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PRECISE ASYMPTOTICS FOR THE MOMENT CONVERGENCE OF MOVING-AVERAGE PROCESS UNDER DEPENDENCE

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Abstract. Let $\{\varepsilon_i : -\infty < i < \infty\}$ be a strictly stationary sequence of linearly positive quadrant dependent random variables and $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. In this paper, we prove the precise asymptotics in the law of iterated logarithm for the moment convergence of moving-average process of the form $X_k = \sum_{i=-\infty}^{\infty} a_{i+k}\varepsilon_i, k \geq 1$.

1. Introduction

We assume that $\{\varepsilon_i : -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed variables. Let $\{a_i : -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and $X_k = \sum_{i=-\infty}^{\infty} a_{i+k}\varepsilon_i, k \geq 1$. Set $S_n = \sum_{k=1}^{n} X_k$, also let $\log y = \log(y \vee e)$, $\log\log y = \log\log(y \vee e^e)$ for all $y > 0$.

When $\{\varepsilon_i : -\infty < i < \infty\}$ is a sequence of independent random variables, many limiting results have been obtained for moving-average process $\{X_k : k \geq 1\}$. For example, Burton and Dehling [1] have obtained a large deviation principle for $\{X_k : k \geq 1\}$ assuming $E\exp t\varepsilon_1 < \infty$ for all t, Ibragimov [4] has established the central limit theorem for $\{X_k : k \geq 1\}$, Li et al. [7] derived convergence rates of moderate deviations and the precise asymptotics in the law of the iterated logarithm.

On the other hand, Gut and Spătaru [3] proved the precise asymptotics of i.i.d random variables. One of their results is as follows.

Theorem A. *Suppose that $\{Y_k : k \geq 1\}$ is a sequence of i.i.d random variables with $EY_1 = 0$ and $EY_1^2 = \sigma^2 < \infty$. Then*

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n\log n} P(|\sum_{k=1}^{n} Y_k| \geq \varepsilon\sqrt{n\log\log n}) = \sigma^2.$$

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Chow [2] discussed the complete moment convergence of i.i.d random variables. He got the following result:

Theorem B. *Let $\{Y, Y_k : k \geq 1\}$ be a sequence of i.i.d random variables with $EY_1 = 0$. Suppose that $p \geq 1, \alpha > \frac{1}{2}, p\alpha > 1, E\{|Y|^p + |Y| \log(1 + |Y|)\} < \infty$. Then for any $\varepsilon > 0$, we have*

$$\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} E\{\max_{j \leq n} |\sum_{k=1}^{j} Y_k| - \varepsilon n^{\alpha}\}_+ < \infty.$$

In this note, we show that the precise asymptotics for the moment convergence holds for moving-average process when $\{\varepsilon_i : -\infty < i < \infty\}$ is a strictly stationary linear positive quadrant dependent sequence. First, we shall give the definition of linear positive quadrant dependent sequence.

Two random variables X and Y are said to be positive quadrant dependent (PQD) if $P(X > x, Y > y) \geq P(X > x)P(Y > y)$ for all $x, \ y \in R$. This notation was first introduced by Lehmann [6], another concept which is stronger than PQD was due to Newman [9]: a sequence $\{\varepsilon_i : -\infty < i < \infty\}$ is said to be linear positive quadrant dependent (LPQD) if for any disjoint finite subsets $A, B \subset \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and any positive real numbers r_j,

$$\sum_{i \in A} r_i \varepsilon_i \ \text{ and } \ \sum_{j \in B} r_j \varepsilon_j \ \text{ are } PQD.$$

2. Main result

Throughout this paper, let $\{\varepsilon_i : -\infty < i < \infty\}$ be a sequence of strictly stationary linear positive quadrant dependent random variables with $E\varepsilon_i = 0$, $0 < E\varepsilon_i^2 < \infty$, and set $0 < \sigma^2 = E\varepsilon_1^2 + 2\sum_{k=2}^{\infty} E\varepsilon_1\varepsilon_k < \infty$ unless it is specially mentioned. Now we state our result as follows.

Theorem 2.1. *Assume*

$$\sum_{i=n+1}^{\infty} E\varepsilon_1\varepsilon_i = O(n^{-\rho}) \quad \textit{for some} \quad \rho > 0,$$

and

$$E|\varepsilon_i|^s < \infty \quad \textit{for some} \quad s > 2.$$

Then for $-1 < b < -1/2$, we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n^{\frac{3}{2}} \log n} E\{|S_n| - \varepsilon\sqrt{2n \log\log n}\}_+ = \frac{2^{-b-1}}{(b+1)(2b+3)} E|Z|^{2b+3},$$

where Z has a normal distribution with mean 0 and variance $\tau^2 = \sigma^2(\sum_{i=-\infty}^{\infty} a_i)^2$.

Remark 2.1. Let $a_{i+k} = 1, i = k; a_{i+k} = 0, i \neq k, 1 \leq k \leq n$. Then $X_k = \varepsilon_k, S_n = \sum_{k=1}^{n} \varepsilon_k$. Thus above result holds under some suitable conditions when $\{X_i : i \geq 1\}$ is a sequence of strictly stationary linear positive quadrant dependent random variables.

The following example comes from Li and Wang [8].

Remark 2.2. A finite family of random variables $\{X_i : 1 \le i \le n\}$ is said to be positively associated (PA) if for every pair of disjoint subsets A and B of $\{1, 2, \dots\}$,

$$\mathrm{Cov}\{f(X_i : i \in A), g(X_j ; j \in B)\} \ge 0,$$

whenever f and g are coordinatewise increasing and the covariance exists. A PA sequence is obviously a LPQD sequence, the following example shows that LPQD does not imply PA: Consider three discrete random variables with joint density $p(x, y, z) := P(X = x, Y = y, Z = z)$.

$$\begin{aligned} p(2,2,1) &= p(3,2,1) = p(2,3,1) = p(3,3,1) = p(1,1,2) \\ &= p(2,1,2) = p(3,1,2) = p(1,2,2) = p(1,3,2) = \frac{1}{17} \quad \text{and} \\ p(1,1,1) &= p(3,3,2) = \frac{4}{17}. \end{aligned}$$

A lengthy verification shows that $\{X, Y, Z\}$ is LPQD. But, $\{X, Y, Z\}$ is not PA since $P(X > 1, Y > 1, Z > 1) = \frac{4}{17} < P(X > 1, Y > 1)P(Z > 1) = \frac{72}{289}$.

3. Some lemmas

First, we give some lemmas which will be used in the proofs. Lemma 3.1 and Lemma 3.2 are from Burton and Dehling [1], Kim [5] respectively.

Lemma 3.1. *Let $\sum_{i=-\infty}^{\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{i=-\infty}^{\infty} a_i$ and $k \ge 1$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \left| \sum_{j=i+1}^{i+n} a_j \right|^k = |a|^k.$$

Lemma 3.2. *Let $\{\varepsilon_i : -\infty < i < \infty\}$ be a sequence of strictly stationary linear positive quadrant dependent random variables with $E\varepsilon_i = 0$, $0 < E\varepsilon_i^2 < \infty$, and set $0 < \sigma^2 = E\varepsilon_1^2 + 2\sum_{k=2}^{\infty} E\varepsilon_1\varepsilon_k < \infty$. Assume*

$$\sum_{i=n+1}^{\infty} E\varepsilon_1\varepsilon_i = O(n^{-\rho}) \quad \text{for some} \quad \rho > 0,$$

and

$$E|\varepsilon_i|^s < \infty \quad \text{for some} \quad s > 2.$$

Then the linear process $\{X_k\}$ fulfills the CLT, that is,

$$\frac{S_n}{\tau\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1), \ \text{where } \tau = \sigma \sum_{i=-\infty}^{\infty} a_i.$$

Throughout the sequel, N represent standard normal variable. C will denote a positive constant although its value may change from one appearance to the next and let $[x]$ indicate the maximum integer not larger than x.

4. Proof of Theorem 2.1

Without loss of generality, we assume $\tau = 1$ in this section. Let $A(\varepsilon) = \exp\left\{\exp\{\frac{M}{\varepsilon^2}\}\right\}$, $M > 1$. Our main result will be proved via the following propositions.

Proposition 4.1. *For any $b > -1$, we have*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}E\{|N|-\varepsilon\sqrt{2\log\log n}\}_+ = \frac{2^{-b-1}}{(b+1)(2b+3)}E|N|^{2b+3}.$$

Proof. By the variable change, we have

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}E\{|N|-\varepsilon\sqrt{2\log\log n}\}_+\\
=&\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}\int_{\varepsilon\sqrt{2\log\log n}}^{\infty}P(|N|\geq x)dx\\
=&\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\int_{e^e}^{\infty}\frac{(\log\log t)^b}{t\log t}\int_{\varepsilon\sqrt{2\log\log t}}^{\infty}P(|N|\geq x)dxdt\\
=&\lim_{\varepsilon\searrow 0}2^{-b}\int_{\varepsilon\sqrt{2}}^{\infty}y^{2b+1}\int_{y}^{\infty}P(|N|\geq x)dxdy\\
=&\lim_{\varepsilon\searrow 0}\frac{2^{-b}}{2(b+1)}\int_{\varepsilon\sqrt{2}}^{\infty}P(|N|\geq x)(x^{2b+2}-\varepsilon^{2b+2}\cdot 2^{b+1})dx\\
=&\lim_{\varepsilon\searrow 0}\frac{2^{-b}}{2(b+1)}\int_{\varepsilon\sqrt{2}}^{\infty}x^{2b+2}P(|N|\geq x)dx\\
=&\frac{2^{-b-1}}{(b+1)(2b+3)}E|N|^{2b+3}.
\end{aligned}$$

Thus the proposition is now proved. □

Proposition 4.2. *For any $b > -1$, we have*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n\leq A(\varepsilon)}\frac{(\log\log n)^b}{n^{\frac{3}{2}}\log n}\left|E\{|S_n|-\varepsilon\sqrt{2n\log\log n}\}_+-\sqrt{n}E\{|N|-\varepsilon\sqrt{2\log\log n}\}_+\right| = 0.$$

Proof. Denote

$$\triangle_n = \sup_x\left|P(\frac{|S_n|}{\sqrt{n}}\geq x)-P(|N|\geq x)\right|,$$

it follows from Lemma 3.2 that $\triangle_n \to 0$ as $n\to\infty$. Then

$$\sum_{n\leq A(\varepsilon)}\frac{(\log\log n)^b}{n^{\frac{3}{2}}\log n}\left|E\{|S_n|-\varepsilon\sqrt{2n\log\log n}\}_+-\sqrt{n}E\{|N|-\varepsilon\sqrt{2\log\log n}\}_+\right|$$

$$\leq \sum_{n\leq A(\varepsilon)} \frac{(\log\log n)^b}{n\log n} \times \int_0^\infty \left| P(\frac{|S_n|}{\sqrt{n}} \geq \varepsilon\sqrt{2\log\log n}+x) - P(|N| \geq \varepsilon\sqrt{2\log\log n}+x)\right| dx$$

$$\leq \sum_{n\leq A(\varepsilon)} \frac{(\log\log n)^b}{n\log n}(\triangle_{n_1}+\triangle_{n_2}) \text{ (say)},$$

where

$$\triangle_{n_1} = \int_0^{\frac{1}{\sqrt{\triangle_n}}} \left| P(\frac{|S_n|}{\sqrt{n}} \geq \varepsilon\sqrt{2\log\log n}+x) - P(|N| \geq \varepsilon\sqrt{2\log\log n}+x)\right| dx,$$

$$\triangle_{n_2} = \int_{\frac{1}{\sqrt{\triangle_n}}}^\infty \left| P(\frac{|S_n|}{\sqrt{n}} \geq \varepsilon\sqrt{2\log\log n}+x) - P(|N| \geq \varepsilon\sqrt{2\log\log n}+x)\right| dx.$$

It is easy to obtain

$$\triangle_{n_1} \leq \sqrt{\triangle_n} \to 0 \quad \text{as } n\to\infty.$$

Next, observe that

$$\sum_{k=1}^n X_k = \sum_{i=-\infty}^\infty \sum_{k=1}^n a_{k+i}\varepsilon_i.$$

Set $a_{ni} = \sum_{k=1}^n a_{k+i}$. Then

$$\sum_{k=1}^n X_k = \sum_{i=-\infty}^\infty a_{ni}\varepsilon_i = \sum_{i=-\infty}^\infty Y_i \text{ (say)}.$$

From Lemma 3.1, we can assume, without loss of generality, that

$$\sum_{i=-\infty}^\infty |a_{ni}| \leq n,\ n\geq 1 \text{ and } \sum_{i=-\infty}^\infty |a_i| \leq 1.$$

And then, by Lemma 3.1 and the stationarity we get

$$\begin{aligned}
\mathrm{Var}(S_n) &= E\varepsilon_1^2 \sum_{i=-\infty}^\infty a_{ni}^2 + 2\sum_{i=-\infty}^\infty \sum_{j=i+1}^\infty a_{ni}a_{nj}E\varepsilon_i\varepsilon_j \\
&\leq nCE\varepsilon_1^2 + 2\sum_{i=-\infty}^\infty \sum_{k=1}^\infty a_{ni}a_{n\ k+i}E\varepsilon_1\varepsilon_{k+1} \\
&\leq nCE\varepsilon_1^2 + \sum_{i=-\infty}^\infty \sum_{k=1}^\infty (a_{ni}^2 + a_{n\ k+i}^2)E\varepsilon_1\varepsilon_{k+1} \qquad (1.1)\\
&\leq nCE\varepsilon_1^2 + \sum_{k=1}^\infty E\varepsilon_1\varepsilon_{k+1}\sum_{i=-\infty}^\infty a_{ni}^2 + \sum_{k=1}^\infty E\varepsilon_1\varepsilon_{k+1}\sum_{i=-\infty}^\infty a_{n\ k+i}^2 \\
&\leq Cn.
\end{aligned}$$

Thus, by virtues of Markov's inequality, we have

$$\triangle_{n_2} \leq \int_{\frac{1}{\sqrt{\triangle_n}}}^\infty \frac{C+1}{(\varepsilon\sqrt{\log\log n}+x)^2}dx \leq (C+1)\sqrt{\triangle_n}.$$

Denote $\triangle_n^{'} = \triangle_{n_1} + \triangle_{n_2}$. It follows that

$$\frac{1}{(\log\log m)^{b+1}} \sum_{n=1}^{m} \frac{\triangle_n^{'} (\log\log n)^b}{n \log n} \to 0 \quad \text{as } m \to \infty.$$

We have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n \le A(\varepsilon)} \frac{(\log\log n)^b}{n^{\frac{3}{2}} \log n} \left| E\{|S_n| - \varepsilon\sqrt{2n\log\log n}\}_+ - \sqrt{n} E\{|N| - \varepsilon\sqrt{2\log\log n}\}_+ \right|$$
$$\le \lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n \le A(\varepsilon)} \frac{(\log\log n)^b}{n \log n} \triangle_n^{'}$$
$$= \lim_{\varepsilon \searrow 0} M^{b+1} \frac{1}{(\log\log[A(\varepsilon)])^{b+1}} \sum_{n \le A(\varepsilon)} \frac{\triangle_n^{'}}{n \log n} (\log\log n)^b \to 0.$$

Hence, the proposition holds. □

Proposition 4.3. *Uniformly for* $0 < \varepsilon < \frac{1}{\sqrt{2}}$, *we have*

$$\lim_{M \longrightarrow \infty} \limsup_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n > A(\varepsilon)} \frac{(\log\log n)^b}{n^{\frac{3}{2}} \log n} \left| E\{|S_n| - \varepsilon\sqrt{2n\log\log n}\}_+ - \sqrt{n} E\{|N| - \varepsilon\sqrt{2\log\log n}\}_+ \right| = 0.$$

Proof. It is sufficient to show

$$\text{(1.2)} \qquad \lim_{M \longrightarrow \infty} \varepsilon^{2(b+1)} \sum_{n > A(\varepsilon)} \frac{(\log\log n)^b}{n \log n} E\{|N| - \varepsilon\sqrt{2\log\log n}\}_+ \Big| = 0$$

uniformly with respect to all sufficient small $0 < \varepsilon < \frac{1}{\sqrt{2}}$, and

$$\text{(1.3)} \quad \lim_{M \longrightarrow \infty} \limsup_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n > A(\varepsilon)} \frac{(\log\log n)^b}{n^{\frac{3}{2}} \log n} E\{|S_n| - \varepsilon\sqrt{2n\log\log n}\}_+ = 0.$$

Note that $A(\varepsilon) - 1 \ge \sqrt{A(\varepsilon)}$ for $M > 1$ and $0 < \varepsilon < \frac{1}{\sqrt{2}}$. Thus

$$\varepsilon^{2(b+1)} \sum_{n > A(\varepsilon)} \frac{(\log\log n)^b}{n \log n} E\{|N| - \varepsilon\sqrt{2\log\log n}\}_+$$
$$\le \varepsilon^{2(b+1)} \int_{A(\varepsilon)-1}^{\infty} \frac{(\log\log y)^b}{y \log y} \int_{\varepsilon\sqrt{\log\log y}}^{\infty} P\{|N| \ge x\} dx dy$$
$$\le \varepsilon^{2(b+1)} \int_{\sqrt{A(\varepsilon)}}^{\infty} \frac{(\log\log y)^b}{y \log y} \int_{\varepsilon\sqrt{\log\log y}}^{\infty} P\{|N| \ge x\} dx dy$$

$$
\begin{aligned}
&= 2\int_{\sqrt{M-\varepsilon^2\log 2}}^{\infty} t^{2b+1}\int_t^{\infty} P\{|N|\geq x\}dxdt\\
&\leq 2\int_{\sqrt{M-\frac{1}{2}\log 2}}^{\infty} t^{2b+1}\int_t^{\infty} P\{|N|\geq x\}dtdx\\
&\leq 2\int_{\sqrt{M-\frac{1}{2}\log 2}}^{\infty} P\{|N|\geq x\}\int_{\sqrt{M-\frac{1}{2}\log 2}}^{x} t^{2b+1}dtdx\\
&\leq C\int_{\sqrt{M-\frac{1}{2}\log 2}}^{\infty} x^{2b+2}P\{|N|\geq x\}dx \longrightarrow 0 \quad \text{as} \quad M\to\infty.
\end{aligned}
$$

Then (1.2) is proved.

Now we turn to prove (1.3). Notice that $E\varepsilon_1^2 < \infty$, which coupled with (1.1), it follows that, for $-1 < b < -1/2$

$$
\begin{aligned}
&\lim_{M\longrightarrow\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n>A(\varepsilon)}\frac{(\log\log n)^b}{n^{\frac{3}{2}}\log n}E\{|S_n|-\varepsilon\sqrt{2n\log\log n}\}_+\\
=&\lim_{M\longrightarrow\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n>A(\varepsilon)}\frac{(\log\log n)^b}{n^{\frac{3}{2}}\log n}\int_{\varepsilon\sqrt{2n\log\log n}}^{\infty}P\Big(\Big|\sum_{i=-\infty}^{\infty}a_{ni}\varepsilon_i\Big|\geq x\Big)dx\\
\leq&\lim_{M\longrightarrow\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n>A(\varepsilon)}\frac{(\log\log n)^b}{n^{\frac{3}{2}}\log n}\int_{\varepsilon\sqrt{2n\log\log n}}^{\infty}\frac{Cn}{x^2}dx\\
\leq&\lim_{M\longrightarrow\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n>A(\varepsilon)}\frac{(\log\log n)^b}{n^{\frac{1}{2}}\log n}(\varepsilon\sqrt{2n\log\log n})^{-1}\\
\leq&\lim_{M\longrightarrow\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{2b+1}[\log\log A(\varepsilon)]^{b+\frac{1}{2}}\\
\leq&\lim_{M\longrightarrow\infty}M^{b+\frac{1}{2}}=0.
\end{aligned}
$$

Then, we complete the proof of this proposition. □

Our main result now follows from the propositions.

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