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LOCAL RESULTS FOR A CONTINUOUS ANALOG OF NEWTON'S METHOD

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Abstract. A local convergence result is provided for the continuous analog of Newton's method in a Banach space setting. The radius of convergence is larger, the error bounds tighter, and under the same or weaker hypotheses than before [12].

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet–differentiable operator defined on a closed subset $\mathcal{D}$ of a real Hilbert space $\mathcal{X}$ with values in a Hilbert space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure,

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they can be introduced and discussed in a general framework.

Let $x_0 \in \mathcal{D}$ be given. Assume:

for $x \in \mathcal{D}$, $F'(x)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ the space of bounded linear operators from $\mathcal{Y}$ into $\mathcal{X}$. Let $U(x_0, r) = \{x \in \mathcal{X},\ \| x - x_0 \| \leq r\}$ for some $r > 0$, such that

$$U(x_0, r) \subseteq \mathcal{D}, \tag{2}$$

$$\sup_{x \in U(x_0,r)} \| F'(x)^{-1}\ F'(x_0) \| \leq m(r), \tag{3}$$

$$\sup_{x \in U(x_0,r)} \| F'(x_0)^{-1}\ F^{(i)}(x) \| \leq M_i, \quad i = 0, 1 \tag{4}$$

and,

there exist a unique solution $x^\star$ of equation (1) in $U(x_0, r)$.

Let us now consider the Newton–type continuous method (NTCM) for solving equation (1):

$$\dot{x} = -\alpha\ F'(x)^{-1}\ F(x), \quad x(0) = x_0, \tag{5}$$

where, α is a given positive constant.

The main local convergence result of this study is:

Theorem 1.1. *If the above assumptions hold, and*

$$m(r)\ \ \| F'(x_0)^{-1}\ F(x_0) \| \leq r, \tag{6}$$

then

(i) *Equation (1) has a unique global solution $x(t) \in U(x^\star, r)$ for any $x_0 \in U(x^\star, r)$;*
(ii) *x_∞ exists, $F(x_\infty) = 0$, $x_\infty = x^\star$, so that*

$$\| x(t) - x^\star \| \leq r\ e^{-\alpha\ t} \quad (t > 0). \tag{7}$$

Theorem 1.1 improves the corresponding one given by the elegant study in [12].

Some special cases and applications are also provided in this study. Other related work can be found in [1]–[11], [13]–[15].

2. Proof of Theorem 1.1

As in [12], we let

$$Z(t) = x(t) - x^\star, \quad g(t) = \| Z(t) \|,$$

$$q(t) = \| F'(x_0)^{-1}\ F(x(t)) \|, \quad \dot{q} = \frac{dq}{dt}.$$

In view of (5), we get:

$$\begin{aligned} q\,\dot q &= -\alpha\;(F'(x)^{-1}\;F'(x)\;F'(x)^{-1}\;F'(x)\;F'(x)^{-1}\;F,\,F'(x)^{-1}\;F) \\ &= -\alpha\;q^2. \end{aligned} \tag{8}$$

Then, we have:

$$\begin{aligned} (8) \;\Longrightarrow\;& \dot q = -\alpha\;q \\ \Longrightarrow\;& q(t) = q(0)\;e^{-\alpha\;t} = \parallel F'(x_0)^{-1}\;F(x_0)\parallel\;\;e^{-\alpha\;t}. \end{aligned} \tag{9}$$

It then follows from (3), (5), and (9):

$$\begin{aligned} \parallel \dot x \parallel \;&\le\; \alpha\;\parallel F'(x)^{-1}\;F'(x_0)\parallel\;\parallel F'(x_0)^{-1}\;F(x)\parallel \\ &\le\; \alpha\;m(r)\;\;\parallel F'(x_0)^{-1}\;F(x_0)\parallel\;\;e^{-\alpha\;t}. \end{aligned} \tag{10}$$

Hence, we deduce x_∞ exists, $\parallel F'(x_0)^{-1}\;F(x_\infty)\parallel = 0$, which implies

$$\parallel F(x_\infty)\parallel = 0,$$

and by the uniqueness hypotheses, we obtain $x^\star = x_\infty$.

Moreover, in view of (3), and (10), we obtain in turn:

$$\begin{aligned} \parallel x(t) - x_\infty \parallel \;&\le\; \parallel \int_t^\infty \dot x(s)\;ds \parallel \\ &=\; \alpha\;\parallel \int_t^\infty (F'(x(s))^{-1}\;F'(x_0))(F'(x_0)^{-1}\;F(x(s)))\;ds \parallel \\ &\le\; \alpha\;m(r)\;\;\parallel F'(x_0)^{-1}\;F(x_0)\parallel\;\;\lim_{p\longrightarrow\infty}\int_t^p e^{-\alpha\;s}\;ds \\ &=\; m(r)\;\;\parallel F'(x_0)^{-1}\;F(x_0)\parallel\;\;\lim_{p\longrightarrow\infty}(e^{-\alpha\;p} - e^{-\alpha\;t}) \\ &=\; m(r)\;\;\parallel F'(x_0)^{-1}\;F(x_0)\parallel\;\;e^{-\alpha\;t} \\ &\le\; r\;e^{-\alpha\;t} \le r, \end{aligned} \tag{11}$$

which implies (7), and $x(t) \in U(x^\star, r)$ $(t > 0)$.

Moreover, as in (11), we get

$$\begin{aligned} \parallel x(t) - x_0 \parallel \;&=\; \parallel \int_0^1 \dot x(t)\;dt \parallel \\ &\le\; m(r)\parallel F'(x_0)^{-1}F(x_0)\parallel (1 - e^{-\alpha\;t}) \le r, \end{aligned} \tag{12}$$

which implies $x(t) \in U(x_0, r)$ $(t > 0)$.

Hence, we conclude $x(t) \in U(x_0, r) \cap U(x^\star, r)$. That completes the proof of Theorem 1.1.

3. Special cases and applications

The results in Theorem 1.1 reduce to the corresponding ones in [12] if $\mathcal{X} = \mathcal{Y} = \mathcal{D}$, and $F'(x_0)^{-1}\;F$ is replaced by F (non–affine invariant form). Otherwise they extend the applicability of Newton's method (5). Note also that the advantages of providing convergence results in affine invariant instead of non–affine invariant form are well known in the literature, and have been given in [6], [7].

Theorem 1.1 can be weakened in some cases

(i) Hypotheses (2)–(4) can be replaced by:

$$x_0 \in U(x^\star, r) \subseteq \mathcal{D}, \tag{13}$$

$$\| F'(x^\star)^{-1} \; (F'(x) - F'(x^\star)) \| \leq L \; \| x - x^\star \|, \quad L > 0 \tag{14}$$

for all $x \in \mathcal{D}$.

If $x \in U^\circ(x^\star, \frac{1}{L}) \subseteq \mathcal{D}$, then, we have

$$\| F'(x^\star)^{-1} \; (F'(x) - F'(x^\star)) \| \leq L \; \| x - x^\star \| < 1. \tag{15}$$

In view of (15), it follows from the Banach lemma on invertible operators [6], [7], that $F'(x)^{-1}$ exists, and

$$\| F'(x)^{-1} \; F'(x^\star) \| \leq (1 - L \; \| x - x^\star \|)^{-1}. \tag{16}$$

Hence, we can set

$$m(r) = \frac{1}{1 - L\, r}, \qquad r \in (0, \frac{1}{L}). \tag{17}$$

In this case (6) is replaced by

$$h = 4\, L \; \| F'(x_0)^{-1} \; F(x_0) \| \leq 1, \tag{18}$$

and

$$r \in [r_1, r_2], \tag{19}$$

for $h \neq 0$, if $r = r_2$, where, r_1, r_2 are the real zeros of quadratic polynomial

$$g(s) = L\, s^2 - s + \| F'(x_0)^{-1} \; F(x_0) \|, \tag{20}$$

given by

$$r_1 = \frac{1 - \sqrt{1 - h}}{2\, L}, \tag{21}$$

$$r_1 = \frac{1 + \sqrt{1 - h}}{2\, L}. \tag{22}$$

(ii) Estimates (18), and (19) can be replaced as follows:

Assume instead of (4):

$$\| F'(x^\star)^{-1} \; (F'(x_0) - F'(x^\star)) \| \leq m_1 \; \| x_0 - x^\star \| . \tag{23}$$

Then, we get:

$$\begin{aligned}\| F'(x_0)^{-1} F(x_0) \| &\leq \| F'(x_0)^{-1} F(x^\star) \| \times \\ &\quad \| F'(x^\star)^{-1} (F'(x_0) - F'(x^\star)) \| \\ &\leq \frac{m_1 r}{1 - L r}.\end{aligned} \tag{24}$$

In view of (11), we should have

$$m(r) \ \| F'(x_0)^{-1} F(x_0) \| \leq r \tag{25}$$

or by (17), and (24)

$$\frac{1}{1 - L r} \frac{m_1 r}{1 - L r} \leq r, \tag{26}$$

or

$$r^\star = \frac{1 - \sqrt{m_1}}{L}, \tag{27}$$

provided that

$$m_1 \in [0, 1). \tag{28}$$

We refer the reader to [5]–[7], where, in the more general setting of a Banach space, we have provided a larger radius of convergence for many interesting examples by using the theoretical approach given in Section 2. The same examples can be used here by introducing the norms using standard inner products, and replacing $\mathcal{C}[a, b]$ by $L^2[a, b]$ in the appropriate places.

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