Bull. Korean Math. Soc. **47** (2010), No. 3, pp. 633–643
DOI 10.4134/BKMS.2010.47.3.633

CHARACTERIZATIONS OF DISTRIBUTIVE LATTICES AND SEMICONTINUOUS LATTICES

Jiang Guanghao and Shi Weixue

Abstract. In this paper, the concept of maximal ideals relative to a filter on posets is introduced and examined. An intrinsic characterization of distributive lattices is obtained. In addition, we also give a characterization of pseudo primes in semicontinuous lattices and a characterization of semicontinuous lattices. Functions of semicontinuous lattices which are order preserving and semicontinuous are studied. A new concept of semiarithmetic lattices is introduced and examined.

1. Introduction and preliminaries

It should be noted that the study of distinctive features of some special elements in continuous lattices, as well as in semicontinuous lattices, is of fundamental importance. And the study of irreducible elements and primes in continuous lattices was begun in [3]. From then on, pseudo primes, weakly primes and weakly irreducible elements were also investigated by many authors (see [1] and [2]). The study of semiprime ideals was begun in [5] by Y. Ray. The theory of semicontinuous lattices was first developed by D. Zhao in [7]. In [6], X. Wu et al. defined the semi-Scott topology and semicontinuous functions. In this paper, the concept of maximal ideals relative to a filter on posets is introduced and examined. The existence of maximal ideals relative to a filter is proved. As an application, we manage to give an intrinsic characterizations of distributive lattices. Characterizations of pseudo primes in semicontinuous lattices and semicontinuous lattices are also obtained. In addition, functions of semicontinuous lattices which are order preserving and semicontinuous are studied and some surprising results are obtained. We show that the strong retract of a stable semicontinuous lattice is stable semicontinuous. Finally, a new concept of lattices, semiarithmetic lattices, is introduced and examined.

Received January 10, 2009.
2000 *Mathematics Subject Classification.* 06A11, 06B35, 54H10.
Key words and phrases. semiprime ideal, maximal ideals relative to a filter, semicontinuous lattice, semicontinuous function, semialgebraic lattice.
Research supported by the NSFC of China (10571081), (10971092) and by the NSF (KJ2009B001) and (KJ2009A050Z) from Education Section of Anhui Province, China.

The following are some basic concepts needed in the sequel, other non-explicitly stated elementary notions please refer to [2] and [7].

An *ideal* on a partially ordered set (in short, *poset*) L means a lower set which is also directed, and a *filter* on a poset can be dually defined. For a semilattice L, a proper ideal I of L is called a *prime ideal* if for any two elements a, b of L, $a \wedge b \in I$ implies $a \in I$ or $b \in I$. For a lattice L, an ideal I of L is called a *semiprime ideal* if for any three elements a, b, c of L, the relations $a \wedge b \in I, a \wedge c \in I$ always imply $a \wedge (b \vee c) \in I$. The set of semiprime ideals of L is denoted by $Rd(L)$. It is easy to see that every prime ideal is semiprime in a lattice.

Recall that in a complete lattice L, for $x, y \in L$, we say that $x \Leftarrow y$, if for any $I \in Rd(L)$, $y \leq \bigvee I$ always implies $x \in I$. If $x \Leftarrow x$, then x is called a $\Leftarrow$*-compact element*. For any $x \in L$, let $\Downarrow x = \{y \in L : y \Leftarrow x\}$ and $\Uparrow x = \{y \in L : x \Leftarrow y\}$. The set of $\Leftarrow$-compact elements is denoted by $SK(L)$. It is easily seen that $SK(L)$ is a sup-semilattice with a minimum element. A complete lattice L is said to be *semicontinuous lattice* if for each $x \in L$, $x \leq \bigvee \Downarrow x$.

A subset U in a complete lattice L is said to be *semi-Scott open* if and only if it satisfies (1) $U = \uparrow U$ (the upper set of U) and (2) for each $I \in Rd(L), I \cap U \neq \emptyset$ whenever $\bigvee I \in U$. The set of semi-Scott open sets form a topology, called the *semi-Scott topology*, denoted by $\sigma_{\Leftarrow}(L)$. Clearly, filters are particular upper sets. We call those that are semi-Scott open in the sense just defined *semi-Scott open filters*.

2. Maximal ideals relative to a filter

In [4], the concept of locally maximal ideals on posets was introduced: an ideal M on a poset L is a *locally maximal ideal* if and only if there is an element $x \in L$ such that M is maximal among the ideals which do not contain x. We generalize this to the concept of maximal ideals relative to a filter.

Definition 2.1. Let M be a proper ideal on a poset L. If there is a filter $F \in$ Filt L such that M is maximal among the ideals which do not intersect F (i.e., for an ideal I on L, $I \cap F = \emptyset$ and $I \supseteq M$ implies $I = M$), then we say M is a maximal ideal relative to the filter F on poset L, or roughly, a maximal ideal relative to a filter.

Remark 2.2. Let M be a proper ideal on a poset L and $x \in L \backslash M$. Then it is easy to see that M is a locally maximal ideal relative to x if and only if M is a maximal ideal relative to filter $\uparrow x$. In particular, every locally maximal ideal of L is a maximal ideal relative to a filter. Counterexamples can be found (cf: [4, Example 2] or the dull poset of the lattice given by the figure above Proposition I-3.3 in [2]) to show that maximal ideals relative to a filter may not be locally maximal ideals.

The following theorem shows the existence of maximal ideals relative to a filter.

Theorem 2.3. *Let L be a poset, $I \in$Idl L, $F \in$Filt L and $I \cap F = \emptyset$. Then there always exists a maximal ideal M relative to filter F such that $M \cap F = \emptyset$ and $M \supseteq I$.*

Proof. Define $\mathcal{A} = \{J : J \text{ is an ideal of } L,\ J \cap F = \emptyset \text{ and } J \supseteq I\}$. By the assumption, we see $I \in \mathcal{A} \neq \emptyset$ and $\mathcal{A}$ is a poset ordered by set inclusion $\subseteq$. Let $\mathcal{B}$ be a chain of $\mathcal{A}$. Let $K = \bigcup_{J \in \mathcal{B}} J$. Claim that $K \in \mathcal{A}$. It is clear that K is a lower set. For $x, y \in K$, there are $J_1, J_2 \in \mathcal{B}$ such that $x \in J_1$ and $y \in J_2$. Since $\mathcal{B}$ is a chain of $\mathcal{A}$, $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$. Suppose $J_1 \subseteq J_2$ without losing generality. Then $x, y \in J_2$ and there is $z \in J_2 \subseteq K$ such that $x, y \leq z$. This shows that K is directed and an ideal of L. By the definition of K and $\mathcal{A}$, we see that $I \subseteq K$ and $K \cap F = \emptyset$. Thus $K \in \mathcal{A}$. By Zorn's Lemma, there is a maximal element $M \in \mathcal{A}$. This M is indeed a maximal ideal relative to filter F which is what we need. □

The following proposition gives characterization of maximal ideals relative to a filter on a sup semilattice.

Proposition 2.4. *Let L be a sup-semilattice. Let $I \in$Idl L, $F \in$Filt L and $I \cap F = \emptyset$. Then I is a maximal ideal relative to filter F if and only if for all $x \in L \backslash I$, there are $y \in F$ and $a \in I$ such that $y \leq x \vee a$.*

Proof. Necessity: Let I be a maximal ideal relative to filter F. Then for all $x \in L\backslash I$, define $I_1 = \bigcup\{\downarrow(x \vee a) : a \in I\}$. It is easy to show that I_1 is an ideal and $I_1 \supseteq I \neq I_1$. By the maximality of I and $I \cap F = \emptyset$, we have that $I_1 \cap F \neq \emptyset$. Pick $y \in I_1 \cap F$. Thus there is $a \in I$ such that $y \leq x \vee a$.

Sufficiency: By Theorem 2.3 and $I \cap F = \emptyset$, there is a maximal ideal M relative to filter F such that $M \cap F = \emptyset$ and $M \supseteq I$. We show that $M = I$. Suppose $M \neq I$. Then there is $x \in M$, $x \notin I$. By the assumption, there are $y \in F$ and $a \in I$ such that $y \leq x \vee a$. Since $I \subseteq M$ and M is an ideal, $a \in M$, $x \vee a \in M$ and $y \in M$. This contradicts to $F \cap M = \emptyset$, as desired. □

Maximal ideals relative to a filter have the typical feature given in the following theorem.

Theorem 2.5. *Maximal ideals relative to a filter on a poset L are all irreducible ideals.*

Proof. Let L be a poset. Let M be a maximal ideal relative to a filter F and $M \cap F = \emptyset$. Suppose there are ideals I and J such that $M = I \cap J$ and $I \supseteq M \neq I$, $J \supseteq M \neq J$, i.e., there are $a \in I\backslash M$ and $b \in J\backslash M$. By Proposition 2.4, there are $u, v \in F$, $c, d \in M$ such that $u \leq a \vee c$, $v \leq b \vee d$, respectively. Since F is a filter, $a \vee c$, $b \vee d \in F$ and $(a \vee c) \wedge (b \vee d) \in F$. Noticing that $c \in I$, $d \in J$ and I, J are ideals, we see that $a \vee c \in I$, $b \vee d \in J$ and $(a \vee c) \wedge (b \vee d) \in I \cap J = M$. This shows that $(a \vee c) \wedge (b \vee d) \in M \cap F \neq \emptyset$, a contradiction. So, M is an irreducible ideal. □

The following two corollaries now can be immediately follow from Theorems 2.3 and 2.5:

Corollary 2.6. *Let L be a lattice. Let $I \in$Idl L, $F \in$Filt L and $I \cap F = \emptyset$. Then there is an irreducible ideal J of L such that $J \cap F = \emptyset$ and $J \supseteq I$.*

Corollary 2.7. *Maximal ideals relative to a filter on a distributive lattice are prime ideals.*

Proposition 2.8. *Let M be a semiprime ideal on a lattice. If M is a maximal ideal relative to a filter, then M is a prime ideal.*

Proof. Let L be a lattice. Let M be a maximal ideal relative to filter F and $M \cap F = \emptyset$. Suppose there are $a, b \in M$ such that $a \wedge b \in M$ but $a \notin M$ and $b \notin M$. By Proposition 2.4, there are $u, v \in F$, $c, d \in M$ such that $u \leq a \vee c$, $v \leq b \vee d$, respectively. Since F is a filter, $a \vee c$, $b \vee d \in F$ and $(a \vee c) \wedge (b \vee d) \in F$. Noticing that $c, d \in M$ and M is a semiprime ideal, we have $a \wedge b \in M, a \wedge d \in M$ and $a \wedge (b \vee d) \in M$; and also $c \wedge b \in M, c \wedge d \in M$ and $c \wedge (b \vee d) \in M$. It follows from M is a semiprime ideal that $(a \vee c) \wedge (b \vee d) \in M$. This shows that $(a \vee c) \wedge (b \vee d) \in M \cap F \neq \emptyset$, a contradiction. Hence, M is a prime ideal. □

This proposition shows that the similar result may be obtained in the non-distributive case. An example is given by Figure 1 to show that maximal ideals relative to a filter may be prime ideals in the non-distributive case, where $I = \{a, b, c, d, 0\}$ is a maximal ideal relative to filter $L \backslash I$ and a prime ideal but L is a non-distributive lattice.

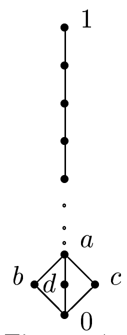

Figure 1

By Figure 1, we find that there exists a maximal ideal $\downarrow b$ relative to filter $\uparrow a$ but not a semiprime ideal in the non-distributive lattice L. By way of contrast, an immediate proposition is obtained.

Proposition 2.9. *If maximal ideals relative to a filter on a lattice L are all semiprime ideals, then L is a distributive lattice.*

Proof. Let L be a lattice and $a, b, c \in L$. Let $x = a \wedge (b \vee c)$ and $y = (a \wedge b) \vee (a \wedge c)$. It is trivial that $x \geq y$. On the other hand, suppose that $x \not\leq y$. Then $\uparrow x \bigcap \downarrow y = \emptyset$. By Theorem 2.3, there exists a maximal ideal M relative to filter $\uparrow x$ such that $M \cap \uparrow x = \emptyset$ and $M \supseteq \downarrow y$. Thus $x \notin M$ and $y \in M$. Since M is a semiprime ideal, $a \wedge b \in M, a \wedge c \in M$ and $x = a \wedge (b \vee c) \in M$, a contradiction. Hence $x \leq y$, and thus L is distributive. □

Corollary 2.10. *If maximal ideals on a lattice L are all semiprime ideals, then L is a distributive lattice.*

Corollaries 2.7, 2.10 and Proposition 2.9 give an intrinsic characterizations of distributive lattices.

Theorem 2.11. *Let L be a lattice. Then the following conditions are equivalent*:

(1) *L is a distributive lattice*;
(2) *Maximal ideals relative to a filter on L are all prime ideals*;
(3) *Maximal ideals relative to a filter on L are all semiprime ideals*;
(4) *Maximal ideals on L are all semiprime ideals*;
(5) *Maximal ideals on L are all prime ideals.*

3. Semicontinuous lattices and functions

In this section, we shall give a characterization of pseudo primes in the case of semicontinuous lattices and a characterization of semicontinuous lattices, respectively. In addition, semicontinuous functions are studied.

Recall that an element p of a poset L is called *pseudo prime element* if $p = \bigvee P$ for some prime ideal P. All the pseudo prime elements of L is denoted by ψPRIME L.

Now we give the following characterization of pseudo primes in semicontinuous lattices.

Lemma 3.1 ([2])**.** *Let L be a distributive lattice, I an ideal and F a filter in L with $I \cap F = \emptyset$. Then there is a prime ideal P in L with $P \supseteq I$ and $P \cap F = \emptyset$.*

Proposition 3.2. *Let L be a complete lattice and $1 \neq p \in L$. Consider the following statements*:

(1) *p is pseudo prime*;

(2) *In any finite collection $x_1, x_2, \ldots, x_n \in L$ with $x_1 \wedge x_2 \wedge \cdots \wedge x_n \Leftarrow p$ there is one of the elements with $x_j \leq p$*;

(3) *The filter generated by $L \backslash \downarrow p$ does not meet $\Downarrow p$.*

Then (1)⇒(2) *and* (2)⇔(3); *if L is in addition distributive semicontinuous, all three statements are equivalent.*

Proof. Condition (2) says that no finite meet of elements from $L \backslash \downarrow p$ is ever $\Leftarrow p$. Therefore (2) and (3) are always equivalent.

(1) implies (2): Let p be pseudo prime and suppose that $x_1 \wedge x_2 \wedge \cdots \wedge x_n \Leftarrow p$. Let P be a prime ideal with $\bigvee P = p$. Since every prime ideal is

semiprime, $P \in Rd(L)$, thus $x_1 \cdots x_n \in P$. Since P is prime, there is one $j \in \{1, 2, \ldots, n\}$ with $x_j \in P \subseteq \downarrow p$. That is, $x_j \leq p$.

(3) implies (1): Suppose that L is semicontinuous. Let F be the filter generated by $L \backslash \downarrow p$. Then $L \backslash \downarrow p \subseteq F$ and $F \cap \Downarrow p = \emptyset$. By Lemma 3.1, there exists a prime ideal P with $P \supseteq \Downarrow p$ and $P \cap F = \emptyset$. Since that $L \backslash \downarrow p \subseteq F$, we have $P \subseteq L \backslash F \subseteq \downarrow p$. Since L is semicontinuous, $p \leq \bigvee \Downarrow p \leq \bigvee P \leq \bigvee \downarrow p = p$. Thus $p = \bigvee P$ is pseudo prime. $\square$

Lemma 3.3 ([6])**.** *Let L be a complete lattice. Then for each $x \in L, \Downarrow x \in Rd(L)$, and if L is semicontinuous lattice, $\Uparrow x \in \sigma_{\Leftarrow}(L)$.*

Lemma 3.4 ([6])**.** *Let L be a complete lattice. Then L is a semicontinuous lattice if and only if for each $y, z \in L$, $y \not\leq z$, there is $x \in L$ with $x \Leftarrow y$ and $x \not\leq z$.*

The following theorem gives a characterization of semicontinuous lattices.

Theorem 3.5. *Let L be a complete lattice. Consider the following statements*:

(1) *L is a semicontinuous lattice*;

(2) *For each $U \in \sigma_{\Leftarrow}(L)$, $U \subseteq \bigcup\{\Uparrow x : x \in U\}$*;

(3) *For each $x \in L, x = \bigvee\{\bigwedge U : x \in U \in \sigma_{\Leftarrow}(L)\}$.*

Then (3)$\Rightarrow$(1)$\Leftrightarrow$(2); *if for each $y, z \in L, y \Rightarrow z$ implies $y \leq z$, then all three statements are equivalent.*

Proof. (1) implies (2): For each $U \in \sigma_{\Leftarrow}(L)$, let $y \in U$, by Lemma 3.3, $\Downarrow y \in Rd(L)$. Since L is a semicontinuous lattice. Then $y \leq \bigvee \Downarrow y$. Thus $\bigvee \Downarrow y \in U$. Since $U \in \sigma_{\Leftarrow}(L)$, $\Downarrow y \bigcap U \neq \emptyset$. Then there is $x \in \Downarrow y \cap U$. Hence $y \in \Uparrow x$, and thus $U \subseteq \bigcup\{\Uparrow x : x \in U\}$.

(2) implies (1): For each $x \in L$, let $y = \bigvee \Downarrow x$. Suppose that $x \not\leq y$. Then, by Lemma 3.4, there is $z \in L$ with $z \Leftarrow x$ and $z \not\leq y$. Then $z \in \Downarrow x \leq \bigvee \Downarrow x = y$, a contradiction. Hence $x \leq \bigvee \Downarrow x$, and thus L is a semicontinuous lattice.

(3) implies (1): Suppose that for each $x \in L$, $x \in U \in \sigma_{\Leftarrow}(L), I \in Rd(L)$ with $x \leq \bigvee I$. Thus $\bigvee I \in U$. Since $U \in \sigma_{\Leftarrow}(L)$, $I \cap U \neq \emptyset$. Then there is $y \in I \cap U$, and thus $\bigwedge U \leq y \in I$. It is trivial that $\bigwedge U \in I$. By the definition of $\Leftarrow$, $\bigwedge U \Leftarrow x$, i.e., $\{\bigwedge U : x \in U \in \sigma_{\Leftarrow}(L)\} \subseteq \Downarrow x$. Therefore, $x = \bigvee\{\bigwedge U : x \in U \in \sigma_{\Leftarrow}(L)\} \leq \bigvee \Downarrow x$.

(1) implies (3): Let $y = \bigvee\{\bigwedge U : x \in U \in \sigma_{\Leftarrow}(L)\}$. It is trivial that $y \leq x$. We shall show that $x \leq y$. Suppose that $x \not\leq y$. Then, by Lemma 3.4, there is $z \in L$ with $z \Leftarrow x$ and $z \not\leq y$. Since $x \in \Uparrow z \in \sigma_{\Leftarrow}(L)$, then $y = \bigvee\{\bigwedge U : x \in U \in \sigma_{\Leftarrow}(L)\} \geq \bigwedge(\Uparrow z)$. That $\Uparrow z \subseteq \uparrow z$ follows immediately from our hypothesis. Thus, $y \geq \bigwedge(\Uparrow z) \geq \bigwedge(\uparrow z) = z$, a contradiction. Therefore, $x = \bigvee\{\bigwedge U : x \in U \in \sigma_{\Leftarrow}(L)\}$. $\square$

Let us now consider functions of semicontinuous lattices which are order preserving and semicontinuous. Recall that a function $f : L \to L_1$ between complete lattices is *semicontinuous* if and only if for each $I \in Rd(L)$, $f(\bigvee I) = \bigvee f(I)$ and $\downarrow f(I) \in Rd(L_1)$.

Lemma 3.6. *Let L_1, L be complete lattices. If there exist order preserving semicontinuous maps $r : L \to L_1$ and $s : L_1 \to L$ such that $r \circ s = id_{L_1}$ and $s(r(A)) \subseteq A$ for each $A \subseteq L$, then $x \Leftarrow y$ if and only if $s(x) \Leftarrow_{S(L_1)} s(y)$ for all $x, y \in L_1$.*

Proof. For each $x, y \in L_1$, assume $x \Leftarrow y$. Let $I \in Rd(S(L_1))$ with $s(y) \leq \bigvee I$. Then $y = r(s(y)) \leq r(\bigvee I)$ by the monotonicity of r. Since r is semicontinuous, we have $y \leq r(\bigvee I) = \bigvee r(I) = \bigvee \downarrow r(I)$ and $\downarrow r(I) \in Rd(L_1)$. It follows from $x \Leftarrow y$ that $x \in \downarrow r(I)$. Hence $s(x) \in s(\downarrow r(I)) \subseteq \downarrow s(r(I)) \subseteq \downarrow I = I$, and thus $s(x) \Leftarrow_{S(L_1)} s(y)$.

Conversely, suppose that $s(x) \Leftarrow_{S(L_1)} s(y)$. Let $I_1 \in Rd(L_1)$ with $y \leq \bigvee I_1$. Then $s(y) \leq s(\bigvee I_1) = \bigvee s(I_1)$ and $\downarrow s(I_1) \in Rd(S(L_1))$ by the monotonicity and semicontinuity of s. Hence $s(x) \in \downarrow s(I)$ and thus $x = r(s(x)) \in r(\downarrow s(I) \subseteq \downarrow r(s(I)) = \downarrow I = I$. So we have $x \Leftarrow y$. □

Theorem 3.7. *Let L be a complete lattice and L_1 be a semicontinuous lattice. If there exist order preserving semicontinuous maps $r : L \to L_1$ and $s : L_1 \to L$ such that $r \circ s = id_{L_1}$ and $s(r(A)) \subseteq A$ for each $A \subseteq L$, then $s(L_1)$ is a semicontinuous lattice.*

Proof. Let $\Downarrow_{s(L_1)} x = \{y \in s(L_1) : y \Leftarrow x \text{ in } s(L_1)\}$. It suffices to show that for each $x \in s(L_1)$, $x \leq \bigvee \Downarrow_{s(L_1)} x$. For each $x \in s(L_1)$, there is $x_1 \in L_1$ with $x = s(x_1)$ and $x_1 \leq \bigvee \Downarrow x_1$ by the semicontinuity of L_1. Since s is order preserving and semicontinuous, $s(x_1) \leq s(\bigvee \Downarrow x_1) = \bigvee s(\Downarrow x_1)$. It follows from Lemma 3.6 that $s(\Downarrow x_1) \subseteq \Downarrow_{s(L_1)} s(x_1)$. Hence $x = s(x_1) \leq \bigvee \Downarrow_{s(L_1)} s(x_1) = \bigvee \Downarrow_{s(L_1)} x$, and thus $s(L_1)$ is semicontinuous. □

Recall that $\Leftarrow$ is *multiplicative* if and only if for all $a, b, x, y \in L$ with $a \Leftarrow x$ and $b \Leftarrow y$ one can deduce that $a \wedge b \Leftarrow x \wedge y$. A semicontinuous lattice having multiplicative $\Leftarrow$ is called a *stable continuous semilattice.*

The final part of this section is devoted to the strong retract of a stable semicontinuous lattice. Recall that a complete lattice L_1 is a *retract* of a complete lattice L if there exist order-preserving semicontinuous functions $r : L \to L_1$ and $s : L_1 \to L$ such that $r \circ s = id_{L_1}$. In [6], X. H. Wu, *et al.* proved that the retract of a semicontinuous lattice is semicontinuous. We shall show that the strong retract of a stable semicontinuous lattice is stable semicontinuous.

Definition 3.8. A complete lattice L_1 is called a strong retract of a complete lattice L if there exist an order preserving semicontinuous map $r : L \to L_1$ and a nonempty-finite-infs-preserving semicontinuous map $s : L_1 \to L$ such that $r \circ s = id_{L_1}$ and $\forall A \subseteq L, s(r(A)) \subseteq A$.

Clearly, any strong retract of a complete lattice is retract with the fact that nonempty finite infs-preserving functions are all order-preserving.

Theorem 3.9. *The strong retract of a stable semicontinuous lattice is stable semicontinuous.*

Proof. Suppose that L_1 is the strong retract of a stable semicontinuous lattice L. By Definition 3.8, there exist an order preserving semicontinuous map $r : L \to L_1$ and a nonempty-finite-infs-preserving semicontinuous map $s : L_1 \to L$ such that $r \circ s = id_{L_1}$. For each $x \in L_1, x = r \circ s(x), s(x) \in L$. Let $x_1, x_2, y_1, y_2 \in L_1$ with $x_1 \Leftarrow y_1$ and $x_2 \Leftarrow y_2$. To complete the proof, by the retract of a semicontinuous lattice is semicontinuous, we must show that $x_1 \wedge x_2 \Leftarrow y_1 \wedge y_2$. By Lemma 3.6, We have $s(x_1) \Leftarrow_{S(L_1)} s(y_1)$ and $s(x_2) \Leftarrow_{S(L_1)} s(y_2)$. Since $\Leftarrow$ is multiplicative in L, $s(x_1) \wedge s(x_2) \Leftarrow_{S(L_1)} s(y_1) \wedge s(y_2)$. Since s preserves nonempty finite infs, we have $s(x_1 \wedge x_2) \Leftarrow_{S(L_1)} s(y_1 \wedge y_2)$. By Lemma 3.6, $x_1 \wedge x_2 \Leftarrow y_1 \wedge y_2$, and thus L_1 is also stable. □

4. Semialgebraic lattices

The theory of semicontinuous lattices was first developed in [7]. In the paper, D. Zhao introduced and studied a new type of lattices, semicontinuous lattices, by using semiprime ideals. In this section, the concept of a new type of lattices, semiarithmetic lattices, is introduced and examined.

Recall that a complete lattice L is called *semialgebraic* if and only if it satisfies the axiom of $\Leftarrow$-compact approximation

$$(\forall x \in L)\ x \leq \bigvee(\downarrow x \cap SK(L)) \ \text{ and } \ \downarrow x \cap SK(L) \in Rd(L).$$

Lemma 4.1 ([7])**.** *In a complete lattice L, $\forall u, x, y, z \in L$, $u \leq x \Leftarrow y \leq z$ implies $u \Leftarrow z$.*

Proposition 4.2. *In a complete lattice L, the following statements are equivalent*:

(1) *L is semialgebraic*;

(2) *L is semicontinuous, and $x \Leftarrow y$ if and only if there is $k \in SK(L)$ with $x \leq k \leq y$.*

Proof. (1) implies (2): Assume (1) and $x, y \in L$. If $x \Leftarrow y$, then, since $y \leq \bigvee D$ with the semiprime ideal $D = \downarrow y \cap SK(L)$ by (1), $x \in D = \downarrow D$. Thus there is $k \in \downarrow y \cap SK(L)$ with $x \leq k$. Hence $x \leq k \leq y$ with $k \in SK(L)$. Conversely, if there is a $\Leftarrow$-compact element k with $x \leq k \leq y$, then $x \leq k \Leftarrow k \leq y$, whence $x \Leftarrow y$ by Lemma 4.1. We claim $D \subseteq \Downarrow y$ for each $y \in L$. Because, for each $u \in D = \downarrow D$, there is $k \in \downarrow y \cap SK(L)$ with $u \leq k$. Hence $u \leq k \leq y$ with $k \in SK(L)$. Therefore $u \Leftarrow y$ by Lemma 4.1, i.e., $u \in \Downarrow y$. The semicontinuity of L now follows directly from the fact that $D \subseteq \Downarrow y$ and $y \leq \bigvee D$.

(2) implies (1): Assume (2) and let $y \in L$. Then $y \leq \bigvee \Downarrow y$ and $\Downarrow y \in Rd(L)$ by Lemma 3.3. We claim that $\Downarrow y = D$. From above $D \subseteq \Downarrow y$. On the other hand, for each $x \in \Downarrow y, x \Leftarrow y$. Then there is a $\Leftarrow$-compact element k with $x \leq k \leq y$ by (2). Since that $k \in \downarrow y \cap SK(L) = D$, we have $x \in D$. Hence $\Downarrow y \subseteq D$, and $D = \Downarrow y \in Rd(L)$. Therefore, L is semialgebraic. □

By the proof of Proposition 4.2 we immediately obtain:

Corollary 4.3. *In a semialgebraic lattice L, $\forall x \in L, \downarrow x \cap SK(L) = \Downarrow x$.*

Definition 4.4. A complete lattice L is called a semiarithmetic lattice if and only if it is a semialgebraic lattice and if $SK(L)$ is a subsemilattice of L, i.e., if $x \wedge y \in SK(L)$ for all $x, y \in SK(L)$. A semiarithmetic lattice is a semialgebraic lattice in which the set of $\Leftarrow$-compact elements is a subsemilattice.

Proposition 4.5. *For a semialgebraic lattice L, the following conditions are equivalent*:

(1) *L is semiarithmetic*;
(2) *the relation $\Leftarrow$ is multiplicative*;
(3) *$SK(L)$ is a semilattice.*

Proof. (1) implies (2): Let $a \Leftarrow x$ and $b \Leftarrow y$. Then there are $p, k \in SK(L)$ with $a \leq p \leq x$ and $b \leq k \leq y$ by Proposition 4.2. Thus $a \wedge b \leq p \wedge k \leq x \wedge y$, and since $p \wedge k \in SK(L)$ by (1), we have $a \wedge b \Leftarrow x \wedge y$ by Proposition 4.2.

(2) implies (1): If $a, b \in SK(L)$, then $a \Leftarrow a$ and $b \Leftarrow b$, hence $a \wedge b \Leftarrow a \wedge b$ by (2). Thus $a \wedge b \in SK(L)$. Therefore, L is semiarithmetic.

(1) implies (3): It is trivial.

(3) implies (1): Let $a, b \in SK(L), c = a \wedge_{SK(L)} b$. Then $c \leq a \wedge b (= a \wedge_L b)$. But if $X = \downarrow (a \wedge b) \cap SK(L)$, then $a \wedge b \leq \bigvee_L X, c \leq \bigvee_{SK(L)} X$, since L is semialgebraic. Noticing that $c = a \wedge_{SK(L)} b \in SK(L)$ by (3), we have $c = \bigvee_{SK(L)} X$. Thus $a \wedge b \leq \bigvee_L X \leq \bigvee_{SK(L)} X = c$, whence $a \wedge b = c \in SK(L)$. Hence, L is semiarithmetic. □

Recall that an element p in a semilattice L is said to be *prime* if and only if for all $x, y \in L, x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. The set of all primes of L is denoted by PRIME L.

Lemma 4.6 ([7])**.** *Let L be a semicontinuous lattice. Then PRIME L = ψPRIME L if and only if $\Leftarrow$ is multiplicative.*

Lemma 4.7 ([7])**.** *Let L be a semicontinuous lattice. If $\Leftarrow$ is multiplicative, then the following statements are equivalent for an element $p \in L$:*

(1) *p is a pseudo prime element*;
(2) *If $a \wedge b \Leftarrow p$, then $a \leq p$ or $b \leq p$ for all $a, b \in L$*;
(3) *p is a prime.*

From Lemmas 4.7, 4.6 and Proposition 4.5 we immediately obtain:

Corollary 4.8. *Every pseudo prime in a semiarithmetic lattice is prime. Conversely, if in a semialgebraic lattice we have PRIME L = ψPRIME L, then L is semiarithmetic.*

Here we give the following characterization of $\Leftarrow$-compact elements.

Proposition 4.9. *For an element k in a complete lattice L, the following statements are equivalent*:

(1) *$\uparrow k$ is a semi-Scott open filter*;
(2) *k is $\Leftarrow$-compact.*

Proof. (1) implies (2): If $I \in Rd(L)$ such that $k \leq \bigvee I$, then $\bigvee I \in \uparrow k$. By (1), there is $d \in I$ such that $d \in \uparrow k$, thus $k \leq d$. Since I is a lower set, we have $k \in I$. Hence, $k \Leftarrow k$.

(2) implies (1): If $u \in \uparrow k$ and $k \Leftarrow k$, then $k \Leftarrow u$ (by Lemma 4.1), that is $u \in \Uparrow k$. Thus $\uparrow k \subseteq \Uparrow k$. If $I \in Rd(L)$ such that $\bigvee I \in \uparrow k$, then $\bigvee I \in \Uparrow k$. Since $k \Leftarrow \bigvee I$, we have $k \in I$. Thus $I \bigcap \uparrow k \neq \emptyset$. Hence $\uparrow k$ is a semi-Scott open filter. □

Two parallel results to Theorem 3.7 for semialgebraic lattices and semiarithmetic lattices read as follows.

Theorem 4.10. *Let L be a complete lattice and L_1 be a semialgebraic lattice. If there exist order preserving semicontinuous maps $r : L \to L_1$ and $s : L_1 \to L$ such that $r \circ s = id_{L_1}$ and $s(r(A)) \subseteq A$ for each $A \subseteq L$, then $s(L_1)$ is a semialgebraic lattice.*

Proof. By Theorem 3.7 and Proposition 4.2, it suffices to show that for all $x, y \in s(L_1)$, if $x \Leftarrow_{S(L_1)} y$, then there is $k \in SK(s(L_1))$ with $x \leq k \leq y$. Let $x_1, y_1 \in L_1$ such that $x = s(x_1)$ and $y = s(y_1)$. Then $s(x_1) \Leftarrow_{S(L_1)} s(y_1)$, by Lemma 3.6, $x_1 \Leftarrow y_1$. Since L_1 is semialgebraic, by Proposition 4.2, there is $k_1 \in SK(L_1)$ with $x_1 \leq k_1 \leq y_1$. Hence $s(x_1) \leq s(k_1) \leq s(y_1)$ by the monotonicity of s and $s(k_1) \in SK(s(L_1))$ by Lemma 3.6. Thus $k = s(k_1)$ as was desired. □

From Proposition 4.5 and Theorem 4.10 we immediately obtain:

Theorem 4.11. *Let L be a complete lattice and L_1 be a semiarithmetic lattice. If there exist an order preserving semicontinuous map $r : L \to L_1$ and a nonempty-finite-infs-preserving semicontinuous map $s : L_1 \to L$ such that $r \circ s = id_{L_1}$ and $s(r(A)) \subseteq A$ for each $A \subseteq L$, then $s(L_1)$ is a semiarithmetic lattice.*

Proof. By Proposition 4.5 and Theorem 4.10, it suffices to show that $\Leftarrow$ is multiplicative in $s(L_1)$. Let $s_1, s_2, t_1, t_2 \in s(L_1)$ with $s_1 \Leftarrow_{S(L_1)} t_1$ and $s_2 \Leftarrow_{S(L_1)} t_2$. Then there are $x_1, x_2, y_1, y_2 \in L$ with $s_1 = s(x_1), s_2 = s(x_2)$ and $t_1 = s(y_1), t_2 = s(y_2)$, i.e., $s(x_1) \Leftarrow_{S(L_1)} s(y_1)$ and $s(x_2) \Leftarrow_{S(L_1)} s(y_2)$. By Lemma 3.6, $x_1 \Leftarrow y_1$ and $x_2 \Leftarrow y_2$. Since L_1 is semiarithmetic, by Proposition 4.5, $\Leftarrow$ is multiplicative in L_1. Thus $x_1 \wedge x_2 \Leftarrow y_1 \wedge y_2$. Applying Lemma 3.6, $s(x_1 \wedge x_2) \Leftarrow_{S(L_1)} s(y_1 \wedge y_2)$. For s is nonempty-finite-infs-preserving, we have $s(x_1) \wedge s(x_2) \Leftarrow_{S(L_1)} s(y_1) \wedge s(y_2)$. Hence $s_1 \wedge s_2 \Leftarrow_{S(L_1)} t_1 \wedge t_2$, and thus $s(L_1)$ is a semiarithmetic lattice. □

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Jiang Guanghao
Department of Mathematics
Huaibei Coal Industry Teachers College
Huaibei 235000, P. R. China
E-mail address: guanghaoj@126.com

Shi Weixue
Department of Mathematics
Nanjing University
Nanjing 210093, P. R. China
E-mail address: wxshi@nju.edu.cn