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ON A POSITIVE SUBHARMONIC BERGMAN FUNCTION

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Abstract. A holomorphic function F defined on the unit disc belongs to $\mathcal{A}^{p,\alpha}(0 < p < \infty, 1 < \alpha < \infty)$ if

$$\int_U |F(z)|^p \frac{1}{1-|z|}\left(1+\log\frac{1}{1-|z|}\right)^{-\alpha} dxdy \; < \; \infty.$$

For boundedness of the composition operator defined by $C_f g = g \circ f$ mapping Blochs into $\mathcal{A}^{p,\alpha}$, the following (1) is a sufficient condition while (2) is a necessary condition.

$$(1) \qquad \int_0^1 \frac{1}{1-r}\left(1+\log\frac{1}{1-r}\right)^{-\alpha} M_p(r,\lambda\circ f)^p \; dr \; < \; \infty,$$

$$(2) \qquad \int_0^1 \frac{1}{1-r}\left(1+\log\frac{1}{1-r}\right)^{-\alpha+p}(1-r)^p \; M_p\left(r,f^\sharp\right)^p dr \; < \; \infty.$$

1. Introduction

We introduce few facts that we need in the sequel, most of which are well known.

Let U denote the open unit disc of the complex plane.

For $1 < \alpha < \infty$ and $0 < p < \infty$, let $\mathcal{A}^{p,\alpha}$ denote the weighted Bergman space of holomorphic functions on U, consisting of those holomorphic f in U for which

$$\|f\|_{\mathcal{A}^{p,\alpha}} := \left(\int_U |f(z)|^p \frac{1}{1-|z|}\left(1+\log\frac{1}{1-|z|}\right)^{-\alpha} dxdy\right)^{1/p} \; < \; \infty.$$

Note that $\mathcal{A}^{p,\alpha}$ is different from $A^{p,\alpha}$, the well-known weighted Bergman space of order $\alpha, \alpha > -1$, consisting of holomorphic functions f in U for which

$$\|f\|_{A^{p,\alpha}} := \left(\int_U |f(z)|^p (1-|z|)^\alpha \; dxdy\right)^{1/p} \; < \; \infty.$$

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For functions defined in U and for $0 < p < \infty$, $0 \le r < 1$, $M_p(r,f)$ is defined as usual by

$$M_p(r,f) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p}.$$

For f with $|f|$ subharmonic in U, we set

$$\|f\|_p := \sup_r \; M_p(r,f).$$

Then the classical Hardy space $H^p = H^p(U)$ is the space of those f holomorphic in U for which $\|f\|_p < \infty$. The Yamashita [8] hyperbolic Hardy class H^p_σ is defined as the set of those holomorphic self-maps f of U for which $||\sigma(f)||_p < \infty$, where $\sigma(z)$ denotes the hyperbolic distance of z and 0 in U, i.e.,

$$\sigma(z) = \frac{1}{2}\log\frac{1+|z|}{1-|z|}.$$

We set, following Yamashita,

$$\lambda(f) = \log\frac{1}{1-|f|^2} \quad \text{and} \quad f^\sharp = \frac{|f'|}{1-|f|^2}$$

for holomorphic self-maps f of U. It obvious that $f \in H^p_\sigma$ if and only if $\|\lambda(f)\|_p < \infty$ and that $f^\sharp$ is $\mathcal{M}$-invariant in the sense that $f^\sharp = (\varphi \circ f)^\sharp$ for any $\varphi \in \mathcal{M}$, where $\mathcal{M}$ is the group of all automorphisms of U.

The Bloch space $\mathcal{B}$ consists of holomorphic functions h in U for which

$$\sup_{z\in U} |h'(z)| \; (1-|z|^2) \; < \; \infty.$$

This is a Banach space, if the norm $\|h\|_{\mathcal{B}}$ of $h \in \mathcal{B}$ is defined to be the sum of $|h(0)|$ and the left side of above inequality. A pair of Bloch functions $h_j,\ j = 1, 2$ are constructed such that

$$(1-|z|^2)(|h_1'(z)| + |h_2'(z)|) \ge 1, \quad z \in U \tag{1.1}$$

([6]). Then it follows that

$$\frac{1}{1-|f|^2} \le |h_1' \circ f| + |h_2' \circ f| \le \frac{C}{1-|f|^2} \tag{1.2}$$

for holomorphic self-maps f, where $C = 2\max\{\|h_1\|_{\mathcal{B}}, \|h_2\|_{\mathcal{B}}\}$. For $h \in \mathcal{B}$, it follows from Schwarz-Pick's Lemma ([2]) that

$$|(h \circ f)'(z)| \le \|h\|_{\mathcal{B}} f^\sharp(z) \le \|h\|_{\mathcal{B}} \frac{1}{1-|z|^2}, \quad z \in U. \tag{1.3}$$

For $f : U \to U$ be holomorphic, the composition operator C_f generated by f is defined by $C_f h = h \circ f,\ h \in \mathcal{B}$. Also, we define C_f^0 by $C_f^0 h = h \circ f - h(0)$. See [7].

Our results in this note are as follows.

Theorem 1. $\lambda(z) = \log \frac{1}{1-|z|^2}$, $|z| < 1$. *Let* $f : U \to U$ *be holomorphic and* $1 < \alpha < \infty, 1 \le p < \infty$. *Then the following* (1) *implies* (3) *with* $\delta = \alpha$; (3) *implies* (2) *with* $\delta = \alpha - p$, *and* (2) *implies* (1).

(1) *The composition operator* $C_f : \mathcal{B} \to \mathcal{A}^{p,\alpha}$ *is bounded.*

$$\int_0^1 \frac{1}{1-r}(1+\log\frac{1}{1-r})^{-\alpha}\ M_p(r,\lambda\circ f)^p\ dr\ <\ \infty. \tag{2}$$

$$\int_0^1 \frac{1}{1-r}(1+\log\frac{1}{1-r})^{-\delta}(1-r)^p\ M_p\left(r,f^\sharp\right)^p dr\ <\ \infty. \tag{3}$$

After introducing simple but useful lemmas in Section 2, we will prove our main results in Section 3.

2. Lemmas

Lemma 1. *Let* $f : U \to U$ *be holomorphic and* $h : U \to C$ *be holomorphic. Then, for* $1 \le p < \infty$ *and* $1 < \alpha < \infty$,

$$\int_U \frac{1}{1-|z|}\big(1+\log\frac{1}{1-|z|}\big)^{-\alpha}(1-|z|)^p\ |(h\circ f)'(z)|^p\ dxdy$$
$$\le C_{\alpha,p}\int_U \frac{1}{1-|z|}\big(1+\log\frac{1}{1-|z|}\big)^{-\alpha}|(h\circ f)(z)-h(0)|^p\ dxdy,$$

where $C_{\alpha,p}$ *is a constant depending on* α *and* p.

Proof. We show that there is a constant $C_{\alpha,p}$ depending only on p and α such that

$$\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-\alpha}(1-r)^p\ rdr\ \int_0^{2\pi}\left|(h\circ f)'(re^{i\theta})\right|^p\ d\theta$$
$$\le C_{\alpha,p}\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-\alpha}\ rdr\ \int_0^{2\pi}\left|(h\circ f)(re^{i\theta})-h(0)\right|^p\ d\theta.$$

Let $h\circ f - h(0) = F$. Then

$$\int_T \left|(h\circ f)'(re^{i\theta})\right|^p\ \frac{d\theta}{2\pi} = M_p(r,F')^p.$$

By [1, p. 80]

$$M_p(r,F') \le \frac{M_p(\frac{1+r}{2},F)}{\big(\frac{1+r}{2}\big)^2-r^2} = \frac{M_p(\frac{1+r}{2},F)}{(\frac{1+3r}{2})(\frac{1-r}{2})} \le 4\frac{M_p(\frac{1+r}{2},F)}{1-r},$$

whence

$$\begin{aligned}
&\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-\alpha}(1-r)^p\ M_p(r,F')^p\ rdr\\
&\le 4^p\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-\alpha}(1-r)^p\ \Big(\frac{M_p(\frac{1+r}{2},F)}{1-r}\Big)^p rdr\\
&\le 4^p\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-\alpha}\ M_p\Big(\frac{1+r}{2},F\Big)^p\ rdr\\
&\le C_{\alpha,p}\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-\alpha}\ M_p(r,F)^p\ rdr,
\end{aligned}$$

where $C_{\alpha,p}$ is a constant depending on α and p. Therefore

$$\begin{aligned}
&\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-\alpha}(1-r)^p\ rdr\ \int_0^{2\pi}\left|(h\circ f)'(re^{i\theta})\right|^p\ d\theta\\
&\le C_{\alpha,p}\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-\alpha}\ rdr\ \int_0^{2\pi}\left|(h\circ f)(re^{i\theta})-h(0)\right|^p\ d\theta.
\end{aligned}$$

□

Lemma 2. *Let $f: U\to U$ be holomorphic. Then*

$$\frac{\partial}{\partial r}\Big(\log\frac{1}{1-|f|^2}\Big)\le 2f^\sharp.$$

Proof. It is easy to see that

$$\frac{\partial}{\partial r}\Big(\log\frac{1}{1-|f|^2}\Big)=\frac{\partial}{\partial r}\Big(-\log(1-f\bar f)\Big)=\frac{f_r\bar f+f\bar f_r}{1-f\bar f},$$

where $f_r=\frac{\partial f}{\partial r}$. But since

$$f_r=\frac{\partial f}{\partial r}=\frac{\partial f}{\partial z}\frac{\partial z}{\partial r}=f'e^{i\theta}$$

and

$$\bar f_r=\frac{\partial \bar f}{\partial r}=\frac{\partial \bar f}{\partial \bar z}\frac{\partial \bar z}{\partial r}=\bar f'e^{-i\theta},$$

it follows that

$$\begin{aligned}
\frac{\partial}{\partial r}\Big(\log\frac{1}{1-|f|^2}\Big)&=\frac{\bar f f'e^{i\theta}+\bar f'e^{-i\theta}f}{1-|f|^2}\\
&=\frac{2Re(\bar f f'e^{i\theta})}{1-|f|^2}\\
&\le\frac{2|f'||f|}{1-|f|^2}\ .
\end{aligned}$$

Noting that $|f|\le 1$, we obtain

$$\frac{\partial}{\partial r}\Big(\log\frac{1}{1-|f|^2}\Big)\le 2f^\sharp\ .$$

□

Lemma 3. *Let* $\lambda(z) = \log \frac{1}{1-|z|^2}$, $|z| < 1$. *Let* $f : U \to U$ *be holomorphic and* $1 < a < \infty, 1 < b < \infty, 1 \le p < \infty$. *Then*

$$\begin{aligned}
&\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b}\ M_p(r,\lambda\circ f)^a\ dr \\
\le\ &\frac{a}{b-1}|\lambda\circ f(0)|^a \\
&+\big(\frac{2a}{b-1}\big)^a\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b+a}(1-r)^a M_p(r,f^\sharp)^a\ dr.
\end{aligned}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b}\ M_p(r,\lambda\circ f)^a\ dr \\
=\ &\frac{1}{-b+1}\big(1+\log\frac{1}{1-r}\big)^{-b+1}\ M_p(r,\lambda\circ f)^a\Big|_0^1 \\
&-\int_0^1 \frac{1}{-b+1}\big(1+\log\frac{1}{1-r}\big)^{-b+1}\ \frac{\partial}{\partial r}M_p(r,\lambda\circ f)^a\ dr \\
=\ &\frac{-1}{-b+1}M_p(0,\lambda\circ f)^a - \int_0^1 \frac{1}{-b+1}\big(1+\log\frac{1}{1-r}\big)^{-b+1}\ \frac{\partial}{\partial r}M_p(r,\lambda\circ f)^a\ dr \\
=\ &\frac{1}{b-1}|\lambda\circ f(0)|^a + \frac{1}{b-1}\int_0^1 \big(1+\log\frac{1}{1-r}\big)^{1-b}\frac{\partial}{\partial r}\ M_p(r,\lambda\circ f)^a\ dr,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b}\ M_p(r,\lambda\circ f)^a\ dr \\
&= \frac{1}{b-1}|\lambda\circ f(0)|^a + \frac{1}{b-1}\int_0^1 \big(1+\log\frac{1}{1-r}\big)^{1-b}\frac{\partial}{\partial r}\ M_p(r,\lambda\circ f)^a\ dr.
\end{aligned} \tag{2.1}$$

But, since

$$\begin{aligned}
\frac{\partial}{\partial r}M_p(r,\lambda\circ f)^a &= aM_p(r,\lambda\circ f)^{a-1}\frac{\partial}{\partial r}M_p(r,\lambda\circ f) \\
&= \frac{a}{p}M_p(r,\lambda\circ f)^{a-p}pM_p(r,\lambda\circ f)^{p-1}\frac{\partial}{\partial r}M_p(r,\lambda\circ f) \\
&= \frac{a}{p}M_p(r,\lambda\circ f)^{a-p}\frac{\partial}{\partial r}M_p(r,\lambda\circ f)^p,
\end{aligned}$$

we have

$$\frac{\partial}{\partial r}M_p(r,\lambda\circ f)^a = \frac{a}{p}M_p(r,\lambda\circ f)^{a-p}\frac{\partial}{\partial r}M_p(r,\lambda\circ f)^p. \tag{2.2}$$

And since

$$\begin{aligned}\frac{\partial}{\partial r}|\lambda\circ f(re^{i\theta})|^p \le \Big|\frac{\partial}{\partial r}(\lambda\circ f(re^{i\theta}))^p\Big| &= \Big|p(\lambda\circ f(re^{i\theta}))^{p-1}\frac{\partial}{\partial r}(\lambda\circ f(re^{i\theta}))\Big|\\ &\le \Big|p(\lambda\circ f(re^{i\theta}))^{p-1}\Big|\Big|\frac{\partial}{\partial r}(\lambda\circ f(re^{i\theta}))\Big|\\ &\le 2p\Big|(\lambda\circ f(re^{i\theta}))^{p-1}\Big|f^\sharp(re^{i\theta}),\end{aligned}$$

where we used Lemma 2 in last inequality, we have

$$\Big|\frac{\partial}{\partial r}|\lambda\circ f(re^{i\theta})|^p\Big| \le 2p|\lambda\circ f(re^{i\theta})|^{p-1}f^\sharp(re^{i\theta}). \tag{2.3}$$

By (2.2) and (2.3),

$$\begin{aligned}\Big|\frac{\partial}{\partial r}M_p(r,\lambda\circ f)^a\Big| &= \Big|\frac{a}{p}\ M_p(r,\lambda\circ f)^{a-p}\frac{\partial}{\partial r}M_p(r,\lambda\circ f)^p\Big|\\ &\le \frac{a}{p}\ M_p(r,\lambda\circ f)^{a-p}\int_0^{2\pi}\Big|\frac{\partial}{\partial r}|\lambda\circ f(re^{i\theta})|^p\Big|\frac{d\theta}{2\pi}\\ &\le \frac{a}{p}\ M_p(r,\lambda\circ f)^{a-p}\int_0^{2\pi}2p|\lambda\circ f(re^{i\theta})|^{p-1}f^\sharp(re^{i\theta})\frac{d\theta}{2\pi}\ .\end{aligned}$$

Thus it follows from Hölder's inequality that

$$\begin{aligned}&\Big|\frac{\partial}{\partial r}M_p(r,\lambda\circ f)^a\Big|\\ &\le \frac{a}{p}\ M_p(r,\lambda\circ f)^{a-p}\int_0^{2\pi}2p|\lambda\circ f(re^{i\theta})|^{p-1}f^\sharp(re^{i\theta})\frac{d\theta}{2\pi}\\ &\le 2a\ M_p(r,\lambda\circ f)^{a-p}\Big(\int_0^{2\pi}\big(|\lambda\circ f(re^{i\theta})|^{p-1}\big)^{p/(p-1)}\frac{d\theta}{2\pi}\Big)^{(p-1)/p}\\ &\quad\times\Big(\int_0^{2\pi}|f^\sharp(re^{i\theta})|^p\frac{d\theta}{2\pi}\Big)^{1/p}\\ &= 2a\ M_p(r,\lambda\circ f)^{a-p}\ M_p(r,\lambda\circ f)^{p-1}\ M_p(r,f^\sharp)\\ &= 2a\ M_p(r,\lambda\circ f)^{a-1}\ M_p(r,f^\sharp)\end{aligned}$$

for $p>1$. This is also true when $p=1$. Consequently, the last integral of (2.1) is dominated by

$$\begin{aligned}&2a\int_0^1\big(1+\log\frac{1}{1-r}\big)^{1-b}\ M_p(r,\lambda\circ f)^{a-1}M_p(r,f^\sharp)\ dr\\ &\le 2a\left\{\int_0^1\frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b}\ M_p(r,\lambda\circ f)^a\ dr\right\}^{1-1/a} \qquad (2.4)\\ &\quad\times\left\{\int_0^1\big(\frac{1}{1-r}\big)^{1-a}\big(1+\log\frac{1}{1-r}\big)^{-b+a}\ M_p(r,f^\sharp)^a\ dr\right\}^{1/a},\end{aligned}$$

where Hölder's inequality has been used again. Therefore by (2.4), (2.1) is dominated by

$$\begin{aligned}
&\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b}\ M_p(r,\lambda\circ f)^a\ dr \\
&= \frac{1}{b-1}|\lambda\circ f(0)|^a + \frac{1}{b-1}\int_0^1 \big(1+\log\frac{1}{1-r}\big)^{1-b}\frac{\partial}{\partial r}\ M_p(r,\lambda\circ f)^a\ dr \\
&\le \frac{1}{b-1}|\lambda\circ f(0)|^a \\
&\quad + \frac{2a}{b-1}\left\{\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b}\ M_p(r,\lambda\circ f)^a\ dr\right\}^{1-1/a} \\
&\quad \times\left\{\int_0^1 \big(\frac{1}{1-r}\big)^{1-a}\big(1+\log\frac{1}{1-r}\big)^{-b+a}\ M_p(r,f^\sharp)^a\ dr\right\}^{1/a}.
\end{aligned} \tag{2.5}$$

In order to obtain the desired inequality, let

$$A = \int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b}\ M_p(r,\lambda\circ f)^a\ dr$$

and

$$B = \int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b+a}(1-r)^a M_p(r,f^\sharp)^a\ dr$$

for a moment. Then (2.5) can be expressed as

$$A \le \frac{1}{b-1}|\lambda\circ f(0)|^a + A^{1-1/a}\times\big[\big(\frac{2a}{b-1}\big)^a B\big]^{1/a}.$$

Using

$$A^\alpha B^{1-\alpha}\ \le\ \alpha A + (1-\alpha)B \quad (0\le\alpha\le 1),$$

we have

$$A \le \frac{1}{b-1}|\lambda\circ f(0)|^a + \Big[(1-\frac{1}{a})A + \big(\frac{2a}{b-1}\big)^a\frac{1}{a}B\Big],$$

equivalently

$$\frac{1}{a}A \le \frac{1}{b-1}|\lambda\circ f(0)|^a + \big(\frac{2a}{b-1}\big)^a\frac{1}{a}B.$$

Therefore

$$A \le \frac{a}{b-1}|\lambda\circ f(0)|^a + \big(\frac{2a}{b-1}\big)^a B.$$

That is,

$$\begin{aligned}
&\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b}\ M_p(r,\lambda\circ f)^a\ dr \\
&\le \frac{a}{b-1}|\lambda\circ f(0)|^a \\
&\quad + \big(\frac{2a}{b-1}\big)^a\int_0^1 \frac{1}{1-r}\big(1+\log\frac{1}{1-r}\big)^{-b+a}(1-r)^a M_p(r,f^\sharp)^a\ dr.
\end{aligned}$$

This completes the proof. □

Lemma 4. *Let $g \in \mathcal{B}$ and $f \in H^\infty(U)$ with $|f| < 1$. Then*

$$|g \circ f(z) - g(0)| \ \le \|g\|_{\mathcal{B}} \ \sigma(f(z)), \ z \in U.$$

Proof. For $w \in U$,

$$\begin{aligned} |g(w) - g(0)| &= \Big| \int_0^w g'(z)\ dz \Big| = \Big| \int_0^{|w|} g'(t\zeta)\ \zeta\ dt \Big| \\ &\le \int_0^{|w|} |g'(t\zeta)|\ dt \\ &\le \int_0^{|w|} \|g\|_{\mathcal{B}} \frac{1}{1-|t\zeta|^2} dt = \|g\|_{\mathcal{B}} \ \int_0^{|w|} \frac{dt}{1-t^2} \\ &= \|g\|_{\mathcal{B}} \ \frac{1}{2} \ \log \frac{1+|w|}{1-|w|} = \|g\|_{\mathcal{B}} \ \sigma(w), \end{aligned}$$

whence

$$|g \circ f(z) - g(0)| \ \le \|g\|_{\mathcal{B}} \ \sigma(f(z)), \ z \in U. \qquad \square$$

3. Proof of the results

Proof of Theorem 1. Suppose (1) holds. Using Minkowski's inequality with those h_j, $j = 1, 2$ of (2.1) and Lemma 1, we obtain

(3.1)

$$\begin{aligned} &\left\{ \int_U \frac{1}{1-|z|} \Big(1 + \log \frac{1}{1-|z|} \Big)^{-\alpha} (1-|z|)^p \ \big(f^\sharp(z)\big)^p \ dxdy \right\}^{1/p} \\ \le &\left\{ \int_U \frac{1}{1-|z|} \Big(1 + \log \frac{1}{1-|z|} \Big)^{-\alpha} (1-|z|)^p \ \Big(\sum_{j=1}^2 |(h_j \circ f)'(z)| \Big)^p \ dxdy \right\}^{1/p} \\ \le &\sum_{j=1}^2 \left\{ \int_U \frac{1}{1-|z|} \Big(1 + \log \frac{1}{1-|z|} \Big)^{-\alpha} (1-|z|)^p \ |(h_j \circ f)'(z)|^p \ dxdy \right\}^{1/p} \\ \le &\, C_{\alpha,p} \sum_{j=1}^2 \left\{ \int_U \frac{1}{1-|z|} \Big(1 + \log \frac{1}{1-|z|} \Big)^{-\alpha} |(h_j \circ f)(z) - h_j(0)|^p \ dxdy \right\}^{1/p}, \end{aligned}$$

where $C_{\alpha,p}$ is a constant depending on α and p.

Since

$$\begin{aligned} &\left\{ \int_U \frac{1}{1-|z|} \Big(1 + \log \frac{1}{1-|z|} \Big)^{-\alpha} |(h_j \circ f)(z) - h_j(0)|^p \ dxdy \right\}^{1/p} \\ \le &\|h_j\|_{\mathcal{B}} \sup_{\substack{h \in \mathcal{B} \\ \|h\|_{\mathcal{B}} \le 1}} \left\{ \int_U \frac{1}{1-|z|} \Big(1 + \log \frac{1}{1-|z|} \Big)^{-\alpha} |(h \circ f)(z) - h(0)|)^p \ dxdy \right\}^{1/p}, \end{aligned}$$

$j = 1, 2$, from (3.1) we have

(3.2)

$$\begin{aligned}
&\left\{ \int_U \frac{1}{1-|z|} \Big(1 + \log \frac{1}{1-|z|}\Big)^{-\alpha} (1-|z|)^p \ \big(f^\sharp(z)\big)^p \ dxdy \right\}^{1/p} \\
&\le C_{\alpha,p} \sup_{\substack{h \in \mathcal{B} \\ \|h\|_{\mathcal{B}} \le 1}} \left\{ \int_U \frac{1}{1-|z|} \Big(1 + \log \frac{1}{1-|z|}\Big)^{-\alpha} |(h \circ f)(z) - h(0)|^p \ dxdy \right\}^{1/p} \\
&= C_{\alpha,p} \sup_{\substack{h \in \mathcal{B} \\ \|h\|_{\mathcal{B}} \le 1}} \|h \circ f - h(0)\|_{\mathcal{A}^{p,\alpha}} \\
&= C_{\alpha,p} \|C_f^0\| \ .
\end{aligned}$$

Thus (3) holds.

Also, (3) with $\delta = \alpha - p \implies$ (2) is proved in Lemma 3. And (2) $\implies$ (1) follows directly from Lemma 4 with an application of Minkowski's inequality: For $g \in \mathcal{B}$,

$$\begin{aligned}
&\int_0^1 \frac{1}{1-r} \Big(1 + \log \frac{1}{1-r}\Big)^{-\alpha} \ M_p(r, g \circ f)^p \ dr \\
&\le \|g\|_{\mathcal{B}}^p \int_0^1 \frac{1}{1-r} \Big(1 + \log \frac{1}{1-r}\Big)^{-\alpha} \ M_p(r, 1 + \sigma \circ f)^p \ dr \ < \ \infty.
\end{aligned}$$

The proof is complete. □

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