East Asian Mathematical Journal
Vol. 26 (2010), No. 3, pp. 389–395

VECTOR F-COMPLEMENTARITY PROBLEMS WITH g-DEMI-PSEUDOMONOTONE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, a class of g-demi-pseudomonotone mappings is introduced and the solvability of a class of generalized vector F-complementarity problems with the mappings in Banach spaces is considered.

1. Introduction and Preliminaries

In the past years, many important generalizations of monotonicity such as quasi monotonicity, pseudo-monotonicity, dense-pseduomonotonicty and semi-monotonicity have been introduced to study the various classes of variational inequalities and complementarity problems [7, 9, 11-14].

In particular, Chen [1] introduced a class of variational inequalities with semi-monotone single-valued mappings, which are continuous in the first variable and monotone in the second variable. In 2003, Fang and Huang [5] considered a class of variational-like inequalities with generalized semi-monotone single-valued mappings. For the cases of set-valued mappings, Kassay and Kolumban [10] considered variational inequalities with semi-pseudomonotonicity, and Kang et al. [8] considered variational-like inequalities with generalized semi-pseudomonotonicity.

On the other hand, Fang and Huang [4] also considered the vector F-complementarity problems with demi-pseudomonotone single-valued mappings, which are vector demicontinuous in the first variable and pseudomonotone in the second variable.

In this paper, we consider the generalized vector F-complementarity problems which generalizing the vector F-complementarity problems considered by Fang and Huang, by adding a continuous convex mapping g as finding $u \in K$

Received November 19, 2009; May 1, 2010.
2000 *Mathematics Subject Classification.* 49J40, 90C33, 47H10.
Key words and phrases. generalized vector variational inequality, generalized vector F-complementarity problem, g-pseudomonotone, g-demi-pseudomonotone, hemicontinuous, vector demicontinuous, g-demi-pseudomonotone, KKM theorem.

such that

$$\langle A(u,u), g(u)\rangle + F(g(u)) \not> 0$$
$$\langle A(u,u), g(v)\rangle + F(g(v)) \not< 0, \text{ for } v \in K,$$

where $A : K \times K \to L(X,Y)$, $F : K \to Y$ and $g : K \to K$ are mappings for a subset K of a reflexive Banach space X, an ordered Banach space $(Y, \leq)$ and a collection $L(X,Y)$ of continuous linear mappings from X into Y.

Definition 1.1. Let (Y, C) be an ordered Banach space, where C is a pointed (i.e., $C \cap \{-C\} = \{0\}$) closed convex cone with a nonempty interior $\operatorname{int} C$. With C we define the order relations $\geq$, $\not\geq$, $<$ and $\not<$ as follows;

$$\begin{aligned}
x \geq y &\Leftrightarrow\ x - y \in C,\\
x \not\geq y &\Leftrightarrow\ x - y \notin C,\\
x < y &\Leftrightarrow\ y - x \in \operatorname{int} C,\\
x \not< y &\Leftrightarrow\ y - x \notin \operatorname{int} C \quad \text{for} \quad x, y \in Y.
\end{aligned}$$

Definition 1.2. A mapping $T : K \to L(X,Y)$ is said to be hemicontinuous if for any fixed x, $y \in K$, the mapping $t \to \langle T(x + t(y-x)), y - x\rangle$ is continuous at 0^+.

Definition 1.3. Let $g : K \to K$ be a single-valued mapping, $T : K \to L(X,Y)$ and $F : K \to Y$ two nonlinear mappings. T is said to be g-pseudomonotone with respect to F if for x, $y \in K$,

$$\begin{aligned}
&\langle T(x),\, g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \ \not<\ 0\\
\text{implies}\quad &\langle T(y),\, g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \ \geq\ 0.
\end{aligned}$$

Definition 1.4. A mapping $G : K \subset X \to 2^X$ is said to be a KKM mapping if for any finite set $\{x_1, x_2, \ldots, x_n\} \subset K$, $\operatorname{Co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$, where 2^X denotes the family of all nonempty subsets of X.

Definition 1.5. A mapping $f : K \to Y$ is said to be convex if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for x, $y \in K$ and $t \in [0,1]$.

F-KKM Theorem ([3]). *Let M be a nonempty subset of a Hausdorff topological vector space E and $G : M \to 2^E$ be a KKM mapping. If $G(x)$ is closed in E for every $x \in M$ and compact for some $x \in M$ then*

$$\bigcap_{x \in M} G(x) \ \neq\ \emptyset.$$

Lemma 1.1. ([1]) *Let $(Y, \leq)$ be an ordered Banach space induced by a pointed closed convex cone C with nonempty* $\operatorname{int} C$. *For a, b, $c \in Y$, the following unifications hold:*

$$\begin{aligned}
c \not< a \quad &\textit{and} \quad a \geq b \quad \textit{implies} \quad b \not> c,\\
c \not> a \quad &\textit{and} \quad a \leq b \quad \textit{implies} \quad b \not< c.
\end{aligned}$$

2. Main results

First we consider the equivalence of Stampacchia-type of g-pseudomonotone vector variational inequalities and Minty-type of g-pseudomonotone vector variational inequalities, and then the existences of solutions to them mentioned.

Next we consider the existences of solutions to the more generalized vector F-complementarity problems with g-demi-pseudomonotone mappings.

In this paper, K is a bounded closed and convex subset of a real reflexive Banach space, $(Y, \leq)$ an ordered Banach space induced by a pointed closed convex cone C with $\mathrm{int}C \neq \emptyset$ and $L(X,Y)$ the space of all the continuous linear mappings from X into Y.

Theorem 2.1. *Let $T : K \to L(X,Y)$ be a hemicontinuous mapping, $g : K \to K$ and $F : K \to Y$ two convex mappings. Suppose that T is g-pseudomonotone with respect to F. Then for any given point $x_0 \in K$, the following are equivalent*

(i) $\langle T(x_0), g(x) - g(x_0)\rangle + F(g(x)) - F(g(x_0)) \not< 0$ *for* $x \in K$*;*
(ii) $\langle T(x), g(x) - g(x_0)\rangle + F(g(x)) - F(g(x_0)) \geq 0$ *for* $x \in K$*.*

Proof. We only prove that (ii) implies (i), the converse is obvious by Definition 1.3.

Suppose that (ii) holds. For any given $x \in K$ and $t \in (0,1)$, let $x_t = x_0 + t(x - x_0)$ then it follows from the convexities of g and F that

$$
\begin{aligned}
&t\langle T(x_0 + t(x - x_0)), g(x) - g(x_0)\rangle + t(F(g(x)) - F(g(x_0)))\\
&\geq \langle T(x_0 + t(x - x_0)), t(g(x) - g(x_0))\rangle + F(g(tx + (1-t)x_0)) - F(g(x_0))\\
&\geq 0.
\end{aligned}
$$

Hence

$$\langle T(x_0 + t(x - x_0)), g(x) - g(x_0)\rangle + F(g(x)) - F(g(x_0)) \geq 0.$$

Since T is hemicontinuous and C is closed, letting $t \to 0^+$ in the above inequality, we have

$$\langle T(x_0),\, g(x) - g(x_0)\rangle + F(g(x)) - F(g(x_0)) \geq 0.$$

Hence

$$\langle T(x_0), g(x) - g(x_0)\rangle + F(g(x)) - F(g(x_0)) \not< 0 \text{ for } x \in K.$$

□

Theorem 2.2. *Let $g : K \to K$, $F : K \to Y$ be continuous convex mappings and $T : K \to L(X,Y)$ a hemicontinuous mapping.*

If T is g-pseudomonotone with respect to F, then there exists $x \in K$ such that

$$\langle T(x), g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \;\not<\; 0 \text{ for } y \in K.$$

Proof. Define two set-valued mappings G_1, $G_2 : K \to 2^K$ as follows:

$$G_1(z) \;=\; \{x \in K :\; \langle T(x),\, g(z) - g(x)\rangle + F(g(z)) - F(g(x)) \;\not<\; 0\}$$

and

$$G_2(z) \;=\; \{x \in K : \; \langle T(z),\, g(z) - g(x)\rangle + F(g(z)) - F(g(x)) \;\geq\; 0\}.$$

Then G_1 is a KKM mapping. In fact, if it is not, then there exist $\{x_1, \ldots, x_n\} \subset K$, $x = \sum_{i=1}^{n} t_i x_i$ with $t_i > 0$ and $\sum_{i=1}^{n} t_i = 1$ such that $x \notin \bigcup_{i=1}^{n} G_1(x_i)$. It follows that

$$\langle T(x), g(x_i) - g(x)\rangle + F(g(x_i)) - F(g(x)) \;<\; 0,\; i = 1, \ldots, n.$$

By the convexities of F and g, we have

$$\begin{aligned} 0 &= \langle T(x),\; g(x) - g(x)\rangle + F(g(x)) - F(g(x)) \\ &\leq \sum_{i=1}^{n} t_i \langle T(x),\; g(x_i) - g(x)\rangle + \sum_{i=1}^{n} t_i F(g(x_i)) - F(g(x)) \\ &= \sum_{i=1}^{n} t_i \Big[\langle T(x),\; g(x_i) - g(x)\rangle + F(g(x_i)) - F(g(x))\Big] \\ &< 0. \end{aligned}$$

Hence $0 \in \operatorname{int} C$, which derives a contradiction. Thus G_1 is a KKM mapping. On the other hand, since T is g-pseudomonotone with respect to F, $G_1(z) \subset G_2(z)$ for $z \in K$ and so G_2 is also a KKM mapping. Also since K is bounded closed and convex, K is weakly compact. Furthermore, it is easy to check that $G_2(z) \subset K$ is closed and convex because F and g are continuous and convex. Hence $G_2(z)$ is weakly compact for each $z \in K$. It follows from F-KKM Theorem and Theorem 2.1 that

$$\bigcap_{z \in K} G_1(z) = \bigcap_{z \in K} G_2(z) \neq \emptyset.$$

Thus there exists $x \in K$ such that

$$\langle T(x),\; g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \;\not<\; 0 \text{ for } y \in K.$$

□

Definition 2.1. Let $g : K \to K$ be a single-valued mapping, $A : K \times K \to L(X, Y)$ and $F : K \to Y$ two nonlinear mappings. A is said to be g-demi-pseudomonotone with respect to F if the following two conditions hold;

(a) for each fixed $u \in K$, $A(u, \cdot)$ is g-pseudomonotone with respect to F. That is,

$$\langle A(u, x),\; g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \;\not<\; 0$$

implies

$$\langle A(u, y),\; g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \;\geq\; 0 \text{ for } x,\, y \in K.$$

(b) for each fixed $v \in K$, $A(v, \cdot)$ is vector demicontinuous, that is, for any net $\{u_\alpha\} \subset K$ and $w \in X$, $\{u_\alpha\}$ converges to u_0 in the weak topology of X implies that $\langle A(u_\alpha, v), w\rangle$ converges to $\langle A(u_0, v), w\rangle$ in the norm topology of Y.

Definition 2.2. A mapping $F : K \to Y$ is said to be completely continuous if for any net $\{u_\alpha\} \subset K$, $\{u_\alpha\}$ converges to u_0 in the weak topology implies that $F(u_\alpha)$ converges to $F(u_0)$ in the norm topology.

Theorem 2.3. *Let $K \subset X$ be a nonempty bounded closed and convex set, $F : K \to Y$ a completely continuous and convex mapping and $g : K \to K$ a continuous and convex mapping. Suppose that*

(i) *A is g-demi-pseudomonotone with respect to F;*
(ii) *for each $x \in K$, $A(x, \cdot) : K \to L(X, Y)$ is finite dimensional continuous, i.e., for any finite dimensional subspace $D \subset X$, $A(x, \cdot) : K \cap D \to L(X, Y)$ is continuous. Then there exists $u \in K$ such that*

$$\langle A(u, u),\ g(v) - g(u)\rangle + F(g(v)) - F(g(u)) \ \not< 0 \text{ for } v \in K.$$

Proof. Let $D \subset X$ be a finite-dimensional subspace with $K_D = D \cap K \neq \emptyset$. For each $w \in K$, consider the following problem:

Find $u_0 \in K_D$ such that

$$\langle A(w, u_0),\ g(v) - g(u_0)\rangle + F(g(v)) - F(g(u_0)) \ \not< 0 \text{ for } v \in K_D. \tag{2.1}$$

Since $K_D \subset D$ is bounded closed and convex, $A(w, \cdot)$ is continuous on K_D and g-pseudomonotone with respect to F for each fixed $w \in K$, from Theorem 2.2, we know that problem (2.1) has a solution $u_0 \in K_D$.

Now we define a set-valued mapping $T : K_D \to 2^{K_D}$ as follows:

$$T(w) = \{u \in K_D : \ \langle A(w, u), g(v) - g(u)\rangle + F(g(v)) - F(g(u)) \not< 0 \text{ for } v \in K_D\}, \quad \text{for } w \in K_D.$$

By Theorem 2.1, for each fixed $w \in K_D$,

$$\begin{aligned} &\{u \in K_D : \ \langle A(w, v), g(v) - g(u)\rangle + F(g(v)) - F(g(u)) \not< 0 \text{ for } v \in K_D\} \\ = &\{u \in K_D : \ \langle A(w, v), g(v) - g(u)\rangle + F(g(v)) - F(g(u)) \geq 0 \text{ for } v \in K_D\}. \end{aligned}$$

Since F is completely continuous and convex, it follows that $T : K_D \to 2^{K_D}$ has nonempty bounded closed and convex values. We also know that T is upper semicontinuous by the vector demicontinuity of $A(\cdot, u)$. By using the Glicksberg fixed point theorem [6], T has a fixed point $w_0 \in K_D$, i.e.,

$$\langle A(w_0, w_0),\ g(v) - g(w_0)\rangle + F(g(v)) - F(g(w_0)) \not< 0 \text{ for } v \in K_D. \tag{2.2}$$

Let $\mathcal{D} = \{D \subset X\colon D \text{ is a finite-dimensional subspace with } D \cap K \neq \emptyset\}$ and

$$W_D = \{u \in K : \ \langle A(u, v), g(v) - g(u)\rangle + F(g(v)) - F(g(u)) \geq 0 \text{ for } v \in K_D\} \quad \text{for } D \in \mathcal{D}.$$

By (2.2) and Theorem 2.1, we know that W_D is nonempty and bounded. Then the weak closure $cl(W_D)$ of W_D is weakly compact in D.

For any $D_i \in \mathcal{D}$, $i = 1, 2, \ldots, n$, we know that $W_{\bigcup_i D_i} \subset \cap W_{D_i}$. So $\{cl(W_D) : D \in \mathcal{D}\}$ has the finite intersection property. It follows that

$$\bigcap_{D \in \mathcal{D}} cl(W_D) \neq \emptyset.$$

Let $u \in \bigcap_{D \in \mathcal{D}} cl(W_D)$. We claim that

$$\langle A(u,u),\ g(v) - g(u)\rangle + F(g(v)) - F(g(u)) \not< 0 \text{ for } v \in K.$$

Indeed, for each $v \in K$, let $D \in \mathcal{D}$ be such that $v \in K_D$ and $u \in K_D$. Since W_D is weakly closed there exists a net $\{u_\alpha\} \subset W_D$ such that $\{u_\alpha\}$ converges to u with respect to the weak topology of X. It follows that

$$\langle A(u_\alpha, v),\ g(v) - g(u_\alpha)\rangle + F(g(v)) - F(g(u_\alpha)) \geq 0.$$

It follows that

$$\langle A(u,v),\ g(v) - g(u)\rangle + F(g(v)) - F(g(u)) \geq 0 \text{ for } v \in K,$$

by the vector demicontinuity of $A(\cdot, v)$ and the continuities of F and g. By Theorem 2.1, we know

$$\langle A(u,u),\ g(v) - g(u)\rangle + F(g(v)) - F(g(u)) \not< 0 \text{ for } v \in K.$$

□

Theorem 2.4. *Suppose that K is a nonempty closed convex cone and all the conditions of Theorem 2.3 hold. Furthermore, if $g(0) = 0$ and $F(0) = 0$, then there exists $u \in K$ such that*

$$\langle A(u,u),\ g(u)\rangle + F(g(u)) \not> 0 \text{ and}$$
$$\langle A(u,u),\ g(v)\rangle + F(g(v)) \not< 0 \text{ for } v \in K.$$

Proof. By Theorem 2.3, there exists $u \in K$ such that

$$\langle A(u,u),\ g(v) - g(u)\rangle + F(g(v)) - F(g(u)) \not< 0, \text{ for } v \in K. \tag{2.3}$$

Since $g(0) = 0$ and $F(0) = 0$, we have

$$\langle A(u,u),\ g(u)\rangle + F(g(u)) \not> 0.$$

On the other hand, any $w \in K$, substituting $v = u + w$ into (2.3), we have

$$\langle A(u,u),\ g(u+w) - g(u)\rangle + F(g(u+w)) - F(g(u)) \not< 0.$$

Since g and F are convex,

$$g(u+w) \leq g(u) + g(w)$$

and

$$F(g(u+w)) \leq F(g(u) + g(w)) \leq F(g(u)) + F(g(w))$$

It follows Lemma 1.1, that

$$\langle A(u,u),\ g(w)\rangle + F(g(w)) \not< 0 \text{ for } w \in K.$$

□

Remark 2.1. By putting $g = I$, the identity in Theorems 2.1, 2.2, 2.3 and 2.4, we obtain the corresponding results in Fang and Huang [4].

References

[1] Y. Q. Chen, *On the semi-monotone operator theory and applications*, J. Math. Anal. Appl. **231** (1999), 177–192.

[2] G. Y. Chen and X. Q. Yang, *The vector complementarity problem and its equivalences with the weak minimal element in ordered spaces*, J. Math. Anal. Appl. **153** (1990), 136–158.

[3] K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519–537.

[4] Y. P. Fang and N. J. Huang, *The vector F-complementarity problems with demipseudomonotone mappings in Banach spaces*, Appl. Math. Lett. **16** (2003), 1019–1024.

[5] Y. P. Fang and N. J. Huang, *Variational-like inequalities with generalized monotone mappings in Banach spaces*, J. Optim. Th. Appl. **118** (2003), 327–338.

[6] I. Glicksberg, *A further generalization of Kakutani fixed point theorem with applications to Nash equilibrium points*, Proc. Amer. Math. Soc. **36** (1952), 170–174.

[7] N. Hadjesavvas and S. Schaible, *A quasimonotone variational inequalities in Banach spaces*, J. Optim. Theo. Appl. **90** (1996), 95–111.

[8] M. K. Kang, N. J. Huang and B. S. Lee, *Generlized pseudomonotone set-valued variational-like inequalities*, Indian J. Math. **45** (2003), 251–264.

[9] S. Karamardian, *Complementarity over cones with monotone and pseudomonotone maps*, J. Optim. Theo. Appl. **18** (1976), 445–454.

[10] G. Kassay and J. Kolumban, *Variational inequalities given by semi-pseudomonotone mappings*, Nonlinear Analysis Forum **5** (2000), 35–50.

[11] D. T. Luc, *Existence results for densely pseudomonotone variational inequalities*, J. Math. Anal. Appl. **254** (2001), 291–308.

[12] R. U. Verma, *Nonlinear variational inequalities on convex subsets of Banach spaces*, Appl. Math. Lett. **10**(4) (1997), 25–27.

[13] H. Y. Yin, C.X. Hu and Z.X. Zhang, *The F-complementarity problems and its equivalence with the least element problem*, Acta Math. Sinica **44**(4) (2001), 679–686.

[14] G. X. Z. Yuan, *KKM Theorem and Applications in Nonlinear Analysis*, Marcel Dekker, New York, 1999.

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