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RATIONAL DIFFERENCE EQUATIONS WITH POSITIVE EQUILIBRIUM POINT

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Abstract. In this note we study positive solutions of the mth order rational difference equation $x_n = (a_0 + \sum_{i=1}^m a_i x_{n-i})/(b_0 + \sum_{i=1}^m b_i x_{n-i})$, where $n = m, m+1, m+2, \dots$ and $x_0, \dots, x_{m-1} > 0$. We describe a sufficient condition on nonnegative real numbers $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_m$ under which every solution x_n of the above equation tends to the limit $(A - b_0 + \sqrt{(A-b_0)^2 + 4a_0B})/2B$ as $n \to \infty$, where $A = \sum_{i=1}^m a_i$ and $B = \sum_{i=1}^m b_i$.

1. Introduction

Consider a sequence of positive numbers $x_0, x_1, x_2, \dots$ defined by the difference equation

$$x_n = \frac{a_0 + \sum_{i=1}^m a_i x_{n-i}}{b_0 + \sum_{i=1}^m b_i x_{n-i}} \tag{1}$$

for $n = m, m+1, m+2, \dots$, where m is a positive integer, $a_0, a_1, \dots, a_m$, $b_0, b_1, \dots, b_m \geqslant 0$ and $x_0, \dots, x_{m-1} > 0$. Suppose $a_i b_i > 0$ for at least one $i \in \{1, \dots, m\}$. Set

$$M = \min_{1 \leqslant i \leqslant m,\, a_i b_i \neq 0} \frac{a_i}{b_i}. \tag{2}$$

We shall prove the following:

Theorem. *Suppose that $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_m \geqslant 0$, where $a_i b_i > 0$ for at least one $i \in \{1, \dots, m\}$, and no index $j \in \{1, \dots, m\}$ exists for which $a_j = 0$ but $b_j \neq 0$. Let $x_0, x_1, x_2, \dots$ be a sequence of positive numbers defined by* (1). *If $a_0/M + A - MB < b_0 \leqslant A$, where $A = \sum_{i=1}^m a_i$, $B = \sum_{i=1}^m b_i$ and M is defined by* (2), *then*

$$\lim_{n \to \infty} x_n = (A - b_0 + \sqrt{(A-b_0)^2 + 4a_0B})/2B$$

for any choice of initial values $x_0, \dots, x_{m-1}$.

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Several partial cases of the equation (1) have been studied on many occasions. One can find numerous references in the monographs [3] and [2] devoted to the cases $m = 2$ and $m = 3$ of (1), respectively. Generally speaking, the positive solution x_n of (1) or, more precisely, the sequence $(x_n)_{n=0}^{\infty}$ can be bounded or unbounded, periodic or not periodic, stable or not stable, etc. In particular, the authors of [3] distinguished 49 special cases of the equation (1) with $m = 2$. Later, 225 different types of (1) with $m = 3$ have been examined in [2].

One should say that the equation (1) often arises not only in pure and applied mathematics but also in various mathematical models of biological systems. Sometimes this is an additional motivation for its study. One of the most natural questions is to determine whether the sequence $(x_n)_{n=1}^{\infty}$, which is a positive solution of (1), has a single finite limit point or not. The difference equation (1) is called *globally stable* if, for any choice of initial values $x_0, \dots, x_{m-1} > 0$, the solution x_n of (1) tends to a finite limit $\overline{x}$, which is called the *equilibrium point*. In this terminology, our theorem gives a sufficient condition on $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_m$ under which the equation (1) is globally stable.

Some sufficient conditions on the coefficients a_i, b_i under which (1) is globally stable have been considered in [1] and [4]. More precisely, Camouzis [1] studied the case $m = 3$, $a_0 = a_2 = b_3 = 0$, $a_1, a_3, b_0, b_1, b_2 \geqslant 0$. Park [4] investigated the case $m = 3$, $a_0 = a_2 = b_2 = 0$, $a_3 = b_3 = 1$, $a_1, b_0, b_1 \geqslant 0$. In the last section, we will show that the main theorem of [4] also follows from our theorem (In fact, the same conclusion follows under even weaker assumptions).

We remark that, by our theorem, every solution x_n tends to a positive equilibrium point if either $b_0 < A$ or $a_0 > 0$. Indeed, since $0 \leqslant b_0 \leqslant A$, $a_0 \geqslant 0$, $A, B > 0$, we have $(A - b_0 + \sqrt{(A - b_0)^2 + 4a_0 B})/2B = 0$ if and only if $b_0 = A$ and $a_0 = 0$.

2. Proof of Theorem

Put

$$\mathcal{I} = \{i \,:\, 1 \leqslant i \leqslant m,\ a_i = Mb_i\} \tag{3}$$

and

$$\mathcal{J} = \{1, \dots, m\} \setminus \mathcal{I}. \tag{4}$$

Note that

$$a_i \geqslant Mb_i \quad \text{for each} \quad i = 1, \dots, m, \tag{5}$$

so

$$\sum_{i=1}^{m} a_i = A \geqslant MB = M \sum_{i=1}^{m} b_i. \tag{6}$$

Let $(x_n)_{n=1}^{\infty}$ be a sequence of positive numbers satisfying (1). We claim that if

$$b_0 > a_0/M + A - MB, \tag{7}$$

where $M > 0$ is given in (2), then

$$x_n \leqslant M \tag{8}$$

for every sufficiently large n.

For the sake of contradiction, assume that $x_n > M$ for infinitely many positive integers n. Take one of those n's satisfying $n \geqslant n_0$, where n_0 is an integer to be chosen later. Then, by (1), we have

$$M < x_n = \frac{a_0 + \sum_{i=1}^m a_i x_{n-i}}{b_0 + \sum_{i=1}^m b_i x_{n-i}}.$$

Multiplying by the denominator $b_0 + \sum_{i=1}^m b_i x_{n-i}$ and using (3), we find that

$$Mb_0 - a_0 < \sum_{i=1}^{m}(a_i - Mb_i)x_{n-i} = \sum_{1\leqslant i\leqslant m,\, i\notin\mathcal{I}} (a_i - Mb_i)x_{n-i}. \tag{9}$$

This cannot happen if $\mathcal{I} = \{1, \dots, m\}$, because then the right hand side of (9) is zero, whereas (6) and (7) imply that $Mb_0 - a_0 > 0$.

So assume that the set $\mathcal{J}$ given in (4) is not empty. Estimating each x_{n-i}, where $i \in \mathcal{J}$, by the maximum of those x_{n-i}, say, $x_{n-i_1} = \max_{i\in\mathcal{J}} x_{n-i}$, from (9) we deduce that

$$Mb_0 - a_0 < x_{n-i_1} \sum_{i\in\mathcal{J}} (a_i - Mb_i). \tag{10}$$

From (3) and (4) it follows that $\sum_{i\in\mathcal{J}}(a_i - Mb_i) = A - MB > 0$. Therefore, the quotient $q = b_0/(A - MB)$ is greater than 1, by (7). Dividing (10) by $A - MB$, we find that

$$x_{n-i_1} > (Mb_0 - a_0)/(A - MB) = Mq - t$$

with $t = a_0/(A - MB) \geqslant 0$. On applying the same argument to $x_{n-i_1} > Mq - t$ (instead of $x_n > M$ as above), we derive that there is an index $i_2 \in \mathcal{J}$ such that $x_{n-i_1-i_2} > (Mq - t)q - t$ and so on. The process stops after, say, k steps, when we have

$$\begin{aligned} x_{n-i_1-\dots-i_k} &> Mq^k - t(q^{k-1} + \dots + 1) = t/(q-1) + (M - t/(q-1))q^k \\ &\geqslant (M - t/(q-1))q^k \end{aligned}$$

and $0 \leqslant n - i_1 - \dots - i_k \leqslant m - 1$. Putting $\mu = \max(x_0, \dots, x_{m-1})$ we thus obtain

$$(M - t/(q-1))q^k < \mu. \tag{11}$$

Note that $M - t/(q - 1) > 0$, because, by the definition of q and t, this is equivalent to the inequality (7).

On the other hand,

$$n_0 \leqslant n \leqslant i_1 + \cdots + i_k + m - 1 \leqslant mk + m - 1 < m(k+1),$$

because each index i_l, $1 \leqslant l \leqslant m$, is at most m. Hence $k > n_0/m - 1$. Select n_0 so large that $(M - t/(q-1))q^{n_0/m-1} > \mu$. Then, by (11), k must be smaller than $n_0/m - 1$, which is a contradiction with $k > n_0/m - 1$. This proves (8).

Next, we will prove that if either $b_0 < A$ or $a_0 > 0$, then there is a positive number u such that

$$x_n \geqslant u \tag{12}$$

for each $n \geqslant 0$. Suppose first that $b_0 < A$. Set $\tau = \min(x_0, \dots, x_{m-1})$ and $\varrho = 1 - b_0/A > 0$. We will prove that then

$$x_n \geqslant u = \min(\tau, \varrho M) \tag{13}$$

for each $n \geqslant 0$.

To prove (13) assume that n is the least index for which $x_n < u = \min(\tau, \varrho M)$. Clearly, $n \geqslant m$. Then, by (1) and (5), we obtain

$$\begin{aligned} a_0 + \sum_{i=1}^{m} a_i x_{n-i} &= x_n b_0 + x_n \sum_{i=1}^{m} b_i x_{n-i} \\ &< u b_0 + \varrho M \sum_{i=1}^{m} b_i x_{n-i} \\ &\leqslant u b_0 + \varrho \sum_{i=1}^{m} a_i x_{n-i}. \end{aligned}$$

Hence

$$u b_0 > a_0 + (1-\varrho) \sum_{i=1}^{m} a_i x_{n-i} \geqslant (1-\varrho) \sum_{i=1}^{m} a_i x_{n-i} \geqslant (1-\varrho) \sum_{i=1}^{m} a_i u = (1-\varrho) A u,$$

because $a_0 \geqslant 0$ and $x_{n-i} \geqslant u$. This yields $b_0 > (1-\varrho)A = b_0$, a contradiction. The proof of (13) is completed.

We now turn to the case $a_0 > 0$. This time, select $u = \min(\tau, M, a_0/(b_0+1))$. Then $a_0 > u b_0$ and $a_i \geqslant u b_i$ for each $i = 1, \dots, m$, by (5). Assume that $x_n < u$ for some $n \geqslant 0$. Then $n \geqslant m$. So (1) implies that

$$a_0 + \sum_{i=1}^{m} a_i x_{n-i} = x_n b_0 + x_n \sum_{i=1}^{m} b_i x_{n-i} < u b_0 + \sum_{i=1}^{m} u b_i x_{n-i} \leqslant u b_0 + \sum_{i=1}^{m} a_i x_{n-i},$$

giving $a_0 < u b_0$, a contradiction. This completes the proof of (12).

Combining (8) and (12), we deduce that $x_n \in [u, M]$ for each $n \geqslant n_0$. Here, $u > 0$ if $b_0 < A$ or $a_0 > 0$. Alternatively, if $b_0 = A$ and $a_0 = 0$, we can trivially take $u = 0$, because all x_n are positive. Put

$$S = \limsup_{n \to \infty} x_n, \qquad I = \liminf_{n \to \infty} x_n.$$

Then $0 \leqslant u \leqslant I \leqslant S \leqslant M$, where $u = 0$ if and only if $b_0 = A$ and $a_0 = 0$.

Let

$z_1 = (A-b_0-\sqrt{(A-b_0)^2+4a_0B})/2B$, $z_2 = (A-b_0+\sqrt{(A-b_0)^2+4a_0B})/2B$

be the solutions of the equation

$$Bz^2 - (A-b_0)z - a_0 = B(z-z_1)(z-z_2) = 0. \tag{14}$$

We shall prove that $S \leqslant z_2$ and $I \geqslant z_2$. This yields $S = I = z_2$, and so the proof of the theorem will be completed.

By the above, the sequence of vectors $(x_n, x_{n-1}, \dots, x_{n-m})$, $n = n_0+m, n_0+m+1, \dots$, belongs the $(m+1)$-dimensional cube $[u, M]^{m+1}$. So, by compactness, there is a sequence of positive integers n_k, $k = 1, 2, 3, \dots$, such that the vector $(x_{n_k}, x_{n_k-1}, \dots, x_{n_k-m})$ tends to the vector $(S, S_1, \dots, S_m)$ as $k \to \infty$, where $S_1, \dots, S_m \leqslant S$ and $u \leqslant S_1, \dots, S_m, S \leqslant M$. From (1) it follows that $S(b_0 + \sum_{i=1}^m b_i S_i) = a_0 + \sum_{i=1}^m a_i S_i$. Hence

$$Sb_0 - a_0 + (Sb_1 - a_1)S_1 + \dots + (Sb_m - a_m)S_m = 0. \tag{15}$$

By (5) and $S \leqslant M$, we obtain $Sb_i - a_i \leqslant 0$ for each $i = 1, \dots, m$. Hence $(Sb_i - a_i)S_i \geqslant (Sb_i - a_i)S$ for $i = 1, \dots, m$. Therefore, on replacing each S_i by S in (15) we will not increase the sum on the left hand side of (15). Hence

$$BS^2 - (A-b_0)S - a_0 = Sb_0 - a_0 + \sum_{i=1}^m (Sb_i - a_i)S \leqslant 0.$$

So $S \in [z_1, z_2]$, by (14), giving $S \leqslant z_2$.

We now consider two cases, $u = 0$ and $u > 0$. In the first case, $u = 0$, we have $z_2 = 0$, because $b_0 = A$ and $a_0 = 0$. In this case also $I = 0$, because $I \geqslant 0$. So $S = I = 0$, which completes the proof of the theorem.

In the second case, $u > 0$, we have $z_1 \leqslant 0 < z_2$ and $S \leqslant z_2$. It remains to prove that $I \geqslant z_2$. The argument is similar to that given above. By compactness, there is a sequence of positive integers ℓ_k, $k = 1, 2, 3, \dots$, such that the vector $(x_{\ell_k}, x_{\ell_k-1}, \dots, x_{\ell_k-m})$ tends to the vector $(I, I_1, \dots, I_m)$ as $k \to \infty$, where $I_1, \dots, I_m \geqslant I \geqslant u > 0$. Now, from (1) it follows that $I(b_0 + \sum_{i=1}^m b_i I_i) = a_0 + \sum_{i=1}^m a_i I_i$. Hence

$$Ib_0 - a_0 + (Ib_1 - a_1)I_1 + \dots + (Ib_m - a_m)I_m = 0. \tag{16}$$

By (5) and $I \leqslant S \leqslant M$, we have $Ib_i - a_i \leqslant 0$ for each $i = 1, \dots, m$. Hence $(Ib_i - a_i)I_i \leqslant (Ib_i - a_i)I$ for $i = 1, \dots, m$. This time, on replacing in (16) each I_i by I we will not decrease the sum on the left hand side of (16), so that

$$B(I-z_1)(I-z_2) = BI^2 - (A-b_0)I - a_0 = Ib_0 - a_0 + \sum_{i=1}^m (Ib_i - a_i)I \geqslant 0.$$

Since $I > 0$, we must have $I \geqslant z_2$ for otherwise $(I - z_1)(I - z_2) < 0$. This proves our assertion $I = S = z_2$.

3. Examples

As the first example we shall consider the case $m = 3$, $a_0 = a_2 = b_2 = 0$, $a_3 = b_3 = 1$, $a_1, b_0, b_1 > 0$. It was proved in [4] that then $\lim_{n\to\infty} x_n = \overline{x} = (a_1 + 1 - b_0)/(b_1 + 1)$ provided that $0 < a_1 \leqslant b_1$ and $1 < b_0 < a_1 + 1$. We shall prove that the same holds under weaker conditions $0 < a_1 \leqslant b_1$ and $1 - a_1/b_1 < b_0 < a_1 + 1$.

Indeed, with the notation of our theorem, we have

$$\begin{aligned} A &= a_1 + a_2 + a_3 = a_1 + 1, \\ B &= b_1 + b_2 + b_3 = b_1 + 1, \\ z_2 &= (A - b_0 + \sqrt{(A - b_0)^2 + 4a_0 B})/2B \\ &= (A - b_0)/B = \overline{x} = (a_1 + 1 - b_0)/(b_1 + 1). \end{aligned}$$

By (2), $M = \min(a_1/b_1, 1) = a_1/b_1$, because $a_1 \leqslant b_1$. The condition of the theorem saying that no index $j \in \{1, \ldots, m\}$ exists for which $a_j = 0$ but $b_j \neq 0$ is satisfied, because $a_2 = b_2 = 0$ and $a_1 b_1, a_3 b_3 > 0$. Since $a_0 = 0$ and $A - MB = 1 - a_1/b_1$, the condition

$$a_0/M + A - MB < b_0 \leqslant A$$

of the theorem is equivalent to $1 - a_1/b_1 < b_0 \leqslant a_1 + 1$. Evidently, the equilibrium point $z_2 = \overline{x} = (a_1 + 1 - b_0)/(b_1 + 1)$ is positive if $b_0 < a_1 + 1$. We conclude that if $0 < a_1 \leqslant b_1$ and $1 - a_1/b_1 < b_0 < a_1 + 1$, then the third order rational difference equation

$$x_n = \frac{a_1 x_{n-1} + x_{n-3}}{b_0 + b_1 x_{n-1} + x_{n-3}},$$

$n = 3, 4, \ldots$, where $x_0, x_1, x_2 > 0$, has a positive solution x_n which converges to the positive equilibrium point $(a_1 + 1 - b_0)/(b_1 + 1)$ as $n \to \infty$.

More generally, suppose that $a_0 = 0$ and $a_1, \ldots, a_m, b_1, \ldots, b_m > 0$. Assume that $a_i \geqslant b_i$ for each $j \in \{1, 2, \ldots, m\}$ with at least one case of equality. Then $M = 1$. Our theorem implies that if $a_0/M + A - MB = A - B < b_0 < A$, then the difference equation

$$x_n = \frac{\sum_{i=1}^m a_i x_{n-i}}{b_0 + \sum_{i=1}^m b_i x_{n-i}},$$

$n = m, m+1, m+2, \ldots$, where $x_0, \ldots, x_{m-1} > 0$, has a positive solution x_n which tends to the positive equilibrium point $(A - b_0)/B$ as $n \to \infty$.

Is the sufficient condition $a_0/M + A - MB < b_0 \leqslant A$ of the theorem sharp for its conclusion $\lim_{n\to\infty} x_n = (A - b_0 + \sqrt{(A - b_0)^2 + 4a_0 B})/2B$? Clearly, inequality $b_0 \leqslant A$ is sharp. Indeed, b_0 cannot be greater than A for $a_0 = 0$, because the limit $(A - b_0)/B$ cannot be negative. A simple example $x_n = x_{n-1}/(x_{n-1} + 1 + \varepsilon)$, where $\varepsilon > 0$, $A = B = 1$, $b_0 = 1 + \varepsilon$, shows that for its every positive solution x_n we have $\lim_{n\to\infty} x_n = 0$ (and not $(A - b_0)/B = -\varepsilon$).

To test the lower bound $b_0 > a_0/M + A - MB$, let us consider the second order difference equation

$$x_n = \frac{\varepsilon x_{n-1} + x_{n-2}}{b_0 + x_{n-1}} \tag{17}$$

with some fixed positive ε. Then $a_0 = 0$, $A = 1 + \varepsilon$, $B = 1$, $M = \varepsilon$. Hence $a_0/M + A - MB = 1$. By the theorem, every positive solution x_n of this equation tends to $1 + \varepsilon - b_0$ provided that $1 < b_0 < 1 + \varepsilon$. We remark that, in this particular case, the same conclusion follows under weaker assumption $1-\varepsilon < b_0 < 1+\varepsilon$ (see Equation #83 on p. 245 in [2]). However, if $b_0 < 1-\varepsilon$, then the positive solution $(x_n)_{n=0}^{\infty}$ of (17) can be even unbounded for some choice of initial values $x_0, x_1 > 0$. According to [2] (see p. 246), the determination of those initial values $x_0, x_1 > 0$, for which the solution $(x_n)_{n=0}^{\infty}$ of (17) is unbounded, is still an open problem.

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