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VISCOSITY APPROXIMATIONS FOR NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

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Abstract. Strong convergence theorem of the explicit viscosity iterative scheme involving the sunny nonexpansive retraction for nonexpansive nonself-mappings is established in a reflexive and strictly convex Banach spaces having a weakly sequentially continuous duality mapping. The main result improves the corresponding result of [19] to the more general class of mappings together with certain different control conditions.

1. Introduction

Let E be a real Banach space and C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0,1)$ such that $\|f(x) - f(y)\| \le k\|x - y\|,\ x,\ y \in C$. We use Σ_C to denote the collection of mappings f verifying the above inequality. That is, $\Sigma_C = \{f : C \to C \mid f \text{ is a contraction with constant } k\}$. Let $T : C \to C$ be a nonexpansive mapping (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|,\ \ x,\ y \in C$) and $F(T)$ denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$.

In 1967, Halpern [5] firstly introduced the following explicit iterative scheme (1.1) in Hilbert space,

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)x_n. \tag{1.1}$$

He pointed out that the control conditions

(C1) $\lim_{n\to\infty} \lambda_n = 0$,
(C2) $\sum_{n=0}^{\infty} \lambda_n = \infty$ or, equivalently, $\prod_{n=0}^{\infty}(1 - \lambda_n) = 0$

are necessary for the convergence of the iteration scheme (1.1) to a fixed point T. In 1992, Wittmann [20], still in Hilbert space, obtained a strong convergence result for the iteration scheme (1.1) under the control conditions (C1), (C2) and

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(C3) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Shioji and Takahashi [18] extended Wittmann's results to a reflexive Banach space having a uniformly Gâteaux differentiable norm such that each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings. For other control conditions, we refer Cho et al. [2], Lions [9] and Reich [17].

On the other hand, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [15]. In 2004, in order to extend Theorem 2.2 of Moudafi [15] to a Banach space setting, Xu [22] considered the the following explicit viscosity iterative scheme in a uniformly smooth Banach space: for $T : C \to C$ nonexpansive mapping, $f \in \Sigma_C$ and $\lambda_n \in (0, 1)$,

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)Tx_n, \quad n \geq 0, \tag{1.2}$$

and under control conditions (C1), (C2) and (C3) or

(C4) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$,

on $\{\lambda_n\}$, he studied the strong convergence of x_n defined by (1.2) to a fixed point of T which is the unique solution of certain variational inequality.

In 2006, using the sunny nonexpansive retraction Q from E onto C and $T : C \to E$ nonexpansive nonself-mapping satisfying the weak inwardness condition, Song and Chen [19] considered the explicit viscosity iterative scheme

$$x_{n+1} = Q(\lambda_n f(x_n) + (1 - \lambda_n)Tx_n), \quad n \geq 0,$$

and improved the results of Xu [22] to the case of nonself-mapping in a reflexive Banach space having a weakly sequentially continuous duality mapping under the control conditions (C1), (C2) and (C3) on $\{\lambda_n\}$.

Very recently, under the control conditions (C1), (C2) and (C3) on $\{\lambda_n\}$, Matsushita and Takahashi [12] studied the following explicit iterative scheme in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm: for $T : C \to E$ nonexpansive mapping, $u \in C, x_0 \in C$, and the sunny nonexpansive retraction Q from E onto C

$$x_{n+1} = Q(\lambda_n u + (1 - \lambda_n)Tx_n), \quad n \geq 0.$$

In this paper, motivated by above-mentioned results, we consider the following explicit viscosity scheme: for $T : C \to E$ nonexpansive mapping, $f \in \Sigma_C$, $\lambda_n \in (0, 1)$, $x_0 \in C$, and the sunny nonexpansive retraction Q from E onto C,

$$x_{n+1} = Q(\lambda_n f(x_n) + (1 - \lambda_n)Tx_n), \quad n \geq 0. \tag{1.3}$$

Under the control conditions (C1), (C2) on $\{\lambda_n\}$ and the weak asymptotic regularity on $\{x_n\}$ instead of the condition (C3) on $\{\lambda_n\}$, we establish the strong convergence of $\{x_n\}$ generated by (1.3) in a reflexive and strictly Banach space having a weakly sequentially continuous duality mapping. The main result improves the corresponding result in Song and Chen [19] to the class of mappings which need't satisfy the weak inwardness condition together with certain different control conditions. Our result also extends the corresponding results of [15, 22] to the case of non-self mappings.

2. Preliminaries and lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$.

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$. It is also said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$.

The norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x,\ y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. Such an E is called a *smooth* Banach space.

By a gauge function we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. The mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{f \in E^* : \langle x, f \rangle = \|x\|\|f\|, \|f\| = \varphi(\|x\|)\}, \quad \text{for all } x \in E$$

is called the *duality mapping* with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t) = t$ denoted by J, is referred to as the *normalized duality mapping*. The following property of duality mapping is well-known:

$$J_\varphi(\lambda x) = \text{sign } \lambda \left(\frac{\varphi(|\lambda| \cdot \|x\|)}{\|x\|} \right) J(x) \quad \text{for all } x \in E \setminus \{0\}, \quad \lambda \in \mathbb{R}, \tag{2.1}$$

where $\mathbb{R}$ is the set of all real numbers; in particular, $J(-x) = -J(x)$ for all $x \in E$ ([3]).

Following Browder [1], we say that a Banach space E has a weakly sequential continuous duality mapping if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J_\varphi(x_n) \stackrel{*}{\rightharpoonup} J_\varphi(x)$. For example, every l^p space $(1 < p < \infty)$ has a weakly sequentially continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$. Set

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \quad \text{for all } t \in \mathbb{R}^+.$$

Then it is known [1] that $J_\varphi(x)$ is the subdifferential of the convex functional $\Phi(\|\cdot\|)$ at x. Thus it is easy to see that the normalized duality mapping $J(x)$ can also be defined as the subdifferential of the convex functional $\Phi(\|x\|) = \|x\|^2/2$, that is, for all $x \in E$

$$J(x) = \partial\Phi(\|x\|) = \{f \in E^* : \Phi(\|y\|) - \Phi(\|x\|) \geq \langle y - x, f\rangle \ \text{ for all } y \in E\}.$$

It is well-known that if E is smooth, then the normalized duality mapping J is single-valued and norm to weak* continuous ([3]).

We need the following well-known lemma for the proof of our main result.

Lemma 2.1. *Let E be a real Banach space and φ a continuous strictly increasing function on $\mathbb{R}^+$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty}\varphi(r) = \infty$. Define*

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \text{for all } t \in \mathbb{R}^+.$$

Then the following inequality holds

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x + y)\rangle, \quad \text{for all } x, y \in E,$$

where $j_\varphi(x + y) \in J_\varphi(x + y)$. In particular, if E is smooth, then one has

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle, \quad \text{for all } x, y \in E.$$

Let μ be a mean on positive integers N, that is, a continuous linear functional on ℓ^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n : n \in N\} \leq \mu(a) \leq \sup\{a_n : n \in N\}$$

for every $a = (a_1, a_2, ...) \in \ell^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, ...) \in \ell^\infty$. Using the Hahn-Banach theorem, we can prove the existence of a Banach limit. If μ is a Banach limit, the following are well-known:

(i) for all $n \geq 1, a_n \leq c_n$ implies $\mu(a_n) \leq \mu(c_n)$,
(ii) $\mu(a_{n+N}) = \mu(a_n)$ for any fixed positive integer N,
(iii) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \cdots) \in l^\infty$.

The following lemma was given in [18, Proposition 2].

Lemma 2.2. *Let $a \in \mathbb{R}$ be a real number and a sequence $\{a_n\} \in l^\infty$ satisfy the condition $\mu_n(a_n) \leq a$ for all Banach limit μ. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

Let D be a subset of C. Then a mapping $Q : C \to D$ is said to be a *retraction* from C onto D if $Qx = x$ for all $x \in D$. A retraction Q is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $t \geq 0$ and $x + t(x - Qx) \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. Sunny nonexpansive retractions are characterized as follows ([4, p. 48]): If E is a smooth Banach space, then

$Q : C \to D$ is a sunny nonexpansive retraction if and only if the following condition holds:

$$\langle x - Qx, J(z - Qx)\rangle \leq 0, \quad x \in C, \quad z \in D. \tag{2.2}$$

(Note that this fact still holds by (2.1) if the normalized duality mapping J is replaced by a general duality mapping J_φ with gauge function φ.)

Let C be a nonempty closed convex subset of a Banach space E. For $x \in C$, let

$$I_C(x) = \{y \in E : y = x + \lambda(z - x), z \in C \text{ and } \lambda \geq 0\}.$$

$I_C(x)$ is called the *inward set* of $x \in C$ with respect to C (for example, see [4]). $I_C(x)$ is a convex set containing C. A mapping $T : C \to E$ is said to be satisfying the *inward condition* if $Tx \in I_C(x)$ for all $x \in C$, and T is also said to be satisfying the *weakly inward condition* if for each $x \in C, Tx \in \overline{I_C(x)}$ ($\overline{I_C(x)}$ is the closure of $I_C(x)$). Every self-mapping is trivially weakly inward.

The following lemmas were given in [12].

Lemma 2.3. *Let C be a closed convex subset of a smooth Banach space E and T be a mapping from C into E. Suppose that C is a sunny nonexpansive retract of E. It T satisfies the nowhere-normal outward condition*

$$Tx \in S_x^c, \quad \text{for all } x \in C, \tag{2.3}$$

where $S_x = \{y \in E : y \neq x, Qy = x\}$ and S_x^c is the complement of S_x, then $F(T) = F(QT)$

Lemma 2.4. *Let C a closed convex subset of a strictly convex Banach space E and T a nonexpansive mapping from C into E. Suppose that C is a sunny nonexpansive retract of E. If $F(T) \neq \emptyset$, then T satisfies the nowhere-normal outward condition* (2.3).

Finally, we need the following lemma, which is essentially Lemma 2 of Liu [10] (see also Xu [21]).

Lemma 2.5. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\beta_n + \gamma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$,
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \lambda_n\beta_n < \infty$,
(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main results

Recall that a mapping T with domain $D(T)$ and range $\mathcal{R}(T)$ in E is called *strongly pseudocontractive* ([13]) if for some constant $k < 1$ and for all $x,\ y \in D(T)$

$$(\lambda - k)\|x - y\| \leq \|(\lambda I - T)(x) - (\lambda I - T)(y)\| \tag{3.1}$$

for $\lambda > k$ (with I denoting the identity mapping), while T is called a *pseudocontraction* if (3.1) holds for $k = 1$. Every nonexpansive mapping is a pseudocontraction. The converse is not true (for example, see [8]).

We need the following result for the existence of solutions of certain variational inequalities which Jung and Sahu [8] established recently.

Theorem JS. ([8, Theorem 3]) *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed subset of E, $A : C \to C$ a continuous strongly pseudocontractive mapping with constant $k \in [0, 1)$ and $T : C \to E$ a demicontinuous pseudocontractive mapping such that the equation*

$$x = tAx + (1 - t)Tx$$

has a solution x_t in C for each $t \in [0, 1)$. Suppose the path $\{x_t\}$ is bounded. Then we have the following:

(a) $\lim_{t\to 0^+} x_t = \tilde{x}$ *exists,*

(b) $\tilde{x}$ *is a fixed point of T and it is the unique solution of the variational inequality:*

$$\langle (I - A)\tilde{x}, J_\varphi(\tilde{x} - v)\rangle \leq 0 \text{ for all } v \in F(T).$$

Remark 1. (1) Theorem JS supplements Theorem 3 of Morales and Jung [14], where $A = u$ is a constant.

(2) Theorem JS also generalizes Theorem 3.10 of O'Hara et al. [16] and Theorems 3.1 of Xu [23] to the viscosity type method for the more general class of nonself-mappings which include the class of nonexpansive mappings.

First, we consider the explicit viscosity iterative scheme: for Q the sunny and nonexpansive retraction of E onto C, $T : C \to E$ nonexpansive non-self-mapping and $f \in \Sigma_C$,

$$\begin{cases} x_0 \in C \\ x_{n+1} = Q(\lambda_n f(x_n) + (1 - \lambda_n)Tx_n). \end{cases} \tag{3.2}$$

Proposition 3.1. *Let E be a reflexive and strictly convex Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $T : C \to E$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that C is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $f \in \Sigma_C$ and let*

$\{x_n\}$ be the sequence generated by (3.2)*. Let $\{\lambda_n\}$ be a sequence in* $(0,1)$ *which satisfies the condition:*

(C1) $\lim_{n\to\infty} \lambda_n = 0$

and μ a Banach limit. Then

$$\mu_n(\langle (I-f)(P(f)), J_\varphi(P(f)-x_n)\rangle) \leq 0,$$

where $P : \Sigma_C \to F(T)$ is defined by $P(f) = \lim_{t\to 0^+} x_t$ and x_t is defined by $x_t = tf(x_t) + QTx_t$, $0 < t < 1$.

Proof. Let $\{x_t\}$ be the net generated by

$$x_t = tf(x_t) + (1-t)QTx_t, \quad 0 < t < 1. \tag{3.3}$$

Since QT is a nonexpansive mapping from C into itself, by Theorem JS with $A = f$ a contraction and Lemmas 2.3 and 2.4, there exists $\lim_{t\to 0} x_t \in F(QT) = F(T)$. Denote it by $P(f)$. This implies that P is a mapping from Σ_C onto $F(T)$. Moreover $P(f)$ is a solution of the variational inequality

$$\langle (I-f)P(f), J_\varphi(P(f)-v)\rangle \leq 0, \quad f \in \Sigma_C, \quad v \in F(T).$$

From (3.3), we have

$$\|x_t - x_{n+1}\| = \|(1-t)(QTx_t - x_{n+1}) + t(f(x_t) - x_{n+1})\|.$$

Applying Lemma 2.1, we have

$$\begin{aligned}\Phi(\|x_t - x_{n+1}\|) \leq \Phi(&(1-t)\|QTx_t - x_{n+1}\|) \\ &+ t\langle f(x_t) - x_{n+1}, J_\varphi(x_t - x_{n+1})\rangle.\end{aligned} \tag{3.4}$$

Let $p \in F$. Now

$$\begin{aligned}\|x_t - p\| \leq& t\|f(x_t) - p\| + (1-t)\|QTx_t - QTp\| \\ \leq& \, t\|f(x_t) - p\| + (1-t)\|x_t - p\|.\end{aligned}$$

This gives that

$$\begin{aligned}\|x_t - p\| &\leq \|f(x_t) - p\| \leq \|f(x_t) - f(p)\| + \|f(p) - p\| \\ &\leq k\|x_t - p\| + \|f(p) - p\|,\end{aligned}$$

and so $\|x_t - p\| \leq \frac{1}{1-k}\|f(p) - p\|$. Hence $\{x_t\}$ is bounded, so are $\{f(x_t)\}$ and $\{QTx_t\}$.

Now we show that $\|x_n - z\| \leq \max\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\}$ for all $n \geq 0$ and all $z \in F(T)$ and so $\{x_n\}$ is bounded. For $p \in F(T)$, we also have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$$

for all $n \geq 0$. Indeed, let $p \in F(T)$ and $d = \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$. Then by the nonexpansivity of T and $f \in \Sigma_C$,

$$\begin{aligned}\|x_1 - p\| &= \|Q(\lambda_0 f(x_0) + (1-\lambda_0)Tx_0) - Qp\| \\ &\leq (1-\lambda_0)\|Tx_0 - Tp\| + \lambda_0\|f(x_0) - p\| \\ &\leq (1-\lambda_0)\|x_0 - p\| + \lambda_0(\|f(x_0) - f(p)\| + \|f(p) - p\|) \\ &\leq (1-\lambda_0)\|x_0 - p\| + \lambda_0(k\|x_0 - p\| + \|f(p) - p\|) \\ &\leq (1-(1-k)\lambda_0)d + \lambda_0(1-k)d = d.\end{aligned}$$

Using an induction, we obtain

$$\|x_{n+1} - p\| \leq d, \quad n \geq 0.$$

Hence, it follows that $\{x_n\}$ is bounded, and so are $\{QTx_n\}$ and $\{f(x_n)\}$. As a consequence with the control condition (C1), we get

$$\|x_{n+1} - QTx_n\| \leq \lambda_{n+1}\|Tx_n - f(x_n)\| \to 0 \quad (n \to \infty).$$

Observe also that

$$\|QTx_t - x_{n+1}\| \leq \|x_t - x_n\| + e_n,$$

where $e_n = \|x_{n+1} - QTx_n\| \to 0$ as $n \to \infty$, and

$$\begin{aligned}\langle f(x_t) - x_{n+1}, J_\varphi(x_t - x_{n+1})\rangle &= \langle f(x_t) - x_t, J_\varphi(x_t - x_{n+1})\rangle \\ &\quad + \|x_t - x_{n+1}\|\varphi(\|x_t - x_{n+1}\|).\end{aligned}$$

Thus it follows from (3.4) that

$$\begin{aligned}\Phi(\|x_t - x_{n+1}\|) \leq \Phi((1-t)&(\|x_t - x_n\| + e_n)) \\ &+ t\langle f(x_t) - x_t, J_\varphi(x_t - x_{n+1})\rangle \\ &+ t\|x_t - x_{n+1}\|\varphi(\|x_t - x_{n+1}\|)\end{aligned} \tag{3.5}$$

Applying the Banach limit μ to (3.5), we have

$$\begin{aligned}\mu_n(\Phi(\|x_t - x_{n+1}\|)) \leq \mu_n(\Phi(&(1-t)(\|x_t - x_n\| + e_n))) \\ &+ t\mu_n(\langle f(x_t) - x_t, J_\varphi(x_t - x_{n+1})\rangle) \\ &+ t\mu_n(\|x_t - x_{n+1}\|\varphi(\|x_t - x_{n+1}\|))\end{aligned} \tag{3.6}$$

and it follows from (3.6) that

$$\begin{aligned}&\mu_n(\langle x_t - f(x_t), J_\varphi(x_t - x_n)\rangle) \\ &\quad \leq \frac{1}{t}\mu_n(\Phi((1-t)\|x_t - x_n\|) - \Phi(\|x_t - x_n\|)) \\ &\qquad + \mu_n(\|x_t - x_{n+1}\|\varphi(\|x_t - x_{n+1}\|)) \\ &\quad = -\frac{1}{t}\mu_n\left\{\int_{(1-t)\|x_t - x_n\|}^{\|x_t - x_n\|}\varphi(\tau)d\tau\right\} \\ &\qquad + \mu_n(\|x_t - x_{n+1}\|\varphi(\|x_t - x_{n+1}\|)) \\ &\quad = \mu_n(\|x_t - x_n\|(\varphi(\|x_t - x_n\|) - \varphi(\tau_n))),\end{aligned} \tag{3.7}$$

for some τ_n satisfying $(1-t)\|x_t - x_n\| \le \tau_n \le \|x_t - x_n\|$. Since φ is uniformly continuous on compact intervals of $\mathbb{R}^+$,

$$\begin{aligned}\|x_t - x_n\| - \tau_n &\le t\|x_t - x_n\| \\ &\le t\left(\frac{2}{1-k}\|f(p) - p\| + \|x_0 - p\|\right) \to 0 \quad (\text{as } t \to 0),\end{aligned}$$

we conclude from (3.7) that

$$\begin{aligned}\mu_n(\langle (I-f)(P(f)), &J_\varphi(P(f) - x_n)\rangle) \\ &\le \limsup_{t\to 0} \mu_n(\langle x_t - f(x_t), J_\varphi(x_t - x_n)\rangle) \le 0,\end{aligned}$$

where $P : \Sigma_C \to F$ is defined by $P(f) = \lim_{t\to 0} x_t$. □

Recall that the sequence $\{x_n\}$ in E is said to be *weakly asymptotically regular* if

$$w - \lim_{n\to\infty}(x_{n+1} - x_n) = 0, \quad \text{that is,} \quad x_{n+1} - x_n \rightharpoonup 0$$

and *asymptotically regular* if

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

respectively.

Using Proposition 3.1, we give the following main result.

Theorem 3.2. *Let E be a reflexive and strictly convex Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $T : C \to E$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that C is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $f \in \Sigma_C$ and let $\{x_n\}$ be the sequence generated by* (3.2). *Let $\{\lambda_n\}$ be a sequence in* $(0, 1)$ *which satisfies the conditions:*

(C1) $\lim_{n\to\infty} \lambda_n = 0$*;*
(C2) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to $P(f) \in F(T)$, where $P(f)$ is the unique solution of the variational inequality

$$\langle (I-f)(P(f)), J_\varphi(P(f) - v)\rangle \le 0, \quad f \in \Sigma_C, \quad v \in F(T).$$

Proof. Let x_t be defined by (3.3), that is, $x_t = tf(x_t) + (1-t)QTx_t$ for $0 < t < 1$ and $\lim_{t\to 0} x_t := P(f) \in F(QT) = F(T)$ (by using Theorem JS with $A = f$ a contraction and Lemmas 2.3 and 2.4). Then $P(f)$ is a solution of a variational inequality

$$\langle (I-f)(P(f)), J_\varphi(P(f) - v)\rangle \le 0 \quad f \in \Sigma_C, \quad v \in F(T).$$

We proceed with the following steps:

Step 1. We show that $\|x_n - z\| \le \max\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\}$ for all $n \ge 0$ and all $z \in F(T)$ as in the proof of Proposition 3.1. Hence $\{x_n\}$ is bounded and so are $\{QTx_n\}$ and $\{f(x_n)\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle (I-f)(P(f)), J_\varphi(P(f)-x_n)\rangle \le 0$. To this end, put

$$a_n := \langle (I-f)(P(f)), J_\varphi(P(f)-x_n)\rangle, \quad n \ge 1.$$

Then Proposition 3.1 implies that $\mu_n(a_n) \le 0$ for any Banach limit μ. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1}-a_n) = \lim_{j\to\infty}(a_{n_j+1}-a_{n_j})$$

and $x_{n_j} \rightharpoonup q \in E$. This implies that $x_{n_j+1} \rightharpoonup q$ since $\{x_n\}$ is weakly asymptotically regular. From the weak sequential continuity of duality mapping J, we have

$$w-\lim_{j\to\infty} J_\varphi(P(f)-x_{n_j+1}) = w-\lim_{j\to\infty} J_\varphi(P(f)-x_{n_j}) = J_\varphi(P(f)-q),$$

and so

$$\begin{aligned}&\limsup_{n\to\infty}(a_{n+1}-a_n)\\ &\quad = \lim_{j\to\infty}\langle (I-f)(P(f)), J_\varphi(P(f)-x_{n_j+1}) - J_\varphi(P(f)-x_{n_j})\rangle = 0.\end{aligned}$$

Then Lemma 2.2 implies that $\limsup_{n\to\infty} a_n \le 0$, that is,

$$\limsup_{n\to\infty}\langle (I-f)P(f), J(P(f)-x_n)\rangle \le 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - P(f)\| = 0$. As a matter of fact, we have

$$\begin{aligned} x_{n+1}-P(f) &= x_{n+1} - (\lambda_n f(x_n) + (1-\lambda_n)P(f)) + \lambda_n(f(x_n)-P(f))\\ &= Q(\lambda_n f(x_n) + (1-\lambda_n)Tx_n) - Q(\lambda_n f(x_n) + (1-\lambda_n)P(f))\\ &\quad + \lambda_n(f(x_n) - f(P(f))) + \lambda_n(f(P(f)) - P(f)).\end{aligned}$$

As a consequence, since Φ is an increasing convex function with $\Phi(0)=0$, by applying Lemma 2.1, we obtain

$$\begin{aligned} &\Phi(\|x_{n+1}-P(f)\|)\\ &\le \Phi(\|Q(\lambda_n f(x_n) + (1-\lambda_n)Tx_n) - Q(\lambda_n f(x_n) + (1-\lambda_n)P(f))\\ &\qquad + \lambda_n(f(x_n) - f(P(f)))\|)\\ &\quad + \lambda_n\langle f(P(f)) - P(f), J_\varphi(x_{n+1}-P(f))\rangle\\ &\le \Phi((1-\lambda_n)\|Tx_n - P(f)\| + k\alpha_n\|x_n - P(f)\|)\\ &\quad + \lambda_n\langle f(P(f)) - P(f), J_\varphi(x_{n+1}-P(f))\rangle\\ &\le (1-(1-k)\lambda_n)\Phi(\|x_n - P(f)\|)\\ &\quad + \lambda_n\langle f(P(f)) - P(f), J_\varphi(x_{n+1}-P(f))\rangle. \end{aligned} \tag{3.8}$$

Put

$$\alpha_n = (1-k)\lambda_n \text{ and } \delta_n = \frac{1}{1-k}\langle (I-f)(P(f)), J_\varphi(P(f)-x_{n+1})\rangle.$$

From (C1), (C2) and Step 2, it follows that $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. Since (3.8) reduces to

$$\Phi(\|x_{n+1} - P(f)\|) \le (1-\alpha_n)\Phi(\|x_n - P(f)\|) + \alpha_n \delta_n,$$

from Lemma 2.5, we conclude that $\lim_{n\to\infty} \Phi(\|x_n - Q(f)\|) = 0$, and thus $\lim_{n\to\infty} x_n = P(f)$. This completes the proof. □

Corollary 3.3. *Let E be a reflexive and strictly convex Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $T : C \to E$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that C is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $f \in \Sigma_C$ and let $\{x_n\}$ be the sequence generated by* (3.2). *Let $\{\lambda_n\}$ be a sequence in* $(0,1)$ *which satisfies the conditions:*

(C1) $\lim_{n\to\infty} \lambda_n = 0$;
(C2) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $P(f) \in F(T)$, where $P(f)$ is the unique solution of the variational inequality

$$\langle (I-f)(P(f)), J_\varphi(P(f)-v)\rangle \le 0, \quad f \in \Sigma_C, \quad v \in F(T).$$

Remark 2. If $\{\lambda_n\}$ in Corollary 3.3 satisfies conditions (C1), (C2) and

(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$; or
(C4) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$ or, equivalently, $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}} = 0$;

or the perturbed control condition:

(C5) $|\lambda_{n+1} - \lambda_n| \le o(\lambda_{n+1}) + \sigma_n, \quad \sum_{n=0}^{\infty} \sigma_n < \infty$,

then the sequence $\{x_n\}$ generated by (3.2) is asymptotically regular. Now we give only the proof in case when $\{\lambda_n\}$ satisfies the conditions (C1), (C2) and (C5). By Step 1 above, there exists a constant $L > 0$ such that for all $n \ge 0$,

$$\|f(x_n)\| + \|Tx_n\| \le L.$$

So we obtain, for all $n \ge 0$,

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ =&\|Q(\lambda_n f(x_n) + (1-\lambda_n)Tx_n) \\ &\quad - Q(\lambda_{n-1} f(x_{n-1}) + (1-\lambda_{n-1})Tx_{n-1})\| \\ \le& \|(1-\lambda_n)(Tx_n - Tx_{n-1}) \\ &+ (\lambda_n - \lambda_{n-1})(f(x_{n-1}) - Tx_{n-1}) + \lambda_n(f(x_n) - f(x_{n-1}))\| \\ \le& (1-\lambda_n)\|x_n - x_{n-1}\| + L|\lambda_n - \lambda_{n-1}| + k\lambda_n\|x_n - x_{n-1}\| \\ \le& (1-(1-k)\lambda_n)\|x_n - x_{n-1}\| + (o(\lambda_n) + \sigma_{n-1})L. \end{aligned} \tag{3.9}$$

By taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\alpha_n = (1-k)\lambda_n$, $\alpha_n\beta_n = o(\lambda_n)L$ and $\gamma_n = \sigma_{n-1}L$, from (3.9) we have

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n.$$

Hence, by (C1), (C2), (C5) and Lemma 2.5,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

In view of this observation, we have the following:

Corollary 3.4. *Let E be a uniformly convex Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $T : C \to E$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that C is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $f \in \Sigma_C$ and let $\{x_n\}$ be the sequence generated by* (3.2). *Let $\{\lambda_n\}$ be a sequence in* $(0,1)$ *which satisfies the conditions* (C1), (C2) *and* (C5) *(or the conditions* (C1), (C2) *and* (C3)*, or the conditions* (C1), (C2) *and* (C4)*). Then $\{x_n\}$ converges strongly to $P(f) \in F(T)$, where $P(f)$ is the unique solution of the variational inequality*

$$\langle (I-f)(P(f)), J_\varphi(P(f) - v)\rangle \leq 0, \quad f \in \Sigma_C, \quad v \in F(T).$$

Remark 3. (1) Theorem 3.2 is a supplement of Theorem 2.4 of Song and Chen [19] by using the weak asymptotic regularity on $\{x_n\}$ instead of the condition (C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ without the weak inwardness condition on T.

(2) Theorem 3.2 also develops Theorem 4.2 of Matsushita and Takahashi [12] to the viscosity iteration method in different Banach spaces together with the weak asymptotic regularity on $\{x_n\}$ instead of the condition (C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

(3) The condition (C5) on $\{\lambda_n\}$ in Corollary 3.4 is independent of condition (C3) or (C4) in Remark 2, which Theorem 2.4 of Song and Chen [19] has used. For this fact, see [2, 6].

(4) Theorem 3.2 generalizes Theorem 3.2 of Xu [22] to the case of nonself-mappings.

Next, we consider the implicit viscosity iterative scheme. Let Q be the sunny nonexpansive retraction of E onto C and $T : C \to E$ nonexpansive mapping and $f \in \Sigma_C$. Following Marino and Trombetta [11], we define the contraction $S_t := S_t^f$ from C into itself by

$$S_t x = Q(tf(x) + (1-t)Tx), \quad x \in C.$$

Then Banach's contraction principle yields a unique point $x_t \in C$ that is fixed by S_t, that is, we have the implicit viscosity iterative scheme

$$x_t = Q(tf(x_t) + (1-t)Tx_t). \tag{3.10}$$

By using directly the proof of Theorem 2.2 in Song and Chen [19] together with Lemma 2.4 and Lemma 2.5, we have the following result:

Theorem 3.5. *Let E be a reflexive and strictly convex Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C a nonempty closed convex subset of E and $T : C \to E$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that C is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. For each $t \in (0, 1)$ and $f \in \Sigma_C$, let $\{x_t\}$ be the net generated by (3.10). Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point of T. If we define $R : \Sigma_C \to F(T)$ by*

$$R(f) := \lim_{t\to 0} x_t, \quad f \in \Sigma_C,$$

then $R(f)$ is the unique solution of the variational inequality

$$\langle (I - f)(R(f)), J_\varphi(R(f) - p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F(T).$$

Proof. Let x_t be defined by (3.10), that is, $x_t = Q(tf(x_t) + (1 - t)Tx_t)$ for $0 < t < 1$. As in the proof of Theorem 2.2 in [19], we have $\|x_t - QTx_t\| \to 0$ as $t \to 0$. Note that $F(T) = F(QT)$ by Lemmas 2.4 and 2.5. Then the remainder of the proof follows from the proof of Theorem 2.2 in [19]. □

Remark 4. (1) Theorem 3.5 is a complement of Theorem 2.2 of Song and Chen [19] without the weak inwardness condition on T.

(2) Theorems 3.5 also generalizes Theorem 4.1 of Xu [22] (and Theorem 2 of Moudafi [15]) to the class of nonself-mappings.

(3) Theorem 3.5 extends Theorem 4 of Jung and Kim [7] and Theorem 3 of Xu and Yin [24] to the viscosity iteration method for the class of mappings which needn't satisfy the weak inwardness condition.

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