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UNIT-REGULARITY AND STABLE RANGE ONE

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Abstract. Let R be a ring, and let $\Psi(R)$ be the ideal generated by the set $\{x \in R \mid 1 + sxt \in R$ is unit-regular for all $s, t \in R\}$. We show that $\Psi(R)$ has "radical-like" property. It is proven that $\Psi(R)$ has stable range one. Thus, diagonal reduction of matrices over such ideal is reduced.

1. Introduction

Let R be an associative ring with an identity. An element $a \in R$ is regular in case there exists $x \in R$ such that $a = axa$. If such x can be chosen a unit, $a \in R$ is said to be unit-regular. A ring R is unit-regular in case every element in R is unit-regular. It is worth noting that a regular ring R is unit-regular if and only if for all finitely generated projective right R-modules A, B and C, $A \oplus B \cong A \oplus C$ implies that $B \cong C$. Many authors studied unit-regular rings, e.g. [5, 7]. As is well known, there exists a largest regular ideal $M(R)$ of a ring R. A natural problem is how to construct a kind of ideal to deal with unit-regularity. The motivation of this article is to extend the known results on unit-regular rings to regular ideals. We always use $ur(R)$ to denote the set of all unit-regular elements in R. Let $\Psi(R)$ be the ideal generated by the set $\{x \in R \mid 1 + sxt \in ur(R)$ for all $s, t \in R\}$. We observe that $\Psi(R)$ has "radical-like" property. That is, $M_n\big(\Psi(R)\big) = \Psi\big(M_n(R)\big)$. An ideal I of a ring R has stable range one provided that $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that $a + by \in R$ is invertible. For general theory of stable range conditions, we refer the reader to [6]. Further, we prove that $\Psi(R)$ has stable range one. As an application, the diagonal reduction of matrices over such ideal is studied.

Throughout, all rings are associative with identity. $U(R)$ and $\mathrm{GL}_n(R)$ denote the set of all units of R and the n-dimensional general linear group over R, respectively. The symbol $\mathbb{N}$ stands for the set of all natural number.

2. Unit-regularity

It is well known that for any $x, y \in R$, $1 + xy \in U(R)$ if and only if $1 + yx \in U(R)$. We extend this simple fact to unit-regularity.

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Lemma 2.1. *Let $x, y \in R$. Then $1+xy$ is unit-regular if and only if so is $1+yx$.*

Proof. Suppose that $1+xy$ is unit-regular. Clearly, one checks that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1+yx \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+xy & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}.$$

Hence $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1+yx \end{smallmatrix}\right) \in M_2(R)$ is unit-regular. Write

$$\begin{pmatrix} 1 & 0 \\ 0 & 1+yx \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1+yx \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1+yx \end{pmatrix},$$

where $\left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$. Then we get $1+yx = (1+yx)c_{22}(1+yx)$. Let $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1+yx \end{smallmatrix}\right)\left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right)$. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & 1+yx \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & c_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1-(1+yx)c_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we get

$$E\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & c_{22} \end{pmatrix}(I_2-E) + \begin{pmatrix} 0 & 0 \\ 0 & 1-(1+yx)c_{22} \end{pmatrix}(I_2-E) = I_2-E.$$

This implies that

$$\begin{aligned} & E + \begin{pmatrix} 0 & 0 \\ 0 & 1-(1+yx)c_{22} \end{pmatrix}(I_2-E) \\ &= I_2 - E\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & c_{22} \end{pmatrix}(I_2-E) \in \mathrm{GL}_2(R). \end{aligned}$$

Set $\left(\begin{smallmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{smallmatrix}\right) = (I_2-E)\left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right)^{-1}$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1+yx \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1-(1+yx)c_{22} \end{pmatrix}\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathrm{GL}_2(R).$$

That is, $\left(\begin{smallmatrix} 1 & 0 \\ * & 1+yx+\big(1-(1+yx)c_{22}\big)z_{22} \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$. As a result, $u := 1+yx+\big(1-(1+yx)c_{22}\big)z_{22} \in U(R)$. Hence, $1+yx = eu$, where $e = (1+yx)c_{22} \in R$ is an idempotent. Therefore $1+yx \in R$ is unit-regular. The converse is symmetric. □

Lemma 2.2. *An element $a \in R$ is unit-regular if and only if* $\mathrm{diag}(a,1,\dots,1) \in M_n(R)$ *is unit-regular.*

Proof. It is an immediate consequence of [7, Theorem 4]. □

Let $\Phi(R) = \{x \in R \mid 1+sxt \in R \text{ is unit-regular for all } s,t \in R\}$, and let $\Psi(R)$ be the ideal generated by the set $\Phi(R)$. We list some examples of such ideal. $\Psi(\mathbb{Z}) = 0$; $\Psi\big(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\big) = 0 \oplus \mathbb{Z}/2\mathbb{Z}$, and that

$$\Psi\left(\begin{pmatrix} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}\right) = \begin{pmatrix} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}.$$

Theorem 2.3. *Let R be a ring. Then $\Psi\big(M_n(R)\big) = M_n\big(\Psi(R)\big)$ for all $n \in \mathbb{N}$.*

Proof. Given any $A \in \Psi\big(M_n(R)\big)$, we have $A = A_1 + \cdots + A_m$ $(m \in \mathbb{N})$ with each $A_i \in \Phi\big(M_n(R)\big)$. Write $A_1 = (a_{ij})$. For any $r \in R$, we have

$$I_n + A_1 \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 + a_{11}r & 0 & \cdots & 0 \\ a_{21}r & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}r & 0 & \cdots & 1 \end{pmatrix}$$

is unit-regular. Clearly, there exists $V \in \mathrm{GL}_n(R)$ such that

$$\begin{pmatrix} 1 + a_{11}r & 0 & \cdots & 0 \\ a_{21}r & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}r & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 + a_{11}r & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} V.$$

Hence $\begin{pmatrix} 1+a_{11}r & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in M_n(R)$ is unit-regular, and then so is $1 + a_{11}r \in R$ from Lemma 2.2. It follows by Lemma 2.1 that $a_{11} \in \Phi(R)$. Likewise, we prove that each $a_{ij} \in \Phi(R)$. Hence, $A_1 \in M_n\big(\Psi(R)\big)$. Similarly, $A_2, \ldots, A_m \in M_n\big(\Psi(R)\big)$. Consequently, $A = A_1 + \cdots + A_m \in M_n\big(\Psi(R)\big)$. We infer that $\Psi\big(M_n(R)\big) \subseteq M_n\big(\Psi(R)\big)$.

Given any $(a_{ij}) \in M_n\big(\Psi(R)\big)$, then each $a_{ij} \in \Psi(R)$. Write $a_{ij} = b_1 + \cdots + b_k$ with each $b_s \in \Phi(R) (1 \leq s \leq k)$. For any $(r_{ij}) \in M_n(R)$, we have

$$I_n + \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} (r_{ij}) = \begin{pmatrix} 1 + b_1 r_{11} & b_1 r_{12} & \cdots & b_1 r_{1n} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Clearly, there is $U \in \mathrm{GL}_n(R)$ such that

$$U\Big(I_n + \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} (r_{ij})\Big) = \begin{pmatrix} 1 + b_1 r_{11} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

As $1 + b_1 r_{11} \in R$ is unit-regular, by Lemma 2.2, we have

$$U\Big(I_n + \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} (r_{ij})\Big) \in M_n(R)$$

is unit-regular, and thus so is

$$I_n + \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} (r_{ij}).$$

Thus, $\begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \Psi\big(M_n(R)\big)$. Likewise, $\begin{pmatrix} b_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \Psi\big(M_n(R)\big)$ for all i. We infer that $\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \Psi\big(M_n(R)\big)$. Analogously,

$$\begin{pmatrix} 0 & a_{12} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in \Psi\big(M_n(R)\big).$$

Therefore $(a_{ij}) \in \Psi\big(M_n(R)\big)$, and so $M_n\big(\Psi(R)\big) \subseteq \Psi\big(M_n(R)\big)$, as required. □

Proposition 2.4. *Let $x, y \in R$. Then $x + y + xy \in ur(R)$ if and only if $x + y + yx \in ur(R)$.*

Proof. Suppose that $x + y + xy \in ur(R)$. Then $(x + 1)(y + 1) - 1 = x + y + xy \in ur(R)$, and then $1 + (x + 1)(-y - 1) \in ur(R)$. In view of Lemma 2.1, $1 + (-y - 1)(x + 1) \in ur(R)$, whence $(y + 1)(x + 1) - 1 \in ur(R)$. That is, $x + y + yx \in ur(R)$. The converse is analogous. □

Let $e \in R$ be an idempotent and $a \in R$. In [7, Theorem 4], Lam and Murray proved that $eae \in ur(eRe)$ if and only if $eae + 1 - e \in ur(R)$. We extend this result as follows.

Proposition 2.5. *Let $e \in R$ be an idempotent of a ring R and $a \in R$. Then the following are equivalent*:

(1) $eae \in ur(eRe)$.
(2) $ae + 1 - e \in ur(R)$.
(3) $ea + 1 - e \in ur(R)$.

Proof. (1) ⇒ (2) Since $eae \in ur(eRe)$, we can find some $u \in U(eRe)$ such that $eae = (eae)(eue)(eae)$. Hence $eae+1-e = (eae+1-e)(eue+1-e)(eae+1-e)$. Clearly, $eue + 1 - e \in U(R)$. This means that $1 + e(ae - e) \in ur(R)$. It follows from Lemma 2.1 that $1 + (ae - e)e \in ur(R)$. That is, $ae + 1 - e \in ur(R)$.

(2) ⇒ (1) As $ae + 1 - e = 1 + (ae - e)e \in ur(R)$, by Lemma 2.1, we have $eae + 1 - e = 1 + e(ae - e) \in ur(R)$. It follows from [7, Theorem 4] that $eae \in ur(eRe)$.

(1) ⇔ (3) is proved in the same manner. □

3. Stable range one

Lemma 3.1. *Suppose that $ax+b=1$ in R. If $a\in ur(R)$, then there exists $z\in R$ such that $x+zb\in U(R)$.*

Proof. Since $a\in ur(R)$, there exist $e=e^2\in R$ and $u\in U(R)$ such that $a=eu$. Hence $eux(1-e)+b(1-e)=1-e$, and then

$$a+b(1-e)u=\big(1-eux(1-e)\big)u=\big(1+eux(1-e)\big)^{-1}u\in U(R).$$

By [4, Lemma 3.1], we can find $z\in R$ such that $x+zb\in U(R)$. □

Theorem 3.2. $\Psi(R)$ *has stable range one.*

Proof. Given $ax+b=1$ with $a\in 1+\Psi(R)$ and $x,b\in R$, we can find $c_1,\dots,c_m\in R$ such that $a=1+c_1+\cdots+c_m$ and each $c_i\in\Phi(R)$. Hence $(1+c_1)x+(c_2+\cdots+c_m)x+b=1$. As $1+c_1\in ur(R)$, by Lemma 3.1, we can find some $z_1\in R$ such that

$$x+z_1(c_2+\cdots+c_m)x+z_1b=u_1\in U(R).$$

Hence

$$(1+z_1c_2)xu_1^{-1}+z_1(c_3+\cdots+c_m)xu_1^{-1}+z_1bu_1^{-1}=1.$$

As $1+z_1c_2\in ur(R)$, by Lemma 3.1 again, we have $z_2\in R$ such that

$$xu_1^{-1}+z_2z_1(c_3+\cdots+c_m)xu_1^{-1}+z_2z_1bu_1^{-1}\in U(R),$$

and then $x+z_2z_1(c_3+\cdots+c_m)x+z_2z_1b\in U(R)$. By iteration of this process, we have $z\in R$ such that $x+zb\in U(R)$. Therefore $\Psi(R)$ has stable range one. □

A right R-module A is said to have the finite exchange property if for every right R-module M and any two decompositions $M=A'\oplus N=\bigoplus_{i\in I}A_i$, where $A'_R\cong A_R$ and the index set I is finite, then there exist submodules $A_i'\subseteq A_i$ such that $M=A'\oplus\big(\bigoplus_{i\in I}A_i'\big)$. A ring R is said to be an exchange ring provided that R has the finite exchange property as a right R-module. As is well known, a ring R is an exchange ring if and only if for any $x\in R$, there exists an idempotent $e\in R$ such that $e\in xR$ and $1-e\in(1-x)R$ (cf. [8, Proposition 28.6]).

Corollary 3.3. *Let R be an exchange ring, and let $A\in M_n\big(\Psi(R)\big)$ be regular. Then A admits a diagonal reduction.*

Proof. By virtue of Theorem 3.2, $\Psi(R)$ has stable range one, and then so has $M_n\big(\Psi(R)\big)$ from [7, Corollary 5.4]. Since $A\in M_n\big(\Psi(R)\big)$ is regular, there exists $E=E^2\in M_n(R)$ such that $A(nR)=E(nR)$. Now we have a split exact sequence

$$0\to \mathrm{Ker}E\to nR\xrightarrow{E}E(nR)\to 0$$

of right R-modules. Thus, $E(nR)\oplus\mathrm{Ker}E\cong nR$, and so $E(nR)$ is a finitely generated projective right R-module. As $A\in M_n\big(\Psi(R)\big)$, we see that $E\in$

$M_n\big(\Psi(R)\big)$; hence, $E(nR) = E(nR)\Psi(R)$. As in the proof of [8, Exercise 29.9], we can find idempotents $e_1, \dots, e_n \in \Psi(R)$ such that

$$E(nR) \cong e_1R \oplus \cdots \oplus e_nR \cong \mathrm{diag}(e_1, \dots, e_n)(nR)$$

as right R-modules. This implies that

$$E(^nR) \bigotimes_R R^n \cong \mathrm{diag}(e_1, \dots, e_n)(^nR) \bigotimes_R R^n,$$

where $^nR = \{\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \mid r_1, \dots, r_n \in R\}$ and $R^n = \{(r_1, \dots, r_n) \mid r_1, \dots, r_n \in R\}$. Obviously, $^nR \bigotimes_R R^n \cong M_n(R)$. Thus, we infer that $\varphi : AM_n(R) \cong \mathrm{diag}(e_1, \dots, e_n)M_n(R)$. One easily checks that

$$M_n(R)A = M_n(R)\varphi(A) \text{ and } \varphi(A)M_n(R) = \mathrm{diag}(e_1, \dots, e_n)M_n(R).$$

Write $A = X\varphi(A)$ and $\varphi(A) = YA$. Without loss of the generality, we may assume that $X, Y \in M_n\big(\Psi(R)\big)$. Since

$$\big(X + (I_n - XY)\big)Y + (I_n - XY)(I_n - Y) = I_n \text{ and } X + (I_n - XY) \in M_n\big(\Psi(R)\big)$$

we have $Z \in M_n(R)$ such that

$$\begin{aligned} U : &= X + (I_n - XY)\big(I_n + (I_n - Y)Z\big) \\ &= X + (I_n - XY) + (I_n - XY)(I_n - Y)Z \in \mathrm{GL}_n(R). \end{aligned}$$

Thus, it follows by [4, Lemma 3.1] that there exists $W \in M_n(R)$ such that $V := Y + W(I_n - XY) \in \mathrm{GL}_n(R)$. This implies that $VA = YA = \varphi(A)$. Likewise, we have $V' \in \mathrm{GL}_n(R)$ such that $\varphi(A)V' = \mathrm{diag}(e_1, \dots, e_n)$. Therefore $VAV' = \mathrm{diag}(e_1, \dots, e_n)$, as desired. □

Corollary 3.4. *Let R be an exchange ring, and let $(a_{ij}) \in M_n(R)$ be regular. If each $1 + a_{ij}r \in R$ is unit-regular for all $r \in R$, then (a_{ij}) admits a diagonal reduction.*

Proof. In view of Lemma 2.1, $\Phi(R) = \{x \in R \mid 1 + xt \in R$ is unit-regular for all $t \in R\}$. If each $1 + a_{ij}r \in R$ is unit-regular for all $r \in R$, then $a_{ij} \in \Phi(R)$. Thus, $(a_{ij}) \in M_n\big(\Psi(R)\big)$. According to Corollary 3.3, we complete the proof. □

As is well known, a regular ring R has stable range one if and only if every element in R is unit-regular. The following result gives an analogue for ideals of a regular ring having stable range one.

Theorem 3.5. *An ideal I of a regular ring R has stable range one if and only if for any $x \in I, r \in R$, $1 + xr$ is unit-regular.*

Proof. Let I be an ideal of a regular ring R. Suppose that I has stable range one. For any $x \in I, r \in R$, $1 + xr \in 1 + I$. Write $1 + xr = (1 + xr)y(1 + xr)$ for

$y \in R$. It follows from $(1+xr)y + \big(1-(1+xr)y\big) = 1$ that there exists $z \in R$ such that

$$u := 1 + xr + \big(1-(1+xr)y\big)z \in U(R).$$

Hence, $1+xr = (1+xr)yu$. Therefore $1+xr = (1+xr)u^{-1}(1+xr)$, and so $1+xr \in R$ is unit-regular.

Conversely, assume that $1+xr$ is unit-regular for all $x \in I, r \in R$. Obviously, $\Phi(R) = \{x \in R \mid 1+xr \in ur(R) \text{ for all } r \in R\}$. Let $x, y \in \Phi(R), s, t \in R$. Then $sxt \in \Phi(R)$. Write $1+(x+y)t = \big(1+(x+y)t\big)c\big(1+(x+y)t\big)$. Then

$$\big(1+(x+y)t\big)c + \big(1-(1+(x+y)t)c\big) = 1.$$

Clearly, $x+y \in \Psi(R)$. According to Theorem 3.2, $\Psi(R)$ has stable range one. As in the proof of [6, Theorem 1.8], we see that $v := c + z\big(1-(1+(x+y)t)c\big) \in U(R)$ for $z \in R$. Therefore

$$1+(x+y)t = \big(1+(x+y)t\big)v\big(1+(x+y)t\big).$$

That is, $1+(x+y)t \in ur(R)$. Thus $x+y \in \Phi(R)$, and so $\Phi(R)$ is an ideal of R. We infer that $\Phi(R) = \Psi(R)$. Hence, $\Phi(R)$ has stable range one by Theorem 3.2. In fact, one easily checks that

$$\begin{aligned}\Phi(R) &= \sum_{I \trianglelefteq R} \{I \mid I \text{ has stable range one}\}\\ &= \{x \in R \mid RxR \text{ has stable range one}\}.\end{aligned}$$

By hypothesis, $I \subseteq \Phi(R)$, and therefore I has stable range one, as asserted. □

Analogously, we deduce that an ideal I of a regular ring R has stable range one if and only if for any $x \in I, r \in R$, $1+rx$ is unit-regular. Let R be a regular ring, and let $(a_{ij}) \in M_n(R)$. If $1+a_{ij}r \in R$ is unit-regular for all $r \in R$, then (a_{ij}) is the product (sum) of an idempotent matrix and an invertible matrix over R. Set $I = \sum_{1\le i,j\le n} Ra_{ij}R$. Then each $a_{ij} \in \Phi(R)$, and so $Ra_{ij}R \subseteq \Psi(R)$. This implies that I has stable range one by Theorem 3.5, and we are done.

4. Ψ-regularity

We say that $a \in R$ is Ψ-regular in case there exists some $u \in \Psi(R)$ such that $a = aua$. Let I be an ideal of a ring R. We say that I is Ψ-regular in case every element in I is Ψ-regular. A ring R is Ψ-regular provided that it is Ψ-regular as an ideal of itself.

Lemma 4.1. *Let $a \in R$. Then $a \in R$ is Ψ-regular if and only if there exists some $x \in \Psi(R)$ such that $a - axa$ is Ψ-regular.*

Proof. Suppose that $a - axa$ is Ψ-regular and $x \in \Psi(R)$. Then we have $u \in \Psi(R)$ such that $a - axa = (a-axa)u(a-axa)$. Hence $a = a\big(x + (1-xa)u(1-ax)\big)a$. As $x, u \in \Psi(R)$, we deduce that $x + (1-xa)u(1-ax) \in \Psi(R)$. So $a \in R$ is Ψ-regular. The converse is obvious. □

Lemma 4.2. *Let I be an ideal of a ring R, and let $e \in R$ be an idempotent. If I is Ψ-regular, then so is eIe.*

Proof. Given any $a \in I$, we have $eae \in I$. Let I be Ψ-regular. Then we can find some $x \in \Psi(R)$ such that $eae = (eae)x(eae)$. In addition, we have $x_1, \dots, x_m \in \Phi(R)$ such that $x = x_1 + \cdots + x_m$. For any $s, t \in R$, we have $1 + (re)(ex_ie)(ese) \in ur(R)$. In view of Proposition 2.5, we get $e + (ere)(ex_ie)(ese) \in ur(eRe)$. This implies that each $ex_ie \in \Psi(eRe)$, and so $exe = ex_1e + \cdots + ex_me \in \Psi(eRe)$. As $eae = (eae)(exe)(eae)$, we prove that $eae \in eIe$ is Ψ-regular, as required. $\square$

Theorem 4.3. *Let I be an ideal of a ring R. If I is Ψ-regular, then so is $M_n(I)$.*

Proof. Given any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(I)$, we have $b' \in \Psi(R)$ such that $b = bb'b$. Write $b' = h_1 + \cdots + h_m (m \in \mathbb{N})$, where each $h_i \in \Phi(R)$. Then $\begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix} = \sum_{i=1}^m \begin{pmatrix} 0 & 0 \\ h_i & 0 \end{pmatrix}$. For any $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in M_2(R)$, we have

$$I_2 + \begin{pmatrix} 0 & 0 \\ h_i & 0 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ h_i r_{11} & 1 + h_i r_{12} \end{pmatrix}.$$

Clearly, we can find $U \in \mathrm{GL}_2(R)$ such that $\begin{pmatrix} 1 & 0 \\ h_i r_{11} & 1+h_i r_{12} \end{pmatrix} = U \begin{pmatrix} 1 & 0 \\ 0 & 1+h_i r_{12} \end{pmatrix}$. So we see that $I_2 + \begin{pmatrix} 0 & 0 \\ h_i & 0 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in ur\big(M_2(R)\big)$. As a result, $\begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix} \in \Psi\big(M_2(R)\big)$. By virtue of Lemma 4.1, it suffices to prove that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(I)$$

is Ψ-regular. That is, it suffices to prove that $\begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in M_2(I)$ is Ψ-regular. Clearly, we have $a'', d'' \in \Psi(R)$ such that $a' = a'a''a'$ and $d' = d'd''d'$. According Theorem 2.3, we see that $\begin{pmatrix} a'' & 0 \\ 0 & d'' \end{pmatrix} \in \Psi\big(M_2(I)\big)$, and so it suffices to prove that

$$\begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} - \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \begin{pmatrix} a'' & 0 \\ 0 & d'' \end{pmatrix} \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c'' & 0 \end{pmatrix} \in M_2(I)$$

is Ψ-regular. Obviously, we have $u \in \Psi(R)$ such that $c'' = c''uc''$. Hence

$$\begin{pmatrix} 0 & 0 \\ c'' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c'' & 0 \end{pmatrix} \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c'' & 0 \end{pmatrix}.$$

As $u \in \Psi(R)$, it follows from Theorem 2.3 that $\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \in \Psi\big(M_2(R)\big)$. We infer that $\begin{pmatrix} 0 & 0 \\ c'' & 0 \end{pmatrix}$ is Ψ-regular. Consequently, $M_2(I)$ is Ψ-regular. By induction, $M_{2^n}(I)$ is Ψ-regular. Choose $E = \mathrm{diag}(I_n, 0, \dots, 0)_{2^n \times 2^n}$. Therefore $M_n(I) \cong EM_{2^n}(I)E$ is Ψ-regular from Lemma 4.2. $\square$

Proposition 4.4. *A ring R is Ψ-regular if and only it is unit-regular.*

Proof. Suppose that R is unit-regular. Given any $a \in R$, there exists $x \in R$ such that $a = axa$. Clearly, $1 + rxs \in ur(R)$ for all $r, s \in R$. Hence $x \in \Psi(R)$. This infers that R is Ψ-regular.

Conversely, assume that R is Ψ-regular. Given any $a \in R$, there exists $x \in \Psi(R)$ such that $a = axa$. Obviously, $\big(x + (1 - xa)\big)a + (1 - xa)(1 - a) = 1$. Since $x + (1 - xa) \in 1 + \Psi(R)$, by Theorem 3.2, there exists $y \in R$ such that $x+(1-xa)+(1-xa)(1-a)y \in U(R)$. That is, $u := x+(1-xa)\big(1+(1-a)y\big) \in U(R)$. Therefore $a = axa = aua$, as desired. □

Corollary 4.5. *If R is unit-regular, then so is $M_n(R)$ for all $n \in \mathbb{N}$.*

Proof. According to Proposition 4.4 and Theorem 4.3, we complete the proof. □

We note that Corollary 4.5 is a well-known result in the theory of unit-regular rings. However, the only known proof of it depends on cancelation of modules (cf. [5, Theorem 4.5]). Our treatment above provided the first element-wise proof of this result.

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