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LOCATING ROOTS OF A CERTAIN CLASS OF POLYNOMIALS

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Abstract. We introduce a special class of real recurrent polynomials f_m ($m \geq 1$) of degree $m+1$, with positive roots s_m, which are decreasing as m increases. The first root s_1, as well as the last one denoted by s_∞ are expressed in closed form, and enclose all s_m ($m > 1$).

This technique is also used to find weaker than before [6] sufficient convergence conditions for some popular iterative processes converging to solutions of equations.

1. Introduction

We introduce a special class of recurrent polynomials f_m ($m \geq 1$) of degree $m+1$ with real coefficients.

Then, we find sufficient conditions under which each polynomial f_m has a positive root s_m, such that $s_{m+1} \leq s_m$ ($m \geq 1$). The first root s_1, as well as the last one denoted by s_∞ are expressed in simple closed form.

Applications are provided. In the first one, we show how to use s_1 and s_∞ to locate any s_m belonging in $(s_\infty, s_1]$ ($m \geq 1$).

In the second one, using this technique on Newton's method (16), we show that the famous for its simplicity and clarity Newton–Kantorovich condition (46) for solving equations can always replaced by a weaker one (49).

We also show how to use our results to generate majorizing sequences appearing in connection for solving abstract equations in a Banach space setting using Newton–type methods [3], [6].

2. Enclosing roots of polynomials

We introduce the main result of this section on enclosing roots of polynomials.

Theorem 2.1. *Assume:*

there exist constants $K \geq 0$, $M \geq 0$, $\mu \geq 0$, $L \geq 0$, $\ell \geq 0$, and $\eta \geq 0$;

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Define a sequence of polynomials $\{f_n\}$ *(*$n \geq 1$*) on* $[0, +\infty)$ *by:*

$$\begin{aligned} f_n(s) = K\ s^n\ \eta + 2\ (M\ (1 + s + s^2 + \cdots + s^{n-1})\ \eta + \mu) \\ + 2\ s\ L\ (1 + s + s^2 + \cdots + s^n)\ \eta + 2\ s\ (\ell - 1). \end{aligned} \tag{1}$$

f_1 *has a minimal root* s_1 *in* $[0, 1)$*, satisfying*

$$s_1 \leq \delta_+, \tag{2}$$

where,

$$\delta_+ = \frac{2\ (K - 2\ M)}{K + \sqrt{K^2 + 8\ L\ (K - 2\ M)}}, \tag{3}$$

$$2\ M < K, \tag{4}$$

and

$$(2\ L + K)\ \eta < 2\ (1 - \ell) \tag{5}$$

Then, each polynomial f_n *(*$n \geq 1$*) has a minimal root* s_n *in* $[0, 1)$.
Moreover, the following estimates hold for all $n \geq 1$*:*

$$s_\infty \leq s^\star \leq s_{n+1} \leq s_n, \tag{6}$$

where,

$$s^\star = \lim_{n \longrightarrow \infty} s_n, \tag{7}$$

and,

s_∞ *is the minimal root of polynomial*

$$\overline{f}_\infty(s) = (1 - \ell)\ s^2 - (1 - \ell - L\ \eta + \mu)\ s + M\ \eta + \mu \tag{8}$$

in $[0, 1)$.

Proof. We need to find a relationship between two consecutive f_m:

$$\begin{aligned} f_{m+1}(s) &= K\ s^{m+1}\ \eta + 2\ \Big(M\ (1 + s + s^2 + \cdots + s^{m-1} + s^m)\ \eta + \mu\Big) \\ &\quad + 2\ s\ L\ (1 + s + \cdots + s^m + s^{m+1})\ \eta + 2\ s\ (\ell - 1) \\ &= K\, s^{m+1}\, \eta - K\ s^m\ \eta + K\ s^m\ \eta \\ &\quad + 2\ \Big(M\ (1 + s + s^2 + \cdots + s^{m-1})\ \eta + \mu\Big) + 2\ M\ s^m\ \eta \\ &\quad + 2\ s\ L\ (1 + s + \cdots + s^m)\ \eta + 2\ s\ L\ s^{m+1}\ \eta + 2\ s\ (\ell - 1) \\ &= f_m(s) + K\ s^{m+1}\ \eta - K\ s^m\ \eta + 2\ M\ s^m\ \eta + 2\ s\ L\ s^{m+1}\ \eta \\ &= f_m(s) + g(s)\ s^m\ \eta, \end{aligned} \tag{9}$$

where,

$$g(s) = 2\ L\ s^2 + K\ s + 2\ M - K. \tag{10}$$

Note that in view of (4), function g has a positive zero δ_+ given by (3), and

$$g(s) < 0 \qquad s \in (0, \delta_+). \tag{11}$$

By hypothesis, the function f_1 has a minimal positive zero s_1. Using (4), it is simple algebra to show $s_1 \in [0, 1)$. It then follows from (9) and (10):

$$\begin{aligned} f_2(s_1) &= f_1(s_1) + g(s_1)\ s_1^m\ \eta \\ &= g(s_1)\ s_1^m\ \eta < 0, \end{aligned} \tag{12}$$

since $f_1(s_1) = 0$, and $g(s_1) < 0$. We also have from (1):

$$f_m(0) = 2\ (M\ \eta + \mu) > 0 \quad (m \geq 1). \tag{13}$$

It follows from the intermediate value theorem that there exists a minimal $s_2 \in (0, s_1)$, such that $f_2(s_2) = 0$. Let us assume: there exists $s_m \in (0, s_{m-1})$, with $f_m(s_m) = 0$. As in (12) we have

$$f_{m+1}(s_m) = f_m(s_m) + g(s_m)\ s_m^m\ \eta < 0. \tag{14}$$

It follows from the intermediate value theorem that there exists a minimal $s_{m+1} \in (0, s_m)$, such that $f_{m+1}(s_{m+1}) = 0$.

We also have for $f_\infty(s) = \lim\limits_{m\to\infty} f_m(s)$, $s \in [0, 1)$:

$$f_\infty(s_\infty) = 2\left(\frac{M}{1 - s_\infty}\ \eta + \mu\right) + \frac{2\ s_\infty\ L}{1 - s_\infty}\ \eta + 2\ s_\infty\ (\ell - 1) = 0,$$

by the choice of s_∞. Note also that by (5), and (8), s_∞ exists in $(0, 1)$.

Sequence $\{s_m\}$ is non–increasing, bounded below by zero, and as such it converges to its unique maximum lowest bound $s^\star$ satisfying $s^\star \geq s_\infty$.

That completes the proof of Theorem 2.1. $\square$

Remark 1. The existence of s_1 can be guaranteed by any of the conditions below:

The discriminant Δ of polynomial f_1 is non–negative, or

$$(K + 4\ L + 2\ M)\ \eta < 2\ (1 - \ell - \mu). \tag{15}$$

In this case, (5) can be dropped, since it is implied by (15). Indeed, if $\Delta \geq 0$, and (5) holds, it is simple algebra to show s_1 exists, and $s_1 \in [0, 1)$. Moreover, by applying the intermediate value theorem on f_1 for $s \in [0, 1)$, we see again that s_1 exists, and $s_1 \in [0, 1)$.

3. Applications

Let us provide an example where s_2 is in $[s_\infty, s_1)$.

Example 3.1. Let $K = 1$, $L = .5$, $M = .25$, and $\eta = \mu = \ell = .01$. Then, using (1), and (8), we obtain:

$$f_1(s) = .01\ s^2 - 1.96\ s + .025, \qquad f_2(s) = .01\ s^3 + .02\ s^2 - 1.965\ s + .025,$$

$$f_\infty(s) = .99\ s^2 - .995\ s + .0125,$$

$$s_1 = .01275, \qquad s_2 = 0.01274, \qquad s_\infty = .01272.$$

Note also that hypotheses (4), (5), and (15) are satisfied with the above choices of K, L, M, η, μ, and ℓ.

We shall show that Theorem 2.1 can be used to provide new sufficient convergence conditions for the semilocal convergence of Newton–type methods (NTM):

$$x_{n+1} = x_n - A(x_n)^{-1} \; (F(x_n) + G(x_n)) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}) \tag{16}$$

to a locally unique solution $x^\star$ of equation

$$F(x) + G(x) = 0. \tag{17}$$

Here, F is a Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ on a Banach space $\mathcal{X}$, with values in a Banach space $\mathcal{Y}$, $G : \mathcal{D} \longrightarrow \mathcal{Y}$ is a continuous operator, and $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators is an approximation to $F'(x)$ [1], [3], [6].

We need a lemma on majorizing sequences for (NTM).

Lemma 3.2. *Under the hypotheses of Theorem 2.1, define δ_0, and δ_∞ by:*

$$\delta_0 = \frac{K \; \eta + 2 \; \mu}{1 - L \; \eta - \ell}, \tag{18}$$

$$\delta_\infty = 2 \; s_\infty, \tag{19}$$

and further, assume:

$$s_1 \leq \delta_+. \tag{20}$$

Choose:

$$\frac{\delta}{2} \in [s_1, \delta_+]. \tag{21}$$

Then, sequence $\{t_n\}$ $(n \geq 0)$ given by

$$\begin{aligned} & t_0 = 0, \quad t_1 = \eta, \\ & t_{n+2} = t_{n+1} + \frac{K \; (t_{n+1} - t_n) + 2 \; (M \; t_n + \mu)}{2 \; (1 - L \; t_{n+1} - \ell)} \; (t_{n+1} - t_n), \quad (n \geq 0), \end{aligned} \tag{22}$$

is well defined, nondecreasing, bounded above by

$$t^{\star\star} = \frac{2 \; \eta}{2 - \delta}, \tag{23}$$

and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$.

Moreover the following estimates hold for all $n \geq 1$:

$$t_{n+1} - t_n \leq \frac{\delta}{2} \; (t_n - t_{n-1}) \leq \left(\frac{\delta}{2}\right)^n \eta, \tag{24}$$

and

$$t^{\star} - t_n \leq \frac{2\ \eta}{2-\delta}\ \left(\frac{\delta}{2}\right)^n.$$

Note that the most appropriate choice for δ seems to be $\delta = 2\ s_1$.

Proof. We shall show using induction on the integer m:

$$\begin{aligned} 0 &< t_{m+2} - t_{m+1} \\ &= \frac{K\ (t_{m+1} - t_m) + 2\ (M\ t_m + \mu)}{2\ (1 - L\ t_{m+1} - \ell)}\ (t_{m+1} - t_m) \\ &\leq \frac{\delta}{2}\ (t_{m+1} - t_m), \end{aligned} \tag{25}$$

and

$$\ell + L\ t_{m+1} < 1. \tag{26}$$

If (25), and (26) hold, we have (24) holds, and

$$\begin{aligned} t_{m+2} &\leq t_{m+1} + \frac{\delta}{2}\ (t_{m+1} - t_m) \\ &\leq t_m + \frac{\delta}{2}\ (t_m - t_{m-1}) + \frac{\delta}{2}\ (t_{m+1} - t_m) \\ &\leq \eta + \left(\frac{\delta}{2}\right)\ \eta + \cdots + \left(\frac{\delta}{2}\right)^{m+1}\ \eta \\ &= \frac{1 - \left(\frac{\delta}{2}\right)^{m+2}}{1 - \frac{\delta}{2}}\ \eta \\ &< \frac{2\ \eta}{2-\delta} = t^{\star\star} \qquad \text{by (23)}. \end{aligned} \tag{27}$$

It will then also follow that sequence $\{t_m\}$ is increasing, bounded above by $t^{\star\star}$, and as such it will converge to some $t^{\star} \in [0, t^{\star\star}]$.

Estimates (25) and (26) hold by the initial conditions for $m = 0$. Indeed (25) and (26) become:

$$\begin{aligned} 0 < t_2 - t_1 &= \frac{K\ (t_1 - t_0) + 2\ (M\ t_0 + \mu)}{2\ (1 - L\ t_1 - \ell)}\ (t_1 - t_0) \\ &= \frac{K\ \eta + 2\ \mu}{2\ (1 - L\ \eta - \ell)}\ (t_1 - t_0) = \frac{\delta_0}{2}\ (t_1 - t_0) \leq \frac{\delta}{2}\ (t_1 - t_0), \end{aligned}$$

$$L\ \eta + \ell < 1,$$

which are true by the choice of δ_0, δ, (5), (22), and the initial conditions. Let us assume (24)–(26) hold for all $m \leq n+1$.

Estimate (25) can be re–written as

$$K\ (t_{m+1}-t_m)+2\ (M\ t_m+\mu)\leq(1-L\ t_{m+1}-\ell)\ \delta$$

or

$$K\ (t_{m+1}-t_m)+2\ (M\ t_m+\mu)+\delta\ L\ t_{m+1}+\delta\ \ell-\ell\leq 0, \tag{28}$$

or

$$K\left(\frac{\delta}{2}\right)^m \eta+2\left(M\ \frac{1-\left(\frac{\delta}{2}\right)^m}{1-\frac{\delta}{2}}\ \eta+\mu\right)+\delta\ L\ \frac{1-\left(\frac{\delta}{2}\right)^{m+1}}{1-\frac{\delta}{2}}\ \eta+\delta\ (\ell-1)\leq 0. \tag{29}$$

Replace $\frac{\delta}{2}$ by s, and define functions f_m on $[0,+\infty)$ $(m\geq 1)$ by (1).
Estimate (29) certainly holds, if:

$$f_m(s)\leq 0 \qquad s\in[s_1,\ \delta_+], \qquad (m\geq 1). \tag{30}$$

Estimate (30) holds by Theorem 2.1. That completes the induction for (25), since (21) holds.

Finally, sequence $\{t_n\}$ is non–decreasing, bounded above by $t^{\star\star}$, and as such that it converges to its unique least upper bound $t^\star$.

That completes the proof of Lemma 3.2.

□

By simply replacing the majorizing sequence in [6] by (22), we obtain the following semilocal convergence theorem for (NTM).

Theorem 3.3. *Let $F\ :\ \mathcal{D}\subseteq\mathcal{X}\longrightarrow\mathcal{Y}$ be a Fréchet–differentiable operator, $G\ :\ \mathcal{D}\longrightarrow\mathcal{Y}$ be a continuous operator, and let $A(x)\in\mathcal{L}(\mathcal{X},\mathcal{Y})$ be an approximation of $F'(x)$. Assume that there exist an open convex subset $\mathcal{D}$ of $\mathcal{X}$, $x_0\in\mathcal{D}$, a bounded inverse $A(x_0)^{-1}$ of $A(x_0)$, and constants $K>0$, $M>0$, $\mu_0\geq 0$, $\mu_1\geq 0$, $L>0$, $\ell\geq 0$, $\eta>0$, such that for all $x,y\in\mathcal{D}$:*

$$\parallel A(x_0)^{-1}\ [F(x_0)+G(x_0)]\parallel\leq\eta, \tag{31}$$
$$\parallel A(x_0)^{-1}\ [F'(x)-F'(y)]\parallel\leq K\ \parallel x-y\parallel, \tag{32}$$
$$\parallel A(x_0)^{-1}\ [F'(x)-A(x)]\parallel\leq M\ \parallel x-x_0\parallel+\mu_0, \tag{33}$$
$$\parallel A(x_0)^{-1}\ [A(x)-A(x_0)]\parallel\leq L\ \parallel x-x_0\parallel+\ell, \tag{34}$$
$$\parallel A(x_0)^{-1}\ [G(x)-G(y)]\parallel\leq\mu_1\ \parallel x-y\parallel, \tag{35}$$
$$\overline{U}(x_0,t^\star)=\{x\in\mathcal{X},\ \parallel x-x_0\parallel\leq t^\star\}\subseteq\mathcal{D},$$

and the hypotheses of Lemma 3.2 hold with $\mu=\mu_0+\mu_1$.

Then, sequence $\{x_n\}$ $(n \geq 0)$ *generated by (NTM) is well defined, remains in* $\overline{U}(x_0, t^\star)$ *for all* $n \geq 0$, *and converges to a solution* $x^\star$ *of equation (17) in* $\overline{U}(x_0, t^\star)$.

Moreover, the following estimates hold for all $n \geq 0$:

$$\| x_{n+1} - x_n \| \leq t_{n+1} - t_n, \tag{36}$$

and

$$\| x_n - x^\star \| \leq t^\star - t_n, \tag{37}$$

where, sequence $\{t_n\}$ $(n \geq 0)$, *and* $t^\star$ *are given in Lemma 3.2.*

Furthermore, the solution $x^\star$ *of equation (17) is unique in* $\overline{U}(x_0, t^\star)$ *provided that:*

$$\left(\frac{K}{2} + M + L \right) t^\star + \mu + \ell < 1.$$

Application 3.4. *Using (31)–(34), and hypothesis*

$$h_K = \sigma\ \eta \leq \frac{1}{2}\ (1-b)^2, \qquad \mu + \ell < 1 \tag{38}$$

where, $\sigma = \max\{K, M+L\}$, *with* $b = \mu + \ell$, *a semilocal convergence theorem was provided in* [5], [8], [9].

(a) *Let us compare the error bounds in this case. The majorizing sequence given in* [5], [8], [9], *is:*

$$\begin{aligned} &v_0 = 0, \quad v_1 = \eta, \\ &v_{n+2} = v_{n+1} + \frac{f(v_{n+1})}{q(v_{n+1})}, \qquad (n \geq 0), \end{aligned} \tag{39}$$

where,

$$f(v) = \frac{\sigma}{2}\ v^2 - (1-b)\ v + \eta,$$

and

$$q(v) = 1 - L\ v - \ell.$$

We now show that the error bounds obtained in Theorem 3.3 are more precise than the corresponding ones in the above references using (38).

Proposition 3.5. *Under the hypotheses of Theorem 3.3, and condition (38), the following error bounds hold:*

$$t_{n+1} \leq v_{n+1} \quad (n \geq 1), \tag{40}$$

$$t_{n+1} - t_n \leq v_{n+1} - v_n \quad (n \geq 1), \tag{41}$$

$$t^\star - t_n \leq v^\star - v_n \quad (n \geq 0), \tag{42}$$

and

$$t^\star \leq v^\star. \tag{43}$$

Moreover strict inequality holds in (40) and (41) if $K < M + L$.

Proof. We use mathematical induction on m to first show (40) and (41). For $n = 0$ in (22) we obtain:

$$\begin{aligned} t_2 - \eta &= \frac{\frac{K}{2}\ \eta^2 + \mu\ \eta}{1 - \ell - L\ \eta} \\ &\leq \frac{\frac{\sigma}{2}\ \eta^2 + (M \cdot 0 + \mu)\ \eta}{1 - \ell - L\ \eta} \\ &\leq \frac{\frac{\sigma}{2}\ \eta^2 + M\ (\eta - 0) + \mu\ (\eta - 0) - q(0)\ (\eta - 0) + f(0)}{q(\eta)} \\ &\leq \frac{\frac{\sigma}{2}\ v_1^2 - (1 - \mu - \ell)\ v_1 + \eta - (\sigma - M - L)\ v_0\ (v_1 - v_0)}{q(v_1)} \\ &\leq \frac{f(v_1)}{q(v_1)} = v_2 - v_1, \end{aligned}$$

and

$$t_2 \leq v_2.$$

Assume:

$$t_{i+1} \leq v_{i+1}, \quad t_{i+1} - t_i \leq v_{i+1} - v_i. \tag{44}$$

Using (22), (39), and (44), we obtain in turn:

$$\begin{aligned} t_{i+2} - t_{i+1} &= \frac{\frac{K}{2}\ (t_{i+1} - t_i)^2 + (M\ t_i + \mu)\ (t_{i+1} - t_i)}{1 - \ell - L\ t_{i+1}} \\ &\leq \frac{\frac{\sigma}{2}\ (v_{i+1} - v_i)^2 + (M\ v_i + \mu)\ (v_{i+1} - v_i)}{q(v_{i+1})} \\ &= \frac{\frac{\sigma}{2}\ (v_{i+1} - v_i)^2 + (M\, v_i + \mu - q(v_i))(v_{i+1} - v_i) + f(v_i)}{q(v_{i+1})} \\ &= \frac{\frac{\sigma}{2}\ v_{i+1}^2 - (1 - \mu - \ell)\ v_{i+1} + \eta - (\sigma - M - L)\ v_i\ (v_{i+1} - v_i)}{q(v_{i+1})} \\ &\leq \frac{f(v_{i+1})}{q(v_{i+1})} \\ &= v_{i+2} - v_{i+1}, \end{aligned}$$

which show (40) and (41) for all ($n \geq 1$).

Let $j \geq 0$, we can get:

$$\begin{aligned} t_{i+j} - t_i &\leq (t_{i+j} - t_{i+j-1}) + (t_{i+j-1} - t_{i+j-2}) + \cdots + (t_{i+1} - t_i) \\ &\leq (v_{i+j} - v_{i+j-1}) + (v_{i+j-1} - v_{i+j-2}) + \cdots + (v_{i+1} - v_i) \\ &\leq v_{i+1} - v_i. \end{aligned} \tag{45}$$

By letting $j \to \infty$ in (45) we obtain (42).

Finally (42) implies (43) (since $t_1 = v_1 = 0$). It can easily be seen from (22), and (39), that strict inequality holds in (40) and (41) if $K < M + L$.

That completes the proof of Proposition 3.5.

□

Note also that the above advantages hold even if hypotheses of Theorem 3.3 are replaced by (38).

(b) *We can now compare our Theorem 3.3 with the corresponding one in* [9] *in the case of Newton's method ($A(x) = F'(x)$, $G(x) = 0$, ($x \in \mathcal{D}$)):*

Hypothesis (38) reduces to the famous for its simplicity and clarity Newton–Kantorovich hypothesis [5], [6] [8], [9], *for solving nonlinear equations*

$$h_K = K\ \eta \leq \frac{1}{2}, \tag{46}$$

since $\sigma = K$, and $\mu_0 = \mu_1 = \ell = M = 0$.

Note that in this case, functions f_m ($m \geq 1$) should be defined by

$$f_m(s) = \Big(K\ s^{m-1} + 2\ L\ (1 + s + s^2 + \cdots + s^m)\Big)\ \eta - 2,$$

and

$$f_{m+1}(s) = f_m(s) + g(s)\ s^{m-1}\ \eta.$$

But this time, the conditions corresponding to Theorem 2.1 and Lemma 3.2 should be:

$$\delta_1 = \max\{\frac{\delta_0}{2},\ \delta_+\} \leq s_\infty = 1 - L\ \eta, \tag{47}$$

whereas,

$$\frac{\delta}{2} \in \Big[\delta_1,\ \delta_\infty\Big]. \tag{48}$$

Howeover, it is simple algebra to show that conditions (47)–(48) reduce to:

$$h_A = \overline{L}\ \eta \leq \frac{1}{2}, \tag{49}$$

where,

$$\overline{L} = \frac{1}{8}\ \Big(K + 4\ L + \sqrt{K^2 + 8\ K\ L}\Big).$$

Note also that

$$L \leq K \tag{50}$$

holds in general, and $\frac{K}{L}$ *can be arbitrarily large.*
In view of (46), (49) and (50), we get

$$h_K \leq \frac{1}{2} \Longrightarrow h_A \leq \frac{1}{2}, \tag{51}$$

but not necessarily vice verca unless if $L = K$.

In the example that follows, we show that $\frac{K}{L}$ can arbitrarily large. Indeed:

Example 3.6. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$, and define scalar functions F and G by

$$F(x) = c_0\ x + c_1 + c_2\ \sin e^{c_3\ x}, \qquad G(x) = 0, \tag{52}$$

where, c_i, $i = 0, 1, 2, 3$ are given parameters. Using (52), it can easily be seen that for c_3 large and c_2 sufficiently small, $\frac{K}{L}$ can be arbitrarily large.

In the next examples, we show (46) is violated but (49) holds.

Example 3.7. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$, $U_0 = \{x\ :\ |x - x_0| \leq 1 - \beta\}$, $\beta \in \left[0, \frac{1}{2}\right)$, and define function F on U_0 by

$$F(x) = x^3 - \beta. \tag{53}$$

Using hypotheses of Theorem 3.3, we get:

$$\eta = \frac{1}{3}\ (1 - \beta), \quad L = 3 - \beta, \quad \text{and} \quad K = 2\ (2 - \beta).$$

The Newton–Kantorovich condition (46) is violated, since

$$\frac{4}{3}\ (1 - \beta)\ (2 - \beta) > 1 \quad \text{for all} \quad \beta \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that Newton's method (16) converges to $x^\star = \sqrt[3]{\beta}$, starting at $x_0 = 1$.

However, our condition (49) is true for all $\beta \in I = \left[.450339002, \frac{1}{2}\right)$. Hence, the conclusions of our Theorem 3.3 can apply to solve equation (53) for all $\beta \in I$.

Example 3.8. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ be the space of real–valued continuous functions defined on the interval $[0, 1]$ with norm

$$\parallel x \parallel = \max_{0 \leq s \leq 1} |x(s)|.$$

Let $\theta \in [0, 1]$ be a given parameter. Consider the "Cubic" integral equation

$$u(s) = u^3(s) + \lambda\, u(s) \int_0^1 q(s,t)\, u(t)\, dt + y(s) - \theta. \tag{54}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1]\times[0,1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. Equations of the form (54) arise in the kinetic theory of gasses [3]. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s,t) = \dfrac{s}{s+t}$, for all $s \in [0,1]$, and $t \in [0,1]$, with $s + t \neq 0$. If we let $\mathcal{D} = U(u_0, 1-\theta)$, and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda\, x(s) \int_0^1 q(s,t)\, x(t)\, dt + y(s) - \theta, \tag{55}$$

for all $s \in [0,1]$, then every zero of F satisfies equation (54).

We have the estimates:

$$\max_{0\le s\le 1} \left| \int \frac{s}{s+t}\, dt \right| = \ln 2.$$

Therefore, if we set $\xi = \parallel F'(u_0)^{-1} \parallel$, then it follows from hypotheses of Theorem 3.3 that

$$\eta = \xi\ (|\lambda|\ \ln 2 + 1 - \theta),$$

$$K = 2\ \xi\ (|\lambda|\ \ln 2 + 3\,(2-\theta)) \quad \text{and} \quad L = \xi\ (2\ |\lambda|\ \ln 2 + 3\,(3-\theta)).$$

It follows from Theorem 3.3 that if condition (49) holds, then problem (54) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis (46).

Note also that $L < K$ for all $\theta \in [0,1]$.

Example 3.9. Consider the following nonlinear boundary value problem [3]

$$\begin{cases} u'' = -u^3 - \gamma\ u^2 \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 Q(s,t)\ (u^3(t) + \gamma\ u^2(t))\ dt \tag{56}$$

where, Q is the Green function:

$$Q(s,t) = \begin{cases} t\ (1-s), & t \le s \\ s\ (1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0\le s\le 1} \int_0^1 |Q(s,t)| = \frac{1}{8}.$$

Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$, with norm

$$\| x \| = \max_{0 \le s \le 1} |x(s)|.$$

Then problem (56) is in the form (17), where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)]\,(s) = x(s) - s - \int_0^1 Q(s,t)\ (x^3(t) + \gamma\ x^2(t))\ dt,$$

and

$$G(x)\,(s) = 0.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v]\,(s) = v(s) - \int_0^1 Q(s,t)\ (3\ x^2(t) + 2\ \gamma\ x(t))\ v(t)\ dt.$$

If we set $u_0(s) = s$, and $\mathcal{D} = U(u_0, R)$, then since $\| u_0 \| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\ \gamma < 5$, then

$$\begin{aligned}
\| I - F'(u_0) \| \quad &\le \quad \frac{3\ \ \| u_0 \|^2 + 2\ \gamma\ \ \| u_0 \|}{8} = \frac{3 + 2\ \gamma}{8}, \\
\| F'(u_0)^{-1} \| \quad &\le \quad \frac{1}{1 - \dfrac{3 + 2\ \gamma}{8}} = \frac{8}{5 - 2\ \gamma}, \\
\| F(u_0) \| \quad &\le \quad \frac{\| u_0 \|^3 + \gamma\ \ \| u_0 \|^2}{8} = \frac{1 + \gamma}{8}, \\
\| F(u_0)^{-1}\ F(u_0) \| \quad &\le \quad \frac{1 + \gamma}{5 - 2\ \gamma}.
\end{aligned}$$

On the other hand, for $x, y \in \mathcal{D}$, we have

$$[(F'(x) - F'(y))v]\,(s) = -\int_0^1 Q(s,t)\ (3\ x^2(t) - 3\ y^2(t) + 2\ \gamma\ (x(t) - y(t)))\ v(t)\ dt.$$

Consequently (see [3]),

$$\| F'(x) - F'(y) \| \le \frac{\gamma + 6\ R + 3}{4}\ \ \| x - y \|,$$

$$\| F'(x) - F'(u_0) \| \le \frac{2\ \gamma + 3\ R + 6}{8}\ \ \| x - u_0 \|.$$

Therefore, conditions of Theorem 3.3 hold with

$$\eta = \frac{1 + \gamma}{5 - 2\ \gamma}, \quad K = \frac{\gamma + 6\ R + 3}{4}, \quad L = \frac{2\ \gamma + 3\ R + 6}{8}.$$

Note also that $L < K$.

References

[1] I. K. Argyros, *On the Newton–Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math. **169** (2004), 315–332.

[2] ———, *A unifying local–semilocal convergence analysis and applications for two–point Newton–like methods in Banach space*, J. Math. Anal. Appl. **298** (2004), 374–397.

[3] ———, *Convergence and applications of Newton–type iterations*, Springer–Verlag Pub., New York, 2008.

[4] ———, *On a class of Newton–like methods for solving nonlinear equations*, J. Comput. Appl. Math. **228** (2009), 115–122.

[5] J. E. Dennis, *Toward a unified convergence theory for Newton–like methods*, in Nonlinear Functional Analysis and Applications (L.B. Rall, ed.), Academic Press, New York, (1971), 425–472.

[6] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.

[7] J. M. McNamee, *Numerical methods for roots of polynomials*, part I, 14, Elsevier, 2007.

[8] F. A. Potra, *Sharp error bounds for a class of Newton–like methods*, Libertas Mathematica **5** (1985), 71–84.

[9] T. Yamamoto, *A convergence theorem for Newton–like methods in Banach spaces*, Numer. Math. **51** (1987), 545–557.

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