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GENERALIZED F-IMPLICIT MULTIVALUED VARIATIONAL INEQUALITY PROBLEMS AND COMPLEMENTARITY PROBLEMS

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Abstract. In this paper, we study generalized F-implicit multivalued variational inequality problems on a real normed vector space setting. As an application, we study generalized F-implicit multivalued complementarity problems.

1. Preliminaries

Let X be a real normed vector space with a dual space X^* and $\langle \cdot, \cdot \rangle$ be the dual pair of X^* and X. Let X and X^* be endowed with their respective norm topologies. Let K be a nonempty closed convex subset of X. A function $F : K \to \mathbb{R}$ and mappings $g : K \to K$, T, $A : K \to 2^{X^*}$ are assumed to be given. The generalized F-implicit multivalued variational inequality problem (in short, GF-IMVIP) is finding an $x^* \in K$ such that

$$\sup_{s \in A(x^*),\, t \in T(x^*)} \langle N(s,t),\ g(x) - g(x^*) \rangle \ \geq\ F(g(x^*)) - F(g(x)) \text{ for } x \in K, \quad (1.1)$$

where $N : X^* \times X^* \to X^*$ be a mapping.

A solution of (1.1) is called a weak solution in the sense that if A and T have compact set-values, then for each $x \in K$ there are $s \in A(x^*)$, $t \in T(x^*)$ (depending on x) such that

$$\langle N(s,t),\ g(x) - g(x^*) \rangle \ \geq\ F(g(x^*)) - F(g(x)).$$

In contrast, we say that x^* is a strong solution of (1.1) if there exist $s^* \in A(x^*)$, $t^* \in T(x^*)$ such that

$$\langle N(s^*,t^*),\ g(x) - g(x^*) \rangle \ \geq\ F(g(x^*)) - F(g(x)) \text{ for } x \in K.$$

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The following generalized F-implicit multivalued complementarity problem (GF-IMCP) corresponding to (GF-IMVIP) is also considered as an applications:

Find $x^* \in K$, $s^* \in A(x^*)$ and $t^* \in T(x^*)$ such that

$$\langle N(s^*, t^*),\ g(x^*)\rangle + F(g(x^*)) \ = \ 0$$

and

$$\langle N(s^*, t^*),\ g(y)\rangle + F(g(y)) \ \geq \ 0 \text{ for } y \in K.$$

Remark 1.1. The following are some special cases of (GF-IMVIP) and (GF-IMCP).

1. If $T \equiv 0$, then (1.1) is equivalent to finding $x^* \in K$ and $s \in A(x^*)$ such that

$$\sup_{s\in A(x^*)} \langle N(s, N(s)),\ g(x) - g(x^*)\rangle \ \geq \ F(g(x^*)) - F(g(x)) \text{ for } x \in K, \quad (1.2)$$

   where $N : X^* \to X^*$ is a mapping.
2. If N is an identity mapping and $g(x) = x$, then (1.2) is collapse to the problem of finding $x^* \in K$, $s \in A(x^*)$ such that

$$\sup_{s\in A(x^*)} \langle s,\ x - g(x^*)\rangle \ \geq \ F(g(x^*)) - F(x) \text{ for } x \in K, \quad (1.3)$$

   introduced by Zeng *et al.* [11].
3. If A is single valued, then (1.3) is equivalent to finding $x^* \in K$ such that

$$\langle T(x^*),\ x - g(x^*)\rangle \ \geq \ F(g(x^*)) - F(x) \text{ for } x \in K,$$

   introduced and studied by Huang and Li [5] in a Banach space setting.
4. If $T \equiv 0$, N is an indentity and $g(y) = y$ for $y \in K$, then (GF-IMCP) reduces to finding $x^* \in K$ and $s^* \in A(x^*)$ such that

$$\langle s^*, g(x^*)\rangle + F(g(x^*)) \ = \ 0$$

   and

$$\langle s^*, y\rangle + F(y) \ \geq \ 0 \text{ for } y \in K,$$

   considered in [11].
5. There are also other special cases in [3, 4, 7-10].

There have been many reseaches on variational inequality problems and their corresponding complementarity problems, for examples, see [4, 7, 8, 11]. In this work, we aim to derive some existence results for weak and strong solutions of (GF-IMVIP) and corresponding results to (GF-IMCP).

The following theorems are essential for our researches.

Berge Theorem ([1]). *Let X, Y be topological spaces, $\phi : X \times Y \to \mathbb{R}$ be an upper semicontinuous function and $A : X \to 2^Y$ be an upper semicontinuous mapping with nonempty compact values. Then a function M defined by*

$M(x) = \max\limits_{s \in A(x)} \phi(x, s)$ *is upper semicontinuous on* X.

Fan's Lemma ([2]). *Let* K *be a nonempty subset of a Hausdorff topological vector space* X. *Let* $G : K \to 2^X$ *be a KKM mapping such that for any* $y \in K$, $G(y)$ *is closed and* $G(y^*)$ *is compact for some* $y^* \in K$. *Then there exists* $x^* \in K$ *such that* $x^* \in G(y)$ *for all* $y \in K$.

2. (GF-IMVIP)

Now, we consider the existence results of solutions for (GF-IMVIP).

Theorem 2.1. *Let a function* $F : K \to \mathbb{R}$ *be lower semicontinuous, a mapping* $g : K \to K$ *be continuous and* A, $T : K \to 2^{X^*}$ *be upper semicontinuous mappings with nonempty compact values. Let a mapping* $N : X^* \times X^* \to X^*$ *and a function* $h : K \times K \to \mathbb{R}$ *be given. Suppose that*

(1) $h(x, x) \geq 0$ *for all* $x \in K$,

(2) *for each* $x \in K$, *there are* $s \in A(x)$ *and* $t \in T(x)$ *such that for all* $y \in K$,
$$h(x, y) - \langle N(s, t), g(y) - g(x)\rangle \;\leq\; F(g(y)) - F(g(x)),$$

(3) *for each* $x \in K$, *the set* $\{y \in K : \; h(x, y) < 0\}$ *is convex,*

(4) *there is a nonempty compact convex subset* C *of* K *such that for every* $x \in K \backslash C$, *there is* $y \in C$ *such that for some* $s \in A(x)$, $t \in T(x)$,
$$\langle N(s, t), g(y) - g(x)\rangle \;<\; F(g(x)) - F(g(y)).$$

Then there exists $x^* \in K$ *which is a solution of (GF-IMVIP).*

Furthermore, the solution set of (GF-IMVIP) is compact.

Proof. Define $\Omega : K \to 2^C$ by
$$\Omega(y) = \Big\{x \in C : \max_{s \in A(x),\, t \in T(x)} \langle N(s, t),\; g(y) - g(x)\rangle \;\geq\; F(g(x)) - F(g(y))\Big\}$$
for all $y \in K$. By the Berge Theorem, we know that the function
$$x \mapsto \max_{s \in A(x),\, t \in T(x)} \langle N(s, t),\; g(y) - g(x)\rangle - F(g(x)) + F(g(y))$$
is upper semicontinuous on K. Hence the set
$$\Big\{x \in K : \max_{s \in A(x),\, t \in T(x)} \langle N(s, t),\; g(y) - g(x)\rangle \;\geq\; F(g(x)) - F(g(y))\Big\}$$
is closed in K and for each $y \in K$, the set
$$\Omega(y) = \Big\{x \in C : \max_{s \in A(x),\, t \in T(x)} \langle N(s, t),\; g(y) - g(x)\rangle \;\geq\; F(g(x)) - F(g(y))\Big\}$$
is compact in C due to the compactness of C.

Next, we claim that a family $\{\Omega(y) : y \in K\}$ has the finite intersection property, then the whole intersection $\bigcap\limits_{y \in K} \Omega(y)$ is nonempty and any element in the intersection $\bigcap_{y\in K} \Omega(y)$ is a solution of (GF-IMVIP). For any given nonempty finite subset L of K, let $C_L = Co(C \cup L)$, the convex hull of $C \cup L$. Then C_L is a compact convex subset of K. Define mappings $P, Q : C_L \to 2^{C_L}$, respectively, by

$$P(y) = \Big\{x \in C_L : \max_{s\in A(x),\, t\in T(x)} \langle N(s,t),\ g(y) - g(x)\rangle \ \geq\ F(g(x)) - F(g(y))\Big\}$$

and

$$Q(y) = \Big\{x \in C_L :\ h(x,y)\ \geq\ 0\Big\} \text{ for } y \in C_L.$$

It is obvious that $y \in P(y)$ for $y \in C_L$. Indeed,

$$0 = \langle N(s,t), g(y) - g(y)\rangle \geq F(g(y)) - F(g(y)) = 0$$

for all $s \in A(x)$, $t \in T(x)$.

It is easily shown that Q has closed set-values in C_L. Since for each $y \in C_L$, $\Omega(y) = P(y) \cap C$, if we prove that the whole intersection of the family $\{P(y) : y \in C_L\}$ is nonempty, then we can deduce that the family $\{\Omega(y) : y \in K\}$ has the finite intersection property from the fact that $L \subset C_L$ and condition (4). In order to deduce the conclusions of our theorem, we apply Fan's lemma by showing that P is a KKM mapping. Indeed, if P is not a KKM mapping, then Q is also not from the fact that $Q(y) \subset P(y)$ for each $y \in C_L$ by condition (2). Then there is a nonempty finite subset M of C_L such that

$$Co\, M \not\subset \bigcup_{u\in M} Q(u).$$

Thus there is an element $u^* \in \mathrm{Co}\, M \subset C_L$ such that $u^* \notin Q(u)$ for all $u \in M$, that is, $h(u^*, u) < 0$ for all $u \in M$. By condition (3), we have

$$u^* \in \mathrm{Co}\, M \subset \{u \in K :\ h(u^*, u) < 0\}$$

and hence $h(u^*, u^*) < 0$, which contradicts condition (1). Hence Q is a KKM-mapping and so is P. Therefore there exists $x^* \in K$, which is a solution of (GF-IMVIP).

Finally, to see that the solution set of (GF-IMVIP) is compact, it is sufficient to show that the solution set is closed, due to the coercivity condition (4). To this end, let B denote the solution set of (GF-IMVIP). Suppose that $\langle x_n\rangle$ is a sequence in B converging to some u. Fix any $x \in K$. For each n, there are $s_n \in A(x_n)$, $t_n \in T(x_n)$ such that

$$\langle N(s_n, t_n),\ g(x) - g(x_n)\rangle \ \geq\ F(g(x_n)) - F(g(x)). \tag{2.1}$$

Since T is an upper semicontinuous mapping with compact set-values and the set $\{x_n : n \in \mathbb{N}\} \cup \{u\}$ is compact, it follows that $T(\{x_n : n \in \mathbb{N}\} \cup \{u\})$ is compact [1]. Therefore without loss of generality, we may assume that the sequences $\langle s_n\rangle$ and $\langle t_n\rangle$ converge to some s and t, respectively. Then $s \in A(u)$, $t \in T(u)$ and by taking the limitinf in (2.1), we obtain

$$\langle N(s,t),\ g(x) - g(u)\rangle \ \geq\ F(g(u)) - F(g(x)).$$

Hence $u \in B$, which shows that B is closed. □

Remark 2.1. If $A \equiv 0$, $N : X^* \to X^*$ is an identity and $g(x) = x$ for all $x \in K$, then Theorem 2.1 reduces to Theorem 2.1 in [11]. Moreover, if T is single valued and X is a Banach space, then Theorem 2.1 reduces to Theorem 3.2 in [5].

Theorem 2.2. *Under the assumptions of Theorem* 2.1 *if, in addition, F is convex and $A(x^*)$, $T(x^*)$ are convex, then x^* is a strong solution of (GF-IMVIP), that is, there exists $s^* \in A(x^*)$, $t^* \in T(x^*)$ such that*

$$\langle N(s^*,t^*),\ g(x) - g(x^*)\rangle \ \geq\ F(g(x^*)) - F(g(x))$$

for all $x \in K$. Furthermore, the set of all strong solutions of (GF-IMVIP) is compact.

Proof. For $x^* \in K$ satisfying (1.1), since $A(x^*)$ and $T(x^*)$ are compact, the supremum is attained. That is,

$$\max_{s\in A(x^*),\, t\in T(x^*)} \langle N(s,t),\ g(x) - g(x^*)\rangle \ \geq\ F(g(x^*)) - F(g(x))$$

for all $x \in K$. Since $A(x^*)$, $T(x^*)$ are convex, by Kneser's minimax theorem [6] we have

$$\max_{s\in A(x^*),\, t\in T(x^*)} \inf_{x\in K} \langle N(s,t),\ g(x) - g(x^*)\rangle - F(g(x^*)) + F(g(x))$$
$$= \inf_{x\in K} \max_{s\in A(x^*),\, t\in T(x^*)} \langle N(s,t),\ g(x) - g(x^*)\rangle - F(g(x^*)) + F(g(x)) \ \geq\ 0.$$

Therefore, there exists $s^* \in A(x^*)$, $t^* \in T(x^*)$ such that

$$\langle N(s^*,t^*),\ g(x) - g(x^*)\rangle \ \geq\ F(g(x^*)) - F(g(x))$$

for all $x \in K$. Hence x^* is a strong solution of (GF-IMVIP). By the same argument shown in the proof of Theorem 2.1, the set of all strong solutions is compact. □

Remark 2.2. If $A \equiv 0$, N is an identity and $g(x) = x$ for all $x \in K$, then Theorem 2.2 reduces to Theorem 3.2 in [11]. Moreover, if T is single valued X is a Banach space, then Theorem 2.2 reduces to Theorem 3.4 in [5].

Theorem 2.3. *Let $F : K \to \mathbb{R}$ be convex and lower semicontinuous on any nonempty compact set, and $g : K \to K$ and $N : X^* \times X^* \to X^*$ be continuous. Let mappings A, $T : K \to 2^{X^*}$ be upper semicontinuous and have nonempty compact set-values. If*

(1) *for each $x \in K$, there are $s \in A(x)$, $t \in T(x)$ such that for all $y \in K$*

$$\langle N(s,t),\ g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \ \geq\ 0,$$

(2) *there is a nonempty compact convex subset C of K such that for every $x \in K\backslash C$, there is a $y \in C$ such that for some $s \in A(x)$, $t \in T(x)$*

$$\langle N(s,t),\ g(y) - g(x)\rangle < F(g(x)) - F(g(y)).$$

Then there exists an $x^* \in K$ which is a solution of (GF-IMVIP). Furthermore, the solution set of (GF-IMVIP) is compact. If in addition, $A(x^*)$, $T(x^*)$ are also convex, then x^* is a strong solution of (GF-IMVIP).

Proof. For a nonempty finite subset L of K, let $C_L = Co(C \cup L)$, then C_L is a nonempty compact convex subset of K. Define $P : C_L \to 2^{C_L}$ as

$$P(y) = \Big\{ x \in C_L : \max_{s \in A(x),\, t \in T(x)} \langle N(s,t),\ g(y) - g(x) \rangle \ \geq\ F(g(x)) - F(g(y)) \Big\}$$

and for each $y \in K$, let

$$\Omega(y) = \Big\{ x \in C : \max_{s \in A(x),\, t \in T(x)} \langle N(s,t),\ g(y) - g(x) \rangle + F(g(y)) - F(g(x)) \geq 0 \Big\}.$$

For each $x \in K$, $P(x)$ is nonempty by condition (1). By the Berge Theorem, we know that for each $y \in C_L$, $P(y)$ is closed in C_L and for each $y \in K$, $\Omega(y)$ is compact in C. Next we claim that P is a KKM-mapping. Indeed, if not, there is a nonempty finite subset M of C_L such that $\mathrm{Co}\, M \not\subset \bigcup_{x \in M} P(x)$. Then there is an $x^* \in \mathrm{Co} M \subset C_L$ such that

$$\max_{s \in A(x^*),\, t \in T(x^*)} \langle N(s,t),\ g(x) - g(x^*) \rangle < F(g(x^*)) - F(g(x)), \text{ for all } x \in M.$$

Since F is convex, the mapping

$$x \mapsto \max_{s \in A(x),\, t \in T(x)} \langle N(s,t),\ g(x) - g(x^*) \rangle + F(g(x))$$

is quasiconvex on C_L. Hence we can deduce that

$$\max_{s \in A(x^*),\, t \in T(x^*)} \langle N(s,t),\ g(x^*) - g(x^*) \rangle < F(g(x^*)) - F(g(x^*)),$$

which contradicts condition (1). Therefore P is a KKM mapping and by Fan's lemma we have $\bigcap_{x \in C_L} P(x) \neq \emptyset$. Let

$$u \in \bigcap_{x \in C_L} P(x),$$

then $u \in C$ by condition (2). Hence we have

$$\bigcap_{y \in L} \Omega(y) \ =\ \bigcap_{y \in L} P(y) \cap C \ \neq\ \emptyset,$$

for any nonempty finite subset L of K. Therefore, the whole intersection $\bigcap_{y \in K} \Omega(y)$ is nonempty. Let $x^* \in \bigcap_{y \in K} \Omega(y)$. Then x^* is a solution of (GF-IMVIP). Since C is compact, the solution set of (GF-IMVIP) is compact. Finally, if $T(x^*)$ is also convex, then by the same argument shown in the proof of Theorem 2.2, we can prove that x^* is a strong solution of (GF-IMVIP). □

3. (GF-IMCP)

We first establish the equivalence between strong solutions of (GF-IMVIP) and solutions of (GF-IMCP) on a closed convex cone K in X. The set K is assumed to be a closed convex cone in X.

Theorem 3.1. (i) *If x^* solves (GF-IMCP), then x^* is a strong solution of (GF-IMVIP);*
(ii) *If $F : K \to \mathbb{R}$ is a positive homogeneous and convex function and x^* is a strong solution of (GF-IMVIP), then x^* solves (GF-IMCP).*

Proof. Let x^* solve (GF-IMCP), then for $x^* \in K$, $s^* \in A(x^*)$ and $t^* \in T(x^*)$, we have

$$\langle N(s^*, t^*),\ g(x^*)\rangle + F(g(x^*)) \ = \ 0$$

and

$$\langle N(s^*, t^*),\ g(x)\rangle + F(g(x)) \ \geq \ 0 \text{ for } x \in K.$$

Hence

$$\langle N(s^*, t^*),\ g(x) - g(x^*)\rangle \ \geq \ F(g(x^*)) - F(g(x)) \text{ for } x \in K.$$

Thus x^* is a strong solution of (GF-IMVIP).
(ii) Let x^* be a strong solution of (GF-IMVIP) then there exist $s^* \in A(x^*)$, $t^* \in T(x^*)$ such that

$$\langle N(s^*, t^*),\ g(x) - g(x^*)\rangle \ \geq \ F(g(x^*)) - F(g(x)) \text{ for } x \in K. \tag{3.1}$$

Since $F : K \to \mathbb{R}$ is a positive homogeneous and convex function and set K is a closed convex cone in X, substituting $g(x) = 2g(x^*)$ and $g(x) = \frac{1}{2}g(x^*)$ in (3.1), we obtain

$$\langle N(s^*, t^*),\ g(x^*)\rangle \ \geq \ -F(g(x^*))$$

and

$$\langle N(s^*, t^*),\ g(x^*)\rangle \ \leq \ -F(g(x^*)),$$

which implies that

$$\langle N(s^*, t^*),\ g(x^*)\rangle + F(g(x^*)) \ = \ 0. \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$\langle N(s^*, t^*),\ g(x)\rangle + F(g(x)) \ \geq \ 0 \text{ for } x \in K.$$

Hence x^* is a solution of (GF-IMCP). □

Remark 3.1. If $T \equiv 0$, N is an indentity and $g(y) = y$ for $y \in K$, then Theorem 3.1 reduces to Theorem 3.1 considered in [11]. Moreover, if A is single-valued and X is a Banach space, then we obtain Theorem 3.1 considered in [5].

Theorem 3.2. *Let the assumptions of Theorem* 2.1 *hold. In addition, if F : $K \to \mathbb{R}$ is a positive homogeneous and convex function and A, T have convex set-values, then (GF-IMCP) has a solution. Furthermore the solution set is compact.*

Proof. Applying Theorems 2.2 and 3.1, we obtain the conclusion. □

Similarly by combining Theorems 2.3 and 3.1, we have the following result.

Theorem 3.3. *Let the assumptions of Theorem* 2.3 *hold. In addition, if* $F : K \to \mathbb{R}$ *is a positive homogeneous function and* A, T *have convex set-values, then (GF-IMCP) has a solution. Furthermore the solution set is compact.*

Remark 3.2. If $A \equiv 0$, N is an identity and $g(x) = x$ for all $x \in K$, then Theorem 3.3 reduces to Theorem 3.3 in [11]. Moreover, if T is single-valued and X is a Banach space, then Theorem 3.3 reduces to Theorem 3.3 in [5].

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