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MAX-INJECTIVE, MAX-FLAT MODULES AND MAX-COHERENT RINGS

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Abstract. A ring R is called left max-coherent provided that every maximal left ideal is finitely presented. $\mathscr{MI}$ (resp. $\mathscr{MF}$) denotes the class of all max-injective left R-modules (resp. all max-flat right R-modules). We prove, in this article, that over a left max-coherent ring every right R-module has an $\mathscr{MF}$-preenvelope, and every left R-module has an $\mathscr{MI}$-cover. Furthermore, it is shown that a ring R is left max-injective if and only if any left R-module has an epic $\mathscr{MI}$-cover if and only if any right R-module has a monic $\mathscr{MF}$-preenvelope. We also give several equivalent characterizations of MI-injectivity and MI-flatness. Finally, $\mathscr{MI}$-dimensions of modules and rings are studied in terms of max-injective modules with the left derived functors of Hom.

1. Introduction

Throughout this article, R is an associative ring with identity and all modules are unitary. For an R-module M, the character module M^+ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ and $\sigma_M : M \to M^{++}$ stands for the evaluation map. For an index set I, $M^{(I)}$ is the direct sum of copies of M indexed by I. The cardinality of an R-module M is denoted by Card(M). We write ${}_R\mathscr{M}$ for the category of all left R-modules. For unexplained concepts and notations, we refer the reader to [1, 4, 11, 14, 15].

The theory of envelopes and covers takes an important part in theory of rings and modules, homological algebra, representation theory of algebras and so on. Since precovers and preenvelopes were introduced by Enochs (see [5]), the existence of precovers and preenvelopes of special modules have been studied extensively by many authors (see [2, 4, 5, 9, 12, 15]). For instance, the well-known Flat Cover Conjecture, asserting that each module admits a flat cover, remained open for almost twenty years and was settled only recently [2]. Let $\mathscr{C}$ be a class of R-modules. For an R-module M, a morphism $g : C \to M$ with $C \in \mathscr{C}$ is called a $\mathscr{C}$-cover of M if: (1) For any homomorphism $g' : C' \to M$

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with $C' \in \mathscr{C}$, there exists a homomorphism $f : C' \to C$ with $g' = gf$. (2) If f is an endomorphism of C with $g = gf$, then f must be an automorphism. If (1) holds but (2) may not, $g : C \to M$ is called a $\mathscr{C}$-precover. Dually, we have the definition of a $\mathscr{C}$-(pre)envelope. $\mathscr{C}$-covers ($\mathscr{C}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism. If every left R-module has a $\mathscr{C}$-precover, every left R-module M has a *left $\mathscr{C}$-resolution*, that is, there is a $\mathrm{Hom}_R(\mathscr{C}, -)$ exact complex $\cdots \to F_1 \to F_0 \to M \to 0$ with each $F_i \in \mathscr{C}$. A left R-module M is said to have *left $\mathscr{C}$-dimension* $\leq n$, denoted left $\mathscr{C}$-dim $M \leq n$, if there is a left $\mathscr{C}$-resolution of the form $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ of M. If there is no such n, we set left $\mathscr{C}$-dim $M = \infty$. In a similar manner, we can define the right $\mathscr{C}$-dimension of left R-module N if every left R-module has a $\mathscr{C}$-preenvelope. The *global left $\mathscr{C}$-dimension of* ${}_R\mathscr{M}$, denoted by gl left $\mathscr{C}$-$\dim_R\mathscr{M}$, is defined to be sup{left $\mathscr{C}$-$\dim M \mid M \in {}_R\mathscr{M}$} and is infinite otherwise. The right versions can be defined similarly.

Let R be a ring. A left R-module M is called *finitely presented* if there is an exact sequence $0 \to K \to F \to M \to 0$ where F is finitely generated free and K is finitely generated. An exact sequence of left R-modules $0 \to A' \to A \to A'' \to 0$ is called *pure exact* provided that $0 \to M \otimes_R A' \to M \otimes_R A \to M \otimes_R A'' \to 0$ is exact for any right R-module M. A' is said to be a pure submodule of A. A left R-module M is called *max-injective* if for any maximal left ideal m, any homomorphism $f : m \to M$ can be extended to $g : R \to M$, if and only if $\mathrm{Ext}^1_R(R/m, M) = 0$ for any maximal left ideal m. A ring R is said to be left max-injective if R is max-injective as a left R-module. Max-injectivity was first introduced in [14], which was used to characterize QF-rings. A right R-module M is called *max-flat* if $\mathrm{Tor}^R_1(M, R/m) = 0$ for any left maximal ideal of R (see [13]). A right R-module M is max-flat if and only if M^+ is max-injective by the standard isomorphism $\mathrm{Ext}^1_R(R/m, M^+) \cong \mathrm{Tor}^R_1(M, R/m)^+$ for any maximal left ideal of R (see [4, Theorem 3.2.1]). In what follows, we write $\mathscr{MI}$ (resp. $\mathscr{MF}$) for the class of all max-injective left (resp. all max-flat right) R-modules. An R-module M is called *reduced* if M has no nonzero injective submodules.

In Section 2 of this article, we introduce the concept of max-coherent rings and give several examples of max-coherent rings. It is shown that, over a left max-coherent ring, every left R-module has an $\mathscr{MI}$-cover, and every right R-module has an $\mathscr{MF}$-preenvelope. Furthermore, we prove that R is left max-injective if and only if any left R-module has an epic $\mathscr{MI}$-cover, if and only if every right R-module has a monic $\mathscr{MF}$-preenvelope. The relation between $\mathscr{MF}$-preenvelopes and $\mathscr{MI}$-precovers is also studied.

In Section 3, the concepts of MI-injective modules and MI-flat modules are introduced. It is proven that a left R-module M is MI-injective if and only if M is the kernel of an $\mathscr{MI}$-precover $f : A \to B$ with A injective. We characterize QF-rings in terms of MI-injective left R-modules.

Section 4 investigates the $\mathscr{MI}$-dimensions of modules and rings. It is shown that if, every maximal left ideal of a ring R is finitely generated and every left R-module has an $\mathscr{MI}$-cover with the unique mapping property, then R is left max-coherent and gl right $\mathscr{MI}$-$\dim_R\mathscr{M} \leq 2$.

2. Max-coherent rings

Definition 2.1. A ring R is called left max-coherent provided that every maximal left ideal is finitely presented. Similarly, we have the concept of right max-coherent rings.

Remark 2.2. Here are some examples of max-coherent rings.

(1) Obviously, Noetherian rings are max-coherent.

(2) Recall that a ring R is said to be *left Kasch* if every simple left R-module embeds in ${}_RR$. R is called *left* Π*-coherent* provided that every finitely generated torsionless left R-module is finitely presented. In view of Lemma 1.40 and Proposition 1.44 in [10], a left Kasch and left Π-coherent ring is left max-coherent.

(3) Left max-coherent rings are left min-coherent, that is, every simple left ideal is finitely presented (cf. [9]). In fact, if every maximal left ideal is finitely presented, then every simple left R-module is finitely presented, so is every simple left ideal. But the converse is not true in general. For example, choose R to be a von Neumann regular ring but not Artinian semisimple. Then R is a left min-coherent ring. However, there exists a simple left R-module which is not finitely presented, hence there is a maximal left ideal which is not finitely generated. Then R is not a max-coherent ring. In particular, left coherent rings need not be max-coherent.

Using the same argument of [9, Theorem 4.5], we can prove the following theorem.

Theorem 2.3. *Let R be a ring. Then the following are equivalent*:

(1) *R is a left max-coherent ring.*

(2) *Any maximal left ideal is finitely generated, and any direct product of copies of R_R is max-flat.*

(3) *Any maximal left ideal is finitely generated, and any direct product of max-flat right R-modules is max-flat.*

(4) *Any maximal left ideal is finitely generated, and any direct limit of max-injective left R-modules is max-injective.*

(5) *Any maximal left ideal is finitely generated, and* $\mathrm{Tor}_1^R(\prod M_i, R/m) \cong \prod \mathrm{Tor}_1^R(M_i, R/m)$ *for any maximal left ideal m and any family $\{M_i\}$ of right R-modules.*

(6) *Any maximal left ideal is finitely generated, and a left R-module M is max-injective if and only if M^+ is max-flat.*

(7) *Any maximal left ideal is finitely generated, and a left R-module M is max-injective if and only if M^{++} is max-injective.*

(8) *Any maximal left ideal is finitely generated, and a right R-module N is max-flat if and only if N^{++} is max-flat.*

Proposition 2.4. *The following are true for an arbitrary ring R :*

(1) *$\mathscr{MI}$ is closed under extensions, direct products and direct summands; $\mathscr{MF}$ is closed under extensions, direct sums and direct summands.*

(2) *If R is a left max-coherent ring, then $\mathscr{MI}$ and $\mathscr{MF}$ are closed under pure submodules. Moreover, $\mathscr{MI}$ is closed under direct sums.*

Proof. (1) is trivial.

(2) Let N be a pure submodule of a max-injective left R-module M. Then we have a pure exact sequence $0 \to N \to M \to M/N \to 0$. So $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$ is a split exact sequence. By Theorem 2.3, M^+ is max-flat. By (1), N^+ is also max-flat, and so N is max-injective by Theorem 2.3 again.

Let L be a pure submodule of a max-flat right R-module F. So $0 \to L \to F \to F/L \to 0$ is a pure exact sequence. Hence $0 \to (F/L)^+ \to F^+ \to L^+ \to 0$ is a split exact sequence. F^+ is max-injective, so is L^+ by (1). Then L is max-flat.

For the latter assertion, it follows from $\oplus M_i$ is a pure submodule of $\prod M_i$ for any family $\{M_i\}$ of left R-modules. □

Theorem 2.5. *If R is a left max-coherent ring, then every right R-module has an $\mathscr{MF}$-preenvelope.*

Proof. Let M be any right R-module. By [4, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ dependent on $\mathrm{Card}(M)$ and $\mathrm{Card}(R)$ such that for any homomorphism $f : M \to F$ with $F \in \mathscr{MF}$, there is a pure submodule G of F such that $f(M) \subseteq G$ and $\mathrm{Card}(G) \leq \aleph_\alpha$. Thus f has a factorization $M \to G \to F$ with G is max-flat by Proposition 2.4 (2). Now let $(f_i)_{i\in I}$ be all such homomorphisms $f_i : M \to G_i$ with $\mathrm{Card}(G_i) \leq \aleph_\alpha$ and $G_i \in \mathscr{MF}$. Then any homomorphism $g : M \to N$ with $N \in \mathscr{MF}$ has a factorization $M \to G_j \to N$ for some $j \in I$. Note that $\prod_{i\in I} G_i$ is max-flat by Theorem 2.3. Thus the homomorphism $M \to \prod_{i\in I} G_i$ induced by all f_i is an $\mathscr{MF}$-preenvelope. □

Proposition 2.6. *Let R be a left max-coherent ring. If $0 \to A \to B \to C \to 0$ is a pure exact sequence of left R-modules with $B \in \mathscr{MI}$, then $C \in \mathscr{MI}$.*

Proof. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of left R-modules with $B \in \mathscr{MI}$. Then we have a split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. By Theorem 2.3, $B^+ \in \mathscr{MF}$, and hence $C^+ \in \mathscr{MF}$ by Proposition 2.4 (1). So $C \in \mathscr{MI}$ by Theorem 2.3 again. □

The next lemma is a special case of [2, Theorem 5].

Lemma 2.7. *Let R be a ring. Then for each cardinal λ, there is a cardinal κ such that any R-module M and for any $L \leq M$ with $\mathrm{Card}(M) \geq \kappa$ and $\mathrm{Card}(M/L) \leq \lambda$, the submodule L contains a nonzero submodule that is pure in M.*

Theorem 2.8. *Let R be a left max-coherent ring. There is a cardinal number κ such that any homomorphism $\varphi : D \to M$ with $D \in \mathscr{MI}$ has a factorization $D \to C \to M$ with $C \in \mathscr{MI}$ and* $\mathrm{Card}(C) \leq \kappa$.

Proof. Let M be a left R-module with $\mathrm{Card}(M) = \lambda$, and let κ be a cardinal as in Lemma 2.7. Take homomorphism $\varphi : D \to M$ with $D \in \mathscr{MI}$, $K = \ker\varphi$. If $\mathrm{Card}(D) \leq \kappa$, then consider the factorization of $D \to M$ as $D \to D \to M$, where the first arrow is the identity.

If $\mathrm{Card}(D) > \kappa$. There is K' maximal with the properties that $K' \subseteq K \subseteq D$ and that K' is a pure submodule of D. So φ has the factorization $D \to D/K' \to M$ in terms of [1, Theorem 3.6]. By Proposition 2.6, D/K' is max-injective. we claim that $\mathrm{Card}(D/K') \leq \kappa$. Otherwise, if $\mathrm{Card}(D/K') > \kappa$, consider $K/K' \subseteq D/K'$. Since D/K is isomorphic to a submodule of M,

$$\mathrm{Card}\left(\frac{D/K'}{K/K'}\right) = \mathrm{Card}(D/K) \leq \mathrm{Card}(M).$$

In view of Lemma 2.7, there exists $0 \neq K''/K' \subseteq K/K' \subseteq D/K'$ such that K''/K' is a pure submodule of D/K'. It is clear that $K' \subsetneqq K'' \subseteq K \subseteq D$. By [7, Proposition 7.2], K'' is a pure submodule of D, contradicting the maximality of K'. So let $C = D/K'$, $\mathrm{Card}(C) \leq \kappa$, as desired. □

Theorem 2.9. *Let R be left max-coherent. Then every left R-module has an $\mathscr{MI}$-precover.*

Proof. It follows from Theorem 2.8 and [4, Proposition 5.2.2]. □

Remark 2.10. (1) $\mathscr{MI}$ is closed under direct limits over max-coherent ring by Theorem 2.3. Then if R is left max-coherent, every left R-module has an $\mathscr{MI}$-cover by [4, Corollary 5.2.7].

(2) In general, $\mathscr{MI}$-cover need not be an epimorphism and $\mathscr{MF}$-preenvelope need not be a monomorphism. In the following theorem, we will consider when every left R-module has an epic $\mathscr{MI}$-cover and when every right R-module has a monic $\mathscr{MF}$-preenvelope.

Theorem 2.11. *Let R be left max-coherent. Then the following are equivalent*:

(1) *R is left max-injective.*
(2) *For any left R-module, there is an epic $\mathscr{MI}$-cover.*
(3) *For any right R-module, there is a monic $\mathscr{MF}$-preenvelope.*

Proof. (1) ⇒ (2). Since R is left max-coherent, every left R-module has an $\mathscr{MI}$-cover $g : D \to M$ by remark 2.10 (1). In addition, for any left R-module M, there is an epimorphism $f : R^{(I)} \to M$. R is max-injective, then $R^{(I)}$ is max-injective in terms of Proposition 2.4 (2). Thus there is a homomorphism $h : R^{(I)} \to D$ such that $gh = f$. Note that f is epic, so is g.

(2) $\Rightarrow$ (1). By assumption, R has an epic $\mathscr{MI}$-cover $\varphi : D \to R$, then we have an exact sequence $0 \to K \to D \xrightarrow{\varphi} R \to 0$ with $K = \ker\varphi$ and D max-injective. Since R is projective, so the sequence splits. Thus R is max-injective as a left R-module by Proposition 2.4.

(1) $\Rightarrow$ (3). Let M be any right R-module. Then M has an $\mathscr{MF}$-preenvelope $f : M \to F$ by Theorem 2.5. Since $({}_RR)^+$ is a cogenerator in the category of right R-modules, there is an exact sequence $0 \to M \xrightarrow{i} \prod({}_RR)^+$. By Theorem 2.3, $\prod({}_RR)^+$ is max-flat. So there exists a homomorphism $g : F \to \prod({}_RR)^+$ such that $gf = i$. Note that i is monic, so is f.

(3) $\Rightarrow$ (1). Note that the injective right R-module $({}_RR)^+$ embeds in a max-flat right R-module by (3). Thus $({}_RR)^+$ is max-flat, and so ${}_RR$ is max-injective by Theorem 2.3. □

We conclude this section with the following result which elaborates the relation between $\mathscr{MI}$-precovers and $\mathscr{MF}$-preenvelopes.

Proposition 2.12. *Let R be a left max-coherent ring. For a right R-module M, $\varphi : M \to F$ is an $\mathscr{MF}$-preenvelope, then $\varphi^+ : F^+ \to M^+$ is an $\mathscr{MI}$-precover of M^+.*

Proof. Since F is max-flat, F^+ is a max-injective left R-module. For any homomorphism $g : D \to M^+$ with D max-injective, we have $g^+ : M^{++} \to D^+$, hence $g^+\sigma_M : M \to D^+$, where $\sigma_M : M \to M^{++}$ is an evaluation map. By Theorem 2.3, D^+ is max-flat since R is left max-coherent. Thus there exists a homomorphism $f : F \to D^+$ such that $f\varphi = g^+\sigma_M$. Whence $\sigma_M^+ g^{++} = \varphi^+ f^+$. Since $g^{++}\sigma_D = \sigma_{M^+} g$. Let $f^+\sigma_D : D \to F^+$, note $\sigma_M^+\sigma_{M^+} = I_{M^+}$, then $\varphi^+ f^+\sigma_D = \sigma_M^+ g^{++}\sigma_D = \sigma_M^+\sigma_{M^+} g = g$. Therefore $\varphi^+ : F^+ \to M^+$ is an $\mathscr{MI}$-precover. □

3. *MI*-injective modules and *MI*-flat modules

The main purpose of this section is to consider the kernel of $\mathscr{MI}$-(pre)covers and the cokernel of $\mathscr{MF}$-(pre)envelopes.

Definition 3.1. A left R-module M is said to be MI-injective if $\mathrm{Ext}_R^1(N, M) = 0$ for any max-injective left R-module N. A right R-module F is called MI-flat if $\mathrm{Tor}_1^R(F, N) = 0$ for any max-injective left R-module N.

Remark 3.2. (1) By using Wakamutsu's Lemma (see [15, Lemma 2.1.1]), any kernel of an $\mathscr{MI}$-cover is MI-injective.

(2) A right R-module M is MI-flat if and only if M^+ is MI-injective by the standard isomorphism $\mathrm{Ext}_R^1(N, M^+) \cong \mathrm{Tor}_1^R(M, N)^+$ for any max-injective left R-module N.

Proposition 3.3. *The following are equivalent for a left R-module M :*

(1) *M is MI-injective.*

(2) *For every exact sequence* $0 \to M \to A \to B \to 0$ *with* A *max-injective,* $A \to B$ *is an* $\mathscr{MI}$*-precover of* B.
(3) M *is the kernel of an* $\mathscr{MI}$*-precover* $f : A \to B$ *with* A *injective.*
(4) M *is injective with respect to every exact sequence* $0 \to K \to A \to C \to 0$ *with* C *max-injective.*

Proof. (1)⇒(2) and (1)⇒(4) are trivial.

(2)⇒(3) is obvious.

(3)⇒(1). Let M be a kernel of an $\mathscr{MI}$-precover $f : A \to B$ with A injective. Then there is an exact sequence $0 \to M \to A \to A/M \to 0$. For any max-injective left R-module N, the sequence $\mathrm{Hom}_R(N, A) \xrightarrow{\pi} \mathrm{Hom}_R(N, A/M) \to \mathrm{Ext}^1_R(N, M) \to 0$ is exact. Note that $A \to A/M$ is $\mathscr{MI}$-precover, so π is epic. Thus $\mathrm{Ext}^1_R(N, M) = 0$, and so M is MI-injective.

(4)⇒(1). For any max-injective left R-module C, there exists an exact sequence $0 \to K \to A \to C \to 0$ with A projective, which induces an exact sequence $\mathrm{Hom}_R(A, M) \xrightarrow{\pi} \mathrm{Hom}_R(K, M) \to \mathrm{Ext}^1_R(C, M) \to 0$. By assumption, π is epic. Thus $\mathrm{Ext}^1_R(C, M) = 0$, and so M is MI-injective. □

By Definition 3.1, every injective left R-module (resp. flat right R-module) is MI-injective (resp. MI-flat). The converse is not true in general. However, over a ring of which every maximal left ideal is projective, we have the following:

Proposition 3.4. *Let* R *be a left max-coherent ring, and every maximal left ideal be projective. Then*

(1) *Every* MI*-injective left* R*-module is injective.*
(2) *Every* MI*-flat right* R*-module is flat.*

Proof. (1) Let M be an MI-injective left R-module. Then there is an exact sequence $0 \to M \to E \to N \to 0$ with E injective. By [13, Proposition 6.7.3], N is max-injective since every maximal left ideal is projective. Hence $\mathrm{Ext}^1_R(N, M) = 0$, and so the exact sequence splits. Then M is injective.

(2) For any MI-flat right R-module F, by Remark 3.2 (2), F^+ is MI-injective left R-module. Thus F^+ is injective by (1). So F is flat. □

Proposition 3.5. *Let* M *be a left* R*-module over a left max-coherent ring* R. *Then the following are equivalent*:

(1) M *is a reduced* MI*-injective left* R*-module.*
(2) M *is the kernel of an* $\mathscr{MI}$*-cover* $f : A \to B$ *with* A *injective.*

Proof. (1)⇒(2). By Proposition 3.3, the natural map $\pi : E(M) \to E(M)/M$ is an $\mathscr{MI}$-precover of $E(M)/M$. But $E(M)/M$ has an $\mathscr{MI}$-cover by Remark 2.10 (1). $E(M)$ has no nonzero direct summand K contained in M since M is reduced. By [15, Corollary 1.2.8], $\pi : E(M) \to E(M)/M$ is an $\mathscr{MI}$-cover of $E(M)/M$.

(2)⇒(1). Let M be the kernel of an $\mathscr{MI}$-cover $f : A \to B$ with A injective. So M is MI-injective by Proposition 3.3. Now let K be an injective submodule of M. Suppose $A = K \oplus L$. $p : A \to L$ is projection and $i : L \to A$ is inclusion. Note that $f(ip) = f$ since $f(K) = 0$. Thus ip is an isomorphism since f is cover. So i is epic, $A = L$. Then $K = 0$, and hence M is reduced. □

Recall that R is a *QF-ring* if R is left Noetherian and ${}_RR$ is injective, if and only if R is left Artinian and right max-injective (see [14, Corollary 3.2]).

Proposition 3.6. *R is a QF-ring if and only if every left R-module is MI-injective.*

Proof. If R is a QF-ring, then R is left Artinian. In view of [14, Theorem 3.1], every max-injective left R-module is injective, and hence every max-injective left R-module is projective by [1, Theorem 31.9]. Thus every left R-module is MI-injective. Conversely, note that any injective left R-module M is max-injective. By assumption, any left R-module N is MI-injective, so $\mathrm{Ext}^1_R(M, N) = 0$, and hence M is projective. Therefore, R is a QF-ring by [1, Theorem 31.9] again. □

Proposition 3.7. *Let R be a left max-coherent ring. Then the following hold*:

(1) *If C is the cokernel of an $\mathscr{MF}$-preenvelope $f : M \to F$ of a right R-module M with F flat, then C is MI-flat.*

(2) *If L is a finitely presented MI-flat right R-module, then L is the cokernel of an $\mathscr{MF}$-preenvelope $g : K \to P$ with P flat.*

Proof. (1) There is an exact sequence of right R-modules $0 \to \mathrm{im}(f) \to F \to C \to 0$. Using functor $- \otimes_R N$ with a max-injective left R-module N, we have an exact sequence

$$0 \to \mathrm{Tor}^R_1(C, N) \to \mathrm{im}(f) \otimes_R N \to F \otimes_R N.$$

Note that $\mathrm{im}(f) \to F$ is also an $\mathscr{MF}$-preenvelope, and N^+ is max-flat by Theorem 2.3. Then the sequence $\mathrm{Hom}_R(F, N^+) \to \mathrm{Hom}_R(\mathrm{im}(f), N^+) \to 0$ is exact. So $(F \otimes_R N)^+ \to (\mathrm{im}(f) \otimes_R N)^+ \to 0$ is exact. Thus we have an exact sequence $0 \to \mathrm{im}(f) \otimes_R N \to F \otimes_R N$, and so $\mathrm{Tor}^R_1(C, N) = 0$. Then C is MI-flat.

(2) Let L be a finitely presented MI-flat right R-module. There is an exact sequence $0 \to K \xrightarrow{i} P \to L \to 0$ with P finitely generated projective and K finitely generated. It is enough to show that $i : K \to P$ is an $\mathscr{MF}$-preenvelope. In fact, for any max-flat right R-module F, we have $\mathrm{Tor}^R_1(L, F^+) = 0$, and so we get the following commutative diagram with the first row exact:

$$\begin{array}{ccccc} 0 & \to & K \otimes F^+ & \xrightarrow{i \otimes 1_{F^+}} & P \otimes F^+ \\ & & \alpha \downarrow & & \beta \downarrow \\ & & \mathrm{Hom}_R(K, F)^+ & \xrightarrow{h} & \mathrm{Hom}_R(P, F)^+. \end{array}$$

Note that α is an epimorphism and β is an isomorphism by [4, Theorem 3.2.11]. Thus h is a monomorphism, and hence $\mathrm{Hom}_R(P, F) \to \mathrm{Hom}_R(K, F)$ is epic, as required. □

4. $\mathscr{MI}$-dimensions

By Theorem 2.9, every left R-module has an $\mathscr{MI}$-precover over a left max-coherent ring. Then every left R-module has a left $\mathscr{MI}$-resolution. According to [12, Theorem 3.5] and Proposition 2.4(2), we can claim that every left R-module has an $\mathscr{MI}$-preenvelope over a left max-coherent ring. Thus every left R-module has a right $\mathscr{MI}$-resolution which is exact since injective R-module is max-injective. So $\mathrm{Hom}_R(-,-)$ is left balanced (see [4]) on ${}_R\mathscr{M} \times_R \mathscr{M}$ by $\mathscr{MI} \times \mathscr{MI}$ if R is a left max-coherent ring. Let $\mathrm{Ext}_n(-,-)$ be the nth left derived functor of $\mathrm{Hom}_R(-,-)$ with respect to the pair $\mathscr{MI} \times \mathscr{MI}$. Then, for two left R-modules M and N, $\mathrm{Ext}_n(M,N)$ can be computed by using a right $\mathscr{MI}$-resolution of M or a left $\mathscr{MI}$-resolution of N.

Let $0 \to M \xrightarrow{d_0} E^0 \xrightarrow{d_1} E^1 \to \cdots$ be a right $\mathscr{MI}$-resolution of M. Applying $\mathrm{Hom}_R(-,N)$, we have the deleted complex

$$(\star) \qquad \cdots \to \mathrm{Hom}_R(E^1,N) \xrightarrow{d_1^*} \mathrm{Hom}_R(E^0,N) \to 0.$$

Then $\mathrm{Ext}_n(M,N)$ is the nth homology of $(\star)$. There is a canonical map

$$\alpha : \mathrm{Ext}_0(M,N) = \mathrm{Hom}_R(E^0,N)/\mathrm{im}(d_1^*) \to \mathrm{Hom}_R(M,N)$$

defined by $\alpha(\beta + \mathrm{im}(d_1^*)) = \beta d_0$ for $\beta \in \mathrm{Hom}_R(E^0,N)$.

Proposition 4.1. *Let R be a left max-coherent ring. The following are equivalent for a left R-module M :*

(1) *M is max-injective.*
(2) *The canonical map $\alpha : \mathrm{Ext}_0(M,N) \to \mathrm{Hom}_R(M,N)$ is an epimorphism for any left R-module N.*
(3) *The canonical map $\alpha : \mathrm{Ext}_0(M,M) \to \mathrm{Hom}_R(M,M)$ is an epimorphism.*

Proof. (1)⇒(2) is clear by letting $E^0 = M$.

(2)⇒(3) is trivial.

(3)⇒(1). By assumption, there is a homomorphism $\beta \in \mathrm{Hom}_R(E^0,M)$ such that $\alpha(\beta + \mathrm{im}(d_1^*)) = \beta d_0 = I_M$. Thus M is isomorphic to a direct summand of E^0, and hence it is max-injective by Proposition 2.4 (1). □

Proposition 4.2. *Let R be a left max-coherent ring. The following are equivalent for a left R-module M :*

(1) *right $\mathscr{MI}$-dim $M \leq 1$.*
(2) *The canonical map $\alpha : \mathrm{Ext}_0(M,N) \to \mathrm{Hom}_R(M,N)$ is a monomorphism for any left R-module N.*

Proof. (1)⇒(2). By assumption, M has an exact right $\mathscr{MI}$-resolution $0 \to M \to E^0 \to E^1 \to 0$. Hence we obtain an exact sequence $0 \to \mathrm{Hom}_R(E^1,N) \to \mathrm{Hom}_R(E^0,N) \to \mathrm{Hom}_R(M,N)$ for any left R-module N. Therefore, α is a monomorphism.

(2)⇒(1). Consider the exact sequence $0 \to M \xrightarrow{\varphi} E^0 \to C \to 0$, where $\varphi : M \to E^0$ is an $\mathscr{MI}$-preenvelope and $C = \mathrm{coker}(\varphi)$. In view of [4, Theorem 8.2.3], we have the commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & \mathrm{Ext}_0(C,C) & \to & \mathrm{Ext}_0(E^0,C) & \to & \mathrm{Ext}_0(M,C) & \to & 0 \\ & & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\ 0 & \to & \mathrm{Hom}_R(C,C) & \to & \mathrm{Hom}_R(E^0,C) & \to & \mathrm{Hom}_R(M,C). & & \end{array}$$

Since E^0 is max-injective, α_2 is epic by Proposition 4.1. By assumption, α_3 is monic. So α_1 is epic by diagram chasing. Thus C is max-injective by Proposition 4.1, as desired. □

Proposition 4.3. *Let R be a left max-coherent ring and an integer $n \geq 2$. The following are equivalent for a left R-module M :*

(1) *right $\mathscr{MI}$-dim $M \leq n$.*
(2) $\mathrm{Ext}_{n+k}(M,N) = 0$ *for all left R-modules N and $k \geq -1$.*
(3) $\mathrm{Ext}_{n-1}(M,N) = 0$ *for all left R-modules N.*

Proof. (1)⇒(2). Let $0 \to M \to E^0 \to E^1 \to \cdots \to E^n \to 0$ be a right $\mathscr{MI}$-resolution of M. Then

$$0 \to \mathrm{Hom}_R(E^n,N) \to \mathrm{Hom}_R(E^{n-1},N) \to \mathrm{Hom}_R(E^{n-2},N)$$

is exact, and so $\mathrm{Ext}_{n-1}(M,N) = \mathrm{Ext}_n(M,N) = 0$. In addition, it is clear that $\mathrm{Ext}_{n+k}(M,N) = 0$ for all $k \geq 1$. Hence (2) follows.

(2)⇒(3) is trivial.

(3)⇒(1). Let $0 \to M \to E^0 \to E^1 \to \cdots$ be a right $\mathscr{MI}$-resolution of M. Let $C = \mathrm{im}(E^{n-2} \to E^{n-1})$. Then $\mathrm{Ext}_{n-1}(M, E^{n-1}/C) = 0$ by assumption. But then $E^{n-1}/C \to E^n$ has a retract. Hence $E^{n-1}/C \in \mathscr{MI}$. So $0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^{n-1}/C \to 0$ be a right $\mathscr{MI}$-resolution of M. □

Recall that a ring R is called *left semiartinian* (or left socular) if any nonzero left R-module contains a simple submodule, or equivalently, every nonzero left R-module has a nonzero left socle (see [6]). It is shown that if R is a left semiartinian ring, then every max-injective left R-module is injective (cf. [14, Theorem 3.1]).

Proposition 4.4. *Let R be a left max-coherent ring and an integer $n \geq 2$. Observe the following statements for a left R-module N :*

(1) *left $\mathscr{MI}$-dim $N \leq n-2$.*
(2) $\mathrm{Ext}_{n+k}(M,N) = 0$ *for all left R-modules M and $k \geq -1$.*
(3) $\mathrm{Ext}_{n-1}(M,N) = 0$ *for all left R-modules M.*

Then (1) ⇒ (2) ⇒ (3)*. If R is left semiartinian,* (3) ⇒ (1)*.*

Proof. (1)⇒(2)⇒(3) are clear.

(3)⇒(1). Let $\cdots \to E_n \xrightarrow{f} E_{n-1} \xrightarrow{g} E_{n-2} \to E_{n-3} \to \cdots \to E_1 \to E_0 \to N \to 0$ be a left $\mathscr{MI}$-resolution of N and $K = \ker(E_{n-1} \to E_{n-2})$, and let $p \in \mathrm{Hom}_R(E_n, K)$. Then $\mathrm{Ext}_{n-1}(K,N) = 0$. So we have an exact sequence

$\mathrm{Hom}_R(K, E_n) \xrightarrow{f_*} \mathrm{Hom}_R(K, E_{n-1}) \xrightarrow{g_*} \mathrm{Hom}_R(K, E_{n-2})$. For inclusion map $i : K \to E_{n-1}$, $g_*(i) = gi = 0$, so $i \in \ker(g_*) = \mathrm{im}(f_*)$. There is $l \in \mathrm{Hom}_R(K, E_n)$ such that $i = f_*(l) = fl$. Thus $i = fl = ipl$, it implies $pl = I_K$ since i is monic. Then $p : E_n \to K$ has a section. Hence $K \in \mathscr{MI}$. Since R is left semiartinian, K is injective, and hence the exact sequence $0 \to K \to E_{n-1} \to E_{n-1}/K \to 0$ splits. So $K_1 = E_{n-1}/K \in \mathscr{MI}$. Similarly, $K_2 = E_{n-2}/K_1 \in \mathscr{MI}$. Thus $0 \to K_2 \to E_{n-3} \to \cdots \to E_1 \to E_0 \to N \to 0$ is also a left $\mathscr{MI}$-resolution of N, as desired. □

Corollary 4.5. *Let R be a left max-coherent ring and an integer $n \geq 2$. Observe the following statements*:

(1) *gl left $\mathscr{MI}$-$\dim_R \mathscr{M} \leq n-2$.*
(2) *gl right $\mathscr{MI}$-$\dim_R \mathscr{M} \leq n$.*
(3) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all left R-modules M, N and $k \geq -1$.*
(4) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all left R-modules M, N.*

Then (1) ⇒ (3) ⇒ (4) ⇒ (2). *If R is left semiartinian,* (2) ⇒ (1).

Proof. It follows from Proposition 4.3 and Proposition 4.4. □

A homomorphism $g : M \to C$ with $C \in \mathscr{C}$ is said to a *$\mathscr{C}$-envelope with the unique mapping property* (see [3]) if for any homomorphism $g' : M \to C'$ with $C' \in \mathscr{C}$, there is a unique homomorphism $f : C \to C'$ such that $fg = g'$. Dually, we have the definition of a *$\mathscr{C}$-cover with the unique mapping property.*

Theorem 4.6. *Let R be a ring of which every maximal left ideal is finitely generated. If every left R-module has an $\mathscr{MI}$-cover with the unique mapping property, then R is left max-coherent and gl right $\mathscr{MI}$-$\dim_R \mathscr{M} \leq 2$.*

Proof. Let $\{F_i, \varphi_j^i\}$ be a direct system with each $F_i \in \mathscr{MI}$. By assumption, $\varinjlim F_i$ has an $\mathscr{MI}$-cover $g : F \to \varinjlim F_i$ with the unique mapping property. Suppose that $\alpha_i : F_i \to \varinjlim F_i$ satisfy $\alpha_i = \alpha_j \varphi_j^i$ whenever $i \leq j$. Then there exists $f_i : F_i \to F$ such that $\alpha_i = gf_i$ for any i, so $gf_i = \alpha_j \varphi_j^i = gf_j \varphi_j^i$. Hence $f_i = f_j \varphi_j^i$ by the unique mapping property of g. Thus there exists $h : \varinjlim F_i \to F$ such that $h\alpha_i = f_i$, and so $(gh)\alpha_i = gf_i = \alpha_i$ for any i. Then $gh = I_{\varinjlim F_i}$ by the definition of direct limits. So $\varinjlim F_i$ is a direct summand of F, and hence $\varinjlim F_i \in \mathscr{MI}$. Then R is a left max-coherent ring by Theorem 2.3.

Now we prove that gl right $\mathscr{MI}$-$\dim_R \mathscr{M} \leq 2$. In fact, for any left R-module M, there exists an $\mathscr{MI}$-cover $f : F \to M$ with the unique mapping property, hence $0 \to F \to M \to 0$ is a left $\mathscr{MI}$-resolution. Thus gl left $\mathscr{MI}$-$\dim_R \mathscr{M} = 0$, so gl right $\mathscr{MI}$-$\dim_R \mathscr{M} \leq 2$ by Corollary 4.5. □

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