 0$ and $ba = b0 = (ba)^n$. In particular, $ab = 0$ implies $ba = b0$.

Proof. (1) Assume that $a^2 = 0$. Then $a^2 \in R_c$, and hence $a \in R_c$. Then we see $a = a0 = a0a = aa = 0$.

(2) First suppose that $ab \in R_c$. Then $(ba)^2 = baba = bab0a = bab0 \in R_c$. Since R is strongly reduced, we have $ba \in R_c$. Then we obtain $xba \in R_c$ for each $x \in R$, whence $(axb)^2 \in R_c$. By the strong reducibility of R, we obtain $axb \in R_c$ for each $x \in R$. Since $ba \in R_c$, we also obtain $bRa \subseteq R_c$. Now suppose $ab^n \in R_c$. Then $(ab)^n \in R_c$ by the above argument. Since R is strongly reduced, this implies $ab \in R_c$. Hence by the first paragraph, the claim is proved.

(3) If $ab^n = 0$ for some $n \geq 1$, then $ab \in R_c$ by (2). Hence $ab = abb^{n-1} = ab^n = 0$. Then $(ba)^2 = baba = b0 \in R_c$. Hence $ba \in R_c$. Therefore we obtain $ba = (ba)^2 = b0$. □

In case, R is a zero-symmetric near-ring, R is strongly reduced if and only if R is reduced. The following example shows that a reduced near-ring is not necessarily strongly reduced.

Exmaple 2.6. Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ with addition modulo 6 and define multiplication as follows:

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	3	1	3	1	1
2	0	0	2	0	2	2
3	3	3	3	3	3	3
4	0	0	4	0	4	4
5	3	3	5	3	5	5

Obviously this is a reduced near-ring. The constant part of $\mathbb{Z}_6$ is $\{0, 3\}$. Since $1^2 = 3$ is a constant element but 1 is not, this near-ring is not strongly reduced. Also note that $1^n \neq 1$ for any integer $n > 1$.

Following Reddy and Murty [6] we say that a near-ring R has the property (*) if it satisfies

(i) for any $a, b \in R$, $ab = 0$ implies $ba = b0$.

(ii) for $a \in R$, $a^3 = a^2$ implies $a^2 = a$.

We obtain equivalent conditions for a near-ring R to be strongly reduced.

Theorem 2.7. *The following statements are equivalent for a near-ring R:*

(1) R is strongly reduced.

(2) For $a \in R$, $a^3 = a^2$ implies $a^2 = a$.

(3) If $a^{n+1} = xa^{n+1}$ for $a, x \in R$ and some nonnegative integer n, then $a = xa = ax$.

Proof. (1) $\Longrightarrow$ (2). Assume that $a^3 = a^2$. Then $(a^2 - a)a = 0$, whence $a(a^2 - a) = a0 \in R_c$ by Proposition 2.5 (3). Then $(a^2 - a)a^2 = (a^3 - a^2)a = 0a = 0$. Again by Proposition 2.5 (3) $a^2(a^2 - a) = a^2 0 \in R_c$. Hence $(a^2 - a)^2 =$

$a^2(a^2-a)-a(a^2-a)=a^20-a0=(a^2-a)0\in R_c$. This implies $a^2-a\in R_c$. Hence $a^2-a=(a^2-a)0=(a^2-a)a=0$.

(2) $\Longrightarrow$ (1). Assume $a^2\in R_c$. Then $a^3=a^2a=a^2$. By hypothesis, this implies $a=a^2\in R_c$.

(1) $\Longrightarrow$ (3). Suppose $a^{n+1}=xa^{n+1}$ for some $n\geq 0$. Then $(a-xa)a^n=0$. Hence $(a-xa)a=0$ by Proposition 2.5 (3), and so $(a-xa)^2\in R_c$ by Proposition 2.5 (2). Since R is strongly reduced, we have $a-xa\in R_c$. Then $a-xa=(a-xa)a=0$, that is $a=xa$. Now $(a-ax)a=a^2-axa=a^2-a^2=0\in R_c$. Hence $(a-ax)^2=a(a-ax)-ax(a-ax)\in R_c$ by Proposition 2.5 (2), and so $a-ax\in R_c$. Therefore $a-ax=(a-ax)a=0$.

(3) $\Longrightarrow$ (2). This is obvious. □

From the Proposition 2.5 (3) and Theorem 2.6, in the property (*) of Reddy and Murty, the condition (i) is dependent on the condition (ii).

The following is a generalization of [6, Theorem 3].

Lemma 2.8. *Let R be a strongly reduced near-ring and let $a,x\in R$. If $a^n=xa^{n+1}$ for some positive integer n, then $a=xa^2=axa$ and $ax=xa$.*

Here we give some characterizations of left regular near-rings.

Theorem 2.9. *Let R be a near-ring. Then the following statements are equivalent:*

(1) R is left regular.

(2) R is strongly reduced and left κ-regular.

(3) For each $a\in R$, there exists $x,y\in R$ such that $a=xa^2ya$.

(4) For each $a\in R$, $a\in<a^2>\cap aRa$.

Proof. (1) $\Longrightarrow$ (2) - (4). By Lemma 2.2 (1), a left regular near-ring is strongly reduced. Hence this follows from Lemma 2.7.

(2) $\Longrightarrow$ (1). This also follows from Lemma 2.7.

(3) $\Longrightarrow$ (1). By hypothesis, R is strongly reduced. If $a=xa^2ya$, then $ya=yxa^2(ya)$. By Theorem 2.6, $ya=yayxa^2$. Thus $a=xa^2yayxa^2$. This implies that R is left regular.

(4) $\Longrightarrow$ (1). Since $a\in<a^2>$ for each $a\in R$, R is strongly reduced by Lemma 2.2 (1). Hence R satisfies (4) in Theorem 2.6. Since $a\in aRa$, there exists $x\in R$ such that $a=axa$. Hence $a=(ax)a=a(ax)=a^2x$. Then we have $a=axa=(a^2x)xa=a^2x^2a$. Then, by the same way as in (3) $\Longrightarrow$ (1), we conclude that R is left regular. □

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