NEW LOOK AT THE CONSTRUCTIONS OF MULTIWAVELET FRAMES

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ABSTRACT. Using the fiberization technique of a shift-invariant space and the matrix characterization of the decomposition of a shift-invariant space of finite length into an orthogonal sum of singly generated shift-invariant spaces, we show that the main result in [13] can be interpreted as a statement about the length of a shift-invariant space, and give a more natural construction of multiwavelet frames from a frame multiresolution analysis of $L^2(\mathbb{R}^d)$.

1. Introduction

For $y \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$, T_y denotes the unitary translation operator such that $T_y f(x) := f(x - y)$ and D denotes the unitary dyadic dilation operator such that $Df(x) := 2^{d/2} f(2x)$. A family of closed subspaces $\{V_k\}_{k \in \mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ is said to be a *frame multiresolution analysis (FMRA)* if

(1) $V_k \subset V_{k+1}, \quad k \in \mathbb{Z}$;
(2) $\overline{\cup_{k \in \mathbb{Z}} V_k} = L^2(\mathbb{R}^d), \cap_{k \in \mathbb{Z}} V_k = \{0\}$;
(3) $D(V_k) = V_{k+1}, \quad k \in \mathbb{Z}$;
(4) There exists a *scaling function* $\varphi \in V_0$ such that $\{T_k \varphi : k \in \mathbb{Z}^d\}$ is a tight frame with the frame bound one for V_0.

For the sake of brevity of presentation we only consider the dyadic dilation even though all of our results can be repeated verbatim if we consider general dilation, and we do not recount the theory and the history of FMRA in the literature [1, 2, 7, 9, 10, 11, 12, 13].

It is well-known that the condition (4) is equivalent to the condition that $\{T_k \varphi : k \in \mathbb{Z}^d\}$ is a frame for V_0 [3, 14]. Moreover, let $W_0 := V_1 \ominus V_0$ and $W_j := D^j(W_0)$. Then it is easy to see that $L^2(\mathbb{R}^d) = \oplus_{j \in \mathbb{Z}} W_j$. Hence, if there exists $\{\psi_1, \psi_2, \ldots, \psi_L\} \subset W_0$ such that $\{T_k \psi_l : k \in \mathbb{Z}^d, 1 \leq l \leq L\}$ is a frame for W_0, then $\{D^j T_k \psi_l : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq l \leq L\}$ is a frame for $L^2(\mathbb{R}^d)$. In this case we say that $\{\psi_1, \psi_2, \ldots, \psi_L\}$ is a (semi-orthogonal) *wavelet set*. It

is interesting to determine the minimal cardinality of a wavelet set, which is called the *index* of an FMRA in [13]. It is well-known and easy to see that the index is $2^d - 1$ if the *shifts* (multi-integer translates) of a scaling function form a Riesz basis of V_0.

We now present the main result in [13]. For $f \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, let $\hat{f}_{||x} := (\hat{f}(x + k))_{k \in \mathbb{Z}^d}$. Here $\wedge$ denotes the Fourier transform defined by $\hat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi i x \cdot t} \, dt$ for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and extended to be a unitary operator on $L^2(\mathbb{R}^d)$ by the Plancherel theorem. Then $\hat{f}_{||x} \in \ell^2(\mathbb{Z}^d)$ for a.e. $x \in \mathbb{R}^d$. Suppose we are given an FMRA with a scaling function φ. Let $\Phi(x) := \|\hat{\varphi}_{||x}\|^2_{\ell^2(\mathbb{Z}^d)}$, $\mathcal{Q} := \{x \in \mathbb{R}^d : \Phi(2x) = 0\}$, and

(1) $$Q := \{q_1, q_2, \ldots, q_{2^d}\} := \{0, 1\}^d,$$

which is a complete set of representatives of the quotient group $\mathbb{Z}^d / 2\mathbb{Z}^d$.

For a Lebesgue measurable subset A of $\mathbb{R}^d$ $|A|$ denotes its Lebesgue measure.

Definition 1.1. (i) For a fixed point $x \in \mathbb{R}^d$, if the sequence $\{\Phi(x + q/2) : q \in Q\}$ has just r nonzero terms, then we say $x \in B_r$.

(ii) If an integer p satisfies both $|B_p| > 0$ and $|B_r| = 0 (r > p)$, then we define the index λ of the underlying FMRA as follows:

$$\lambda = \begin{cases} p, & \text{if } |\mathcal{Q} \cap B_p| > 0; \\ p - 1, & \text{if } |\mathcal{Q} \cap B_p| = 0. \end{cases}$$

The following is the main result in [13].

Theorem 1.2 ([13]). *Let $\{V_j\}_{j \in \mathbb{Z}}$ be an FMRA for $L^2(\mathbb{R}^d)$ with its index λ. Then there exists a wavelet set $\{\psi_l\}_{l=1}^n \subset W_0$ such that $\{T_k \psi_l : 1 \leq l \leq n, k \in \mathbb{Z}^d\}$ is a tight frame for W_0 with the frame bound one if and only if $n \geq \lambda$. In this case, $\{D^j T_k \psi_l : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq l \leq n\}$ is a tight frame for $L^2(\mathbb{R}^d)$ with the frame bound one.*

Theorem 1.2 has a most natural interpretation in the language of the theory of shift-invariant spaces [3, 4, 5, 6, 14, 15]. A closed subspace S of $L^2(\mathbb{R}^d)$ is said to be a *shift-invariant space* if it is invariant under each shift. In this case, $\hat{S}_{||x} := \{\hat{f}_{||x} : f \in S\}$, called the *fiber* of S at x, is a closed subspace of $\ell^2(\mathbb{Z}^d)$ for a.e. $x \in \mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d \equiv [0, 1]^d \equiv [-1/2, 1/2]^d$. The *spectrum* of S is defined to be $\sigma(S) := \{x \in \mathbb{T}^d : \hat{S}_{||x} \neq \{0\}\}$. It is known that any shift-invariant space S has a generating set $F \subset L^2(\mathbb{R}^d)$, which is at most countable, such that $S = \mathcal{S}(F) := \overline{\text{span}}\{T_k f : k \in \mathbb{Z}^d, f \in F\}$. The *length* of S is defined to be $\text{len } S := \min\{\#F : S = \mathcal{S}(F)\}$. Moreover, we have ([3])

(2) $$\text{len } S = \text{ess-sup} \left\{ \dim \hat{S}_{||x} : x \in \mathbb{T}^d \right\}.$$

Now, suppose $L := \text{len } S$ is finite. Then, a careful examination of [4, Theorem 3.3] shows the following fact: For any $F \subset L^2(\mathbb{R}^d)$ with $\#F < L$, $S \neq \mathcal{S}(F)$, and for any $n \geq L$ there exists $F \subset L^2(\mathbb{R}^d)$ with $\#F = n$ such

that $S = \oplus_{f \in F} \mathcal{S}(\{f\})$, and $\{T_k f : k \in \mathbb{Z}^d, f \in F\}$ is a tight frame for S with the bound one. Here $\oplus$ denotes the orthogonal sum. Note that V_0 is a shift-invariant space with $\operatorname{len} V_0 = 1$. It is a standard fact that V_1 is also a shift-invariant space with $\operatorname{len} V_1 \leq 2^d$ [7, 11] (see Section 2), and hence W_0 is also a shift-invariant space with $\operatorname{len} W_0 \leq 2^d$. Therefore, once we show that the index λ and $\operatorname{len} W_0$ coincide (Theorem 2.4), then Theorem 1.2 follows. We also give a more natural construction of a wavelet set in Theorem 2.5 using the fiberization technique of shift-invariant space theory, the newly found matrix characterization theorem of the decomposition of a shift-invariant space of finite length into an orthogonal sum of singly generated shift-invariant spaces (Theorem 2.2) and elementary linear algebra.

2. Main results

In this section, we prove that the index of an FMRA is $\operatorname{len} W_0$, and construct a semi-orthogonal wavelet set from an FMRA using the language of shift-invariant spaces.

Suppose that φ is a scaling function of an FMRA $\{V_j\}_{j \in \mathbb{Z}}$. We assume that $\sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(x + k)|^2 = \chi_{\sigma(V_0)}(x)$ a.e. $x \in \mathbb{T}^d$, which is equivalent to the condition that the shifts of φ form a tight frame with the frame bound one for V_0 [3, 14]. Since $V_0 \subset V_1 := D(V_0)$, φ is refinable, i.e., there exists $m_0 \in L^2(\mathbb{T}^d)$ such that

(3) $$\hat{\varphi}(2x) = m_0(x)\hat{\varphi}(x) \quad \text{a.e. } x \in \mathbb{R}^d.$$

We note that each $k \in \mathbb{Z}^d$ can be written uniquely as $k = 2k' + q$ for some $k' \in \mathbb{Z}^d$ and $q \in Q$. Note also that $DT_y = T_{y/2}D$ for each $y \in \mathbb{R}^d$. Therefore, $\{DT_k\varphi : k \in \mathbb{Z}^d\} = \{T_{k'}DT_q\varphi : k' \in \mathbb{Z}^d, q \in Q\}$. Hence $V_1 = \mathcal{S}(\Pi)$, where $\Pi := \{DT_q\varphi : q \in Q\}$. This shows that $\operatorname{len} V_1 \leq 2^d$.

We recall the known facts about the fibers of the shift-invariant subspaces V_0, W_0 and V_1 in [7, 8, 11] that will be used throughout this article. For $q \in Q$, define $P_q : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ via

$$(P_q a)(k) := \begin{cases} a(k), & \text{if } k \in 2\mathbb{Z}^d + q, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\ell^2(\mathbb{Z}^d) = \oplus_{q \in Q} P_q(\ell^2(\mathbb{Z}^d))$. Define, for each $x \in \mathbb{T}^d$ and each $q \in Q$,

(4) $$a_{x,q} := P_q\left(\left(\hat{\varphi}\left(\frac{x + k}{2}\right)\right)_{k \in \mathbb{Z}^d}\right).$$

Note that $a_{x,q}$ is the 'up-sampled' version of $\hat{\varphi}_{||(x+q)/2}$, i.e.,

$$\begin{cases} a_{x,q}(2k + q) & = \hat{\varphi}_{||(x+q)/2}(k), & \text{if } k \in \mathbb{Z}^d, \\ a_{x,q}(k) & = 0, & \text{if } k \notin 2\mathbb{Z}^d + q. \end{cases}$$

Therefore,

(5) $$\|a_{x,q}\|_{\ell^2(\mathbb{Z}^d)} = \|\hat{\varphi}_{||(x+q)/2}\|_{\ell^2(\mathbb{Z}^d)}.$$

With this notation, we get

$$\hat{\varphi}_{||x} = (\hat{\varphi}(x+k))_{k\in\mathbb{Z}^d} = \left(m_0\left(\frac{x+k}{2}\right) \hat{\varphi}\left(\frac{x+k}{2}\right) \right)_{k\in\mathbb{Z}^d}$$

$$(6) \qquad = \sum_{q\in Q} m_0\left(\frac{x+q}{2}\right) a_{x,q}.$$

Then, we have, by (6) and the orthogonality of $\{a_{x,q} : q \in Q\}$,

$$(7) \qquad \chi_{\sigma(V_0)}(x) = \sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(x+k)|^2 = \sum_{q\in Q} \left| m_0\left(\frac{x+q}{2}\right) \right|^2 \|a_{x,q}\|_{\ell^2(\mathbb{Z}^d)}^2.$$

The basic facts about the fibers of the shift-invariant spaces V_0, W_0 and V_1 are concisely represented in the following lemma whose proof can be found in [7, 8, 11].

Lemma 2.1 ([7, 8, 11]). *For a.e. $x \in \mathbb{T}^d$*

$$(8) \qquad \hat{V}_{0||x} = \mathrm{span}\left\{ \sum_{q\in Q} m_0\left(\frac{x+q}{2}\right) a_{x,q} \right\},$$

$$(9) \qquad \hat{V}_{1||x} = \mathrm{span}\left\{ a_{x,q} : q \in Q \right\}.$$

In particular, for a.e. $x \in \mathbb{T}^d$,

$$(10) \qquad \dim \hat{V}_{1||x} = \#\{a_{x,q} : a_{x,q} \neq 0\}.$$

Now, suppose that $\{\psi_1, \psi_2, \ldots, \psi_n\} \subset V_1$. Let $W_0^{(i)} := \mathcal{S}(\{\psi_i\})$ for $i = 1, 2, \ldots, n$. For the sake of notation let $\psi_0 := \varphi$ and $W_0^{(0)} := V_0$. Then there exist $m_1, m_2, \ldots, m_n \in L^2(\mathbb{T}^d)$ such that

$$(11) \qquad \hat{\psi}_i(2x) := m_i(x)\hat{\varphi}(x), \ 1 \leq i \leq n.$$

Then for each $i = 1, 2, \ldots, n$,

$$(12) \qquad \hat{\psi}_{i||x} = (\hat{\psi}_i(x+k))_{k\in\mathbb{Z}^d} = \sum_{q\in Q} m_i\left(\frac{x+q}{2}\right) a_{x,q}.$$

Let M be the matrix-valued function defined by

$$(13) \qquad M(x) := \begin{pmatrix} m_0\left(x+\frac{q_1}{2}\right) & m_0\left(x+\frac{q_2}{2}\right) & \cdots & m_0\left(x+\frac{q_{2d}}{2}\right) \\ m_1\left(x+\frac{q_1}{2}\right) & m_1\left(x+\frac{q_2}{2}\right) & \cdots & m_1\left(x+\frac{q_{2d}}{2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ m_n\left(x+\frac{q_1}{2}\right) & m_n\left(x+\frac{q_2}{2}\right) & \cdots & m_n\left(x+\frac{q_{2d}}{2}\right) \end{pmatrix}$$

and let $M(x)^{k_1,\ldots,k_l}$ denote the matrix-valued function obtained from $M(x)$ by deleting the $k_1,\ldots,k_l$-th columns of $M(x)$. Note that M satisfies

$$(14) \qquad \begin{pmatrix} \hat{\varphi}_{\|x} \\ \hat{\psi}_{1\,\|x} \\ \vdots \\ \hat{\psi}_{n\,\|x} \end{pmatrix} = M\left(\frac{x}{2}\right) \begin{pmatrix} a_{x,q_1} \\ a_{x,q_2} \\ \vdots \\ a_{x,q_{2^d}} \end{pmatrix}.$$

Let us define, for $0 \le i \le 2^d$,

$$(15) \qquad \Delta_i := \{x \in \mathbb{T}^d : \dim \hat{V}_{1\|x} = i\},$$

where $\dim \hat{V}_{1\|x}$ can be computed by (10). Then $\sigma(V_1) = \biguplus_{i=1}^{2^d} \Delta_i$, where $\uplus$ denotes the disjoint union. And, for each $1 \le l \le 2^d - 1$ and each $k_1, k_2, \ldots, k_l$ such that $1 \le k_1 < k_2 < \cdots < k_l \le 2^d$, define

$$\Delta_{2^d-l}^{k_1,\ldots,k_l} := \left\{ x \in \Delta_{2^d-l} : a_{x,q_{k_1}} = \cdots = a_{x,q_{k_l}} = 0 \right\},$$

which is clearly a Lebesgue measurable set. Then we have, by (10) and the orthogonality of $\{a_{x,q} : q \in Q\}$,

$$\Delta_{2^d-l} = \biguplus_{1 \le k_1 < k_2 < \cdots < k_l \le 2^d} \Delta_{2^d-l}^{k_1,k_2,\ldots,k_l}.$$

The following matrix characterization theorem of the decomposition of a shift-invariant space of finite length into an orthogonal sum of singly generated shift-invariant spaces is used repeatedly in the construction of a wavelet set.

Theorem 2.2. *Let* $\psi_0 := \varphi \in L^2(\mathbb{R}^d)$ *be refinable and its shifts form a tight frame with the frame bound one for* $V_0 := W_0^{(0)} := \mathcal{S}(\{\varphi\})$. *Let* $m_1, \ldots, m_n$ *be in* $L^2(\mathbb{T}^d)$ *and define* $\psi_1, \ldots, \psi_n$ *and* M *as in* (11) *and* (13), *respectively. Then the shifts of* ψ_i *form a tight frame with the frame bound one for* $W_0^{(i)} := \mathcal{S}(\{\psi_i\})$ *for each* $i = 1, 2, \ldots, n$ *and*

$$(16) \qquad V_1 := DV_0 = W_0^{(0)} \oplus W_0^{(1)} \oplus \cdots \oplus W_0^{(n)}$$

if and only if

(1) *For a.e.* $x \in \Delta_{2^d}$, 2^d *rows of* $M(x/2)$ *are orthonormal vectors and the remaining rows are zero vectors in* $\mathbb{C}^{2^d}$;

(2) *For each* $l = 1, 2, \ldots, 2^d - 1$ *and for each choice of* $1 \le k_1 < k_2 < \cdots < k_l \le 2^d$, $2^d - l$ *rows of* $M(x/2)^{k_1,\ldots,k_l}$ *are orthonormal vectors and the remaining rows are zero vectors in* $\mathbb{C}^{2^d-l}$ *for a.e.* $x \in \Delta_{2^d-l}^{k_1,\ldots,k_l}$.

We can check $\psi_j \in L^2(\mathbb{R}^d), 1 \le j \le n$, by (7) and (11). For the proof of Theorem 2.2, we need the following lemma, whose 'only if' part is mentioned in [3].

Lemma 2.3. *Let* A, B *and* C *be shift-invariant subspaces of* $L^2(\mathbb{R}^d)$. *Then* $A = B \oplus C$ *if and only if* $\hat{A}_{\|x} = \hat{B}_{\|x} \oplus \hat{C}_{\|x}$ *for a.e.* $x \in \mathbb{T}^d$.

Proof. ($\Rightarrow$): Let $b \in B$ and $c \in C$. Then, for any $k \in \mathbb{Z}^d$,

$$0 = \langle b, T_k c \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{T}^d} \left\langle \hat{b}_{||x}, \hat{c}_{||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} e^{2\pi i k \cdot x} \, dx.$$

This shows that, for any $b \in B$ and $c \in C$, $\left\langle \hat{b}_{||x}, \hat{c}_{||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} = 0$ for a.e. $x \in \mathbb{T}^d$. Now, let $a \in A$. Then there exist $b \in B$ and $c \in C$ such that $a = b \oplus c$. Hence, for a.e. $x \in \mathbb{T}^d$, $\hat{a}_{||x} = \hat{b}_{||x} \oplus \hat{c}_{||x}$.

($\Leftarrow$): The assumption implies that $B \perp C$ by a calculation similar to the previous one. Since B and C are shift-invariant subspaces, so is $D := B \oplus C$. Now, the fiber spaces of A and D coincide a.e.. Hence $A = D$ by a standard result (see, for example, [4, Proposition 1.5]). □

Proof of Theorem 2.2. ($\Rightarrow$): The tight frame assumption implies that

$$\sum_{k \in \mathbb{Z}} |\hat{\psi}_i(x + k)|^2 = \chi_{\sigma(W_0^{(i)})}(x) \text{ a.e. for each } i = 0, 1, \ldots, n.$$

In particular, the tight frame assumption for $\psi_0 = \varphi$ implies that $||a_{x,q}||^2_{\ell^2(\mathbb{Z}^d)} = 1$ or 0 a.e. by (5) for each $q \in Q$. We have

$$(17) \quad \chi_{\sigma(W_0^{(i)})}(x) = \sum_{q \in Q} \left| m_i \left(\tfrac{x+q}{2} \right) \right|^2 ||a_{x,q}||^2_{\ell^2(\mathbb{Z}^d)} = \sum_{q \in Q, a_{x,q} \neq 0} \left| m_i \left(\tfrac{x+q}{2} \right) \right|^2$$

for a.e. $x \in \mathbb{T}^d$ and for each $i = 0, 1, \ldots, n$ by Equations (6) and (12) and by the orthogonality of $\{a_{x,q} : q \in Q\}$. On the other hand, Equations (16), (6) and (12) imply that

$$(18) \quad 0 = \sum_{q \in Q} m_i \left(\tfrac{x+q}{2} \right) \overline{m_j} \left(\tfrac{x+q}{2} \right) ||a_{x,q}||^2_{\ell^2(\mathbb{Z}^d)} = \sum_{q \in Q, a_{x,q} \neq 0} m_i \left(\tfrac{x+q}{2} \right) \overline{m_j} \left(\tfrac{x+q}{2} \right)$$

for a.e. $x \in \mathbb{T}^d$ and for each $0 \leq i \neq j \leq n$. Now, by Lemma 2.3, (16) is equivalent to the fact that, for a.e. $x \in \mathbb{T}^d$,

$$(19) \quad \hat{V}_{1||x} = (W_0^{(0)})^\wedge_{||x} \oplus (W_0^{(1)})^\wedge_{||x} \oplus \cdots \oplus (W_0^{(n)})^\wedge_{||x}.$$

Therefore, by Lemma 2.1 and (19), we have

$$\dim \hat{V}_{1||x} = \#\{q \in Q : a_{x,q} \neq 0\} = \# \left\{ i \in \{0, 1, \ldots, n\} : x \in \sigma \left(W_0^{(i)} \right) \right\}.$$

Suppose that $x \in \Delta^{k_1,\ldots,k_l}_{2^d-l}$. For a.e. such x, $\dim \hat{V}_{1||x} = 2^d - l$ and $a_{x,q} = 0$ if $q \in \{q_{k_1}, \ldots, q_{k_l}\}$ and $a_{x,q} \neq 0$ if $q \in Q \backslash \{q_{k_1}, \ldots, q_{k_l}\}$. Hence there exist exactly $2^d - l$ number of $i \in \{0, 1, \ldots, n\}$ such that $x \in \sigma(W_0^{(i)})$. Now, Equations (17) and (18) imply that those $2^d - l$ rows of $M(x/2)^{k_1,\ldots,k_l}$ is orthonormal vectors in $\mathbb{C}^{2^d-l}$ and the remaining rows are zero vectors. This proves Condition (2). Condition (1) can be proved similarly.

($\Leftarrow$): Note that Condition (11) implies that $\{\psi_i\}^n_{i=1} \subset V_1$. We show that (19) holds a.e. $x \in \mathbb{T}^d$ and that for $i = 1, \ldots, n$, $||\hat{\psi}_{i||x}||^2_{\ell^2(\mathbb{Z}^d)} = 0$ or 1 a.e., which is equivalent to the condition that the shifts of ψ_i form a tight frame.

If $x \in \mathbb{T}^d \setminus \sigma(V_1)$, then (19) holds trivially. Now let $x \in \Delta_{2^d}$. For a.e. such x, (14) holds. The elements of the right-most vector in the right-hand side of (14) form an orthonormal basis of $\hat{V}_{1||x}$. Condition (1) now implies that 2^d elements of the vector in the left-hand side of (14) also form an orthonormal basis of $\hat{V}_{1||x}$ and the remaining elements are zero vectors of $\ell^2(\mathbb{Z}^d)$. Hence (19) holds a.e. $x \in \Delta_{2^d}$. Moreover, $||\hat{\psi}_{i||x}||^2_{\ell^2(\mathbb{Z}^d)} = 0$ or 1 for a.e. $x \in \Delta_{2^d}$. On the other hand, for a.e. $x \in \Delta_{2^d-l}^{k_1,\ldots,k_l}$, (14) also holds. For those $x \in \Delta_{2^d-l}^{k_1,\ldots,k_l}$, $\{a_{x,q} : q \in Q \setminus \{q_{k_1}, q_{k_2}, \ldots, q_{k_l}\}\}$ is an orthonormal basis for $\hat{V}_{1||x}$, and $a_{x,q} = 0$ if $q \in \{q_{k_1}, q_{k_2}, \ldots, q_{k_l}\}$. Hence, (14) now takes a simpler form:

$$\left((\hat{\psi}_{i||x})_{i=0}^n \right)^t = M(x/2)^{k_1,\ldots,k_l} \left((a_{x,q})_{q \in Q \setminus \{q_{k_1}, q_{k_2}, \ldots, q_{k_l}\}} \right)^t,$$

where t denotes the transpose of a matrix or a vector. Now (19) holds by the same argument for $x \in \Delta_{2^d}$, and $||\hat{\psi}_{i||x}||^2_{\ell^2(\mathbb{Z}^d)} = 0$ or 1 for a.e. $x \in \Delta_{2^d}^{k_1,\ldots,k_l}$. $\square$

We now show that len W_0 is equal to the index of the FMRA in Definition 1.1.

Theorem 2.4. *Let* $\lambda_1 := \operatorname{len} V_1$. *Then*

(20)
$$\operatorname{len} W_0 = \begin{cases} \lambda_1, & \text{if } |\Delta_{\lambda_1} \setminus \sigma(V_0)| > 0, \\ \lambda_1 - 1, & \text{if } |\Delta_{\lambda_1} \setminus \sigma(V_0)| = 0. \end{cases}$$
$$= \text{the index } \lambda \text{ of the FMRA.}$$

Proof. Note that, by (5) and (10), $\lambda_1 = p$ where p is defined as in Definition 1.1(ii). Equations (2) and (15) imply that $\lambda_1 = \max\{i : |\Delta_i| > 0\}$, where $|\Delta_i|$ denotes the Lebesgue measure of Δ_i. Recall that $\hat{V}_{1||x} = \hat{V}_{0||x} \oplus \hat{W}_{0||x}$ a.e. and that $\dim \hat{V}_{0||x} = \chi_{\sigma(V_0)}(x)$. Hence $\dim \hat{W}_{0||x} = \dim \hat{V}_{1||x} - \dim \hat{V}_{0||x} = \dim \hat{V}_{1||x} - \chi_{\sigma(V_0)}(x)$, which implies the first equality of (20). The second equality of (20) follows by the facts that

$$2\mathcal{Q} \cap \mathbb{T}^d = \sigma(V_0)^c, \quad 2B_p \cap \mathbb{T}^d = \Delta_{\lambda_1},$$

where $\mathcal{Q}$ and B_p are defined as in Definition 1.1. $\square$

We now give a concrete construction of such $\{\psi_i : 1 \le i \le \lambda\}$ based on Theorem 2.2, which recovers Theorem 1.2.

Theorem 2.5. *Let* $\{V_j\}_{j \in \mathbb{Z}}$ *be an FMRA for* $L^2(\mathbb{R}^d)$ *and* $\lambda := \operatorname{len} W_0$. *Then there exist* $\{\psi_l\}_{l=1}^n \subset W_0$ *such that* $\{T_k \psi_l : 1 \le l \le n, k \in \mathbb{Z}^d\}$ *is a tight frame for* W_0 *with the frame bound one if and only if* $n \ge \lambda$.

Proof. If $n < \operatorname{len} W_0$, then the shifts of n functions are not even complete in W_0 by the very definition of the length of the shift-invariant space. On the other hand, we now suppose that $n \ge \lambda$. Our goal is to construct a 1-periodic matrix-valued measurable mapping $M(x)$ of size $(\lambda + 1) \times 2^d$ such that its first row of $M(x/2)$ is $(m_0(\frac{x+q_j}{2}))_{j=1}^{2^d}$ and it satisfies Conditions (1) and (2)

of Theorem 2.2. The remaining entries of $M(x/2)$ will be measurable simple
functions in our constructions below. Recall

$$\sigma(V_1) = \Delta_{2^d} \uplus \biguplus_{l=1}^{2^d-1} \biguplus_{1 \leq k_1 < k_2 < \cdots < k_l \leq 2^d} \Delta_{2^d-l}^{k_1,k_2,\ldots,k_l},$$

where some of the sets may be null sets. Recall also that either $a_{x,q} = 0$ or
$\|a_{x,q}\|_{\ell^2(\mathbb{Z}^d)} = 1$ for each $q \in Q$ by (5) since we assumed that the shifts of φ
form a tight frame with the frame bound one for V_0.

Now for any $x \in \mathbb{T}^d \setminus \sigma(V_1)$, let the remaining λ rows be 0. It suffices to
define the remaining λ rows of $M(x/2)$ for a.e. $x \in \sigma(V_1)$. Now we divide into
two cases.

Case I: $\lambda = \lambda_1$, i.e., $|\Delta_{\lambda_1} \setminus \sigma(V_0)| > 0$. First, suppose that $\lambda = \lambda_1 = 2^d$. Then,
$\|a_{x,q}\|_{\ell^2(\mathbb{Z}^d)} = 1$ for each $q \in Q$ and for a.e. $x \in \Delta_{2^d}$ by (9). If $x \in \Delta_{2^d} \setminus \sigma(V_0)$,
then $m(\frac{x+q}{2}) = 0$ a.e. for each $q \in Q$ by (7). Hence the first row of $M(x/2)$
is 0. We define the remaining $\lambda = 2^d$ rows to be those of the $2^d \times 2^d$ identity
matrix. Note that for a.e. x in such set, whose Lebesgue measure is positive,
the last λ rows of $M(x/2)$ are linearly independent. Hence, $M(x/2)$ with the
first row deleted times $(a_{x,q})_{q \in Q}^t$ gives λ linearly independent vectors of $\hat{V}_{1\|x}$.
On the other hand, if $x \in \Delta_{2^d} \cap \sigma(V_0)$, then the norm of the first row is 1 by
(7) since $\|a_{x,q}\|_{\ell^2(\mathbb{Z}^d)}^2 = 1$ for each $q \in Q$. We define the last row of $M(x/2)$ to
be 0 and define the in-between $\lambda - 1$ rows so that the first λ rows of $M(x/2)$
is a unitary matrix. This definition satisfies Condition (1) of Theorem 2.2.

Next, suppose that $\lambda = \lambda_1 = 2^d - l < 2^d$ for some $l \in \{1, 2, \ldots, 2^d - 1\}$.
Then

$$\Delta_{\lambda_1} = \biguplus_{1 \leq k_1 < k_2 < \cdots < k_l \leq 2^d} \Delta_{2^d-l}^{k_1,k_2,\ldots,k_l}.$$

If $x \in \Delta_{2^d-l}^{k_1,k_2,\ldots,k_l}$, then $a_{x,q} = 0$ if $q \in \{q_{k_1}, \ldots, q_{k_l}\}$ and $\|a_{x,q}\|_{\ell^2(\mathbb{Z}^d)} = 1$ if $q \in$
$Q \setminus \{q_{k_1}, \ldots, q_{k_l}\}$. For a.e. $x \in \Delta_{2^d-l}^{k_1,k_2,\ldots,k_l} \setminus \sigma(V_0)$, the first row of $M(x/2)$ is 0.
We define the remaining λ rows of $M(x/2)$ so that the k_i-th column is 0 for each
$i = 1, 2, \ldots, l$, and the deletion of the first row of $M(x/2)^{k_1,\ldots,k_l}$ is the $\lambda_1 \times \lambda_1$
identity matrix. Since $|\Delta_{\lambda_1} \setminus \sigma(V_0)| > 0$, at least one of $\Delta_{2^d-l}^{k_1,k_2,\ldots,k_l} \setminus \sigma(V_0)$ is of
positive Lebesgue measure. For this set the last λ rows of $M(x/2)$ are linearly
independent. Moreover, $M(x/2)$ with the first row deleted times $(a_{x,q})_{q \in Q}^t$
gives λ linearly independent vectors of $\hat{V}_{1\|x}$. Now if $x \in \Delta_{2^d-l}^{k_1,k_2,\ldots,k_l} \cap \sigma(V_0)$,
the norm of the first row of $M(x/2)$ with each the k_i-th element deleted is 1
by (7). We define the last row of $M(x/2)$ to be 0 and the $\lambda - 1$ number of
in-between rows so that all the k_i-th columns of $M(x/2)$ are 0 and the first λ
rows of $M(x/2)^{k_1,\ldots,k_l}$ is a unitary matrix. This definition satisfies Condition
(2) of Theorem 2.2.

We have defined $M(x/2)$ for $x \in \Delta_{\lambda_1}$. In particular, for at least one choice
of $1 \leq k_1 < k_2 < \cdots < k_l \leq 2^d$, $\Delta_{2^d-l}^{k_1,k_2,\ldots,k_l} \setminus \sigma(V_0)$ has positive Lebesgue

measure, and for a.e. x in such set $M(x/2)$ with the first row deleted times $(a_{x,q})_{q \in Q}^t$ gives λ linearly independent vectors of $\hat{V}_{1||x}$.

Now our job is to define $M(x/2)$ on

$$\Delta_{\lambda_1 - l} = \biguplus_{1 \le k_1 < k_2 < \cdots < k_{2^d - (\lambda_1 - l)}} \Delta_{\lambda_1 - l}^{k_1, \ldots, k_{2^d - (\lambda_1 - l)}}$$

for $l = 1, 2, \ldots, \lambda_1 - 1$ so that it satisfies Condition (2) of Theorem 2.2. This can be done by the above idea in the following way. First, define the last l rows of $M(x/2)$ to be 0. We only need to define the second to the $(\lambda_1 - l + 1 = \lambda - l + 1)$-st rows of $M(x/2)$. If $x \in \Delta_{\lambda_1 - l}^{k_1, \ldots, k_{2^d - (\lambda_1 - l)}} \setminus \sigma(V_0)$, then the first row is 0. Define the $k_1, \ldots, k_{2^d - (\lambda_1 - l)}$-th columns of $M(x/2)$ to be 0 and define the remaining rows so that $\lambda - l$ in-between rows with the $k_1, \ldots, k_{2^d - (\lambda_1 - l)}$-th columns deleted form the identity matrix. On the other hand, if $x \in \Delta_{\lambda_1 - l}^{k_1, \ldots, k_{2^d - (\lambda_1 - l)}} \cap \sigma(V_0)$, then the norm of the first row with $k_1, \ldots, k_{2^d - (\lambda_1 - l)}$-th elements deleted is 1. Define the $k_1, \ldots, k_{2^d - (\lambda_1 - l)}$-th columns of $M(x/2)$ to be 0 and define $\lambda - l + 1$-st row to be also 0. Then we define the remaining $\lambda - l - 1$ in-between rows so that the first $\lambda - l$ rows with the $k_1, \ldots, k_{2^d - (\lambda_1 - l)}$-th columns deleted form a unitary matrix.

Case II: $\lambda = \lambda_1 - 1$, i.e., $|\Delta_{\lambda_1} \setminus \sigma(V_0)| = 0$. The proof is identical to Case I except for the fact that when we deal with Δ_{λ_1}, the first row with suitable elements deleted is almost surely of norm 1. Hence the construction yields a subset of Δ_{λ_1} of positive measure such that for a.e. elements of the set the first $\lambda_1 = \lambda + 1$ rows of $M(x/2)$ with suitable columns deleted form a unitary matrix. The construction is identical for $\Delta_{\lambda_1 - l}, 1 \le l \le \lambda_1 - 1$.

It is routine to see that $M(x/2)$ satisfies Conditions (1) and (2) of Theorem 2.2 in either cases for a.e. $x \in \mathbb{T}^d$. Hence we have defined $M(x/2)$ for a.e. $x \in \mathbb{T}^d$. If we scrutinize (13), then we see that we have defined the filters $m_0, m_1, \ldots, m_\lambda$ on $\mathbb{T}^d$. If we extend the filters 1-periodically, then we define a 1-periodic M. Now, finally, define $\psi_1, \psi_2, \ldots, \psi_\lambda$ so that (14) is satisfied. It is easy to see that $\psi_1, \psi_2, \ldots, \psi_\lambda \in L^2(\mathbb{R}^d)$. Then, Theorem 2.2 yields that $V_1 = V_0 \oplus W_0$, and $W_0 = \oplus_{i=1}^\lambda \mathcal{S}(\{\psi_i\})$. This completes the proof in the case that $n = \lambda$.

We now consider the case $n > \lambda$. Note that $|\Delta_\lambda| > 0$. Then there exist positive measurable subsets $\{I_i\}_{i=0}^{n-\lambda}$ such that

$$\Delta_\lambda = \uplus_{i=0}^{n-\lambda} I_i.$$

Define $M(x)$ as in Case I. Let $\tilde{M}(x)$ be the $n \times 2^d$ matrix-valued function defined by

$$(21) \qquad \left(\tilde{M}(x)\right)_{i,j} = \begin{cases} (M(x))_{i,j}, & \text{if } 1 \le i \le \lambda - 1; \\ (M(x))_{\lambda, j}\, \chi_{I_{i-\lambda}}(x), & \text{if } \lambda \le i \le n. \end{cases}$$

Define $\psi_1, \psi_2, \ldots, \psi_n$ so that (14) is satisfied by replacing $M(x)$ by $\tilde{M}(x)$. Using the argument similar to Case I, we can show that $\psi_1, \psi_2, \ldots, \psi_n$ satisfy that $V_1 = V_0 \oplus W_0$, and $W_0 = \oplus_{i=1}^n \mathcal{S}(\{\psi_i\})$. $\qquad\square$

We now give another interesting corollary to Theorem 2.2, which is related to the concept of the unitary extension principle (UEP) of Ron and Shen [15].

Corollary 2.6. *Let $\psi_0 := \varphi \in L^2(\mathbb{R}^d)$ be refinable and its shifts form a tight frame with the frame bound one for $V_0 := W_0^{(0)} := \mathcal{S}(\{\varphi\})$. Suppose there exist $n(\geq 2^d - 1)$ functions $\psi_1, \psi_2, \ldots, \psi_n \in V_1 := D(V_0)$ such that the shifts of each ψ_l form a tight frame with the frame bound one for $W_0^{(l)} := \mathcal{S}(\{\psi_l\})$ and (16) is satisfied. Then there exists a 1-periodic $(n+1) \times 2^d$ matrix-valued mapping M satisfying (14) such that $M(x)^* M(x) = I_{2^d}$ for a.e. $x \in \sigma(V_0)$. In particular, if $\lim_{x \to 0} \hat{\varphi}(x) = 1$, then $\{D^j T_k \psi_l : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq l \leq n\}$ is a tight frame associated with an FMRA defined by the UEP.*

Proof. The hypotheses guarantee a 1-periodic $(n+1) \times 2^d$ matrix-valued mapping $M(x)$ satisfying (14). We change the entries of $M(x)$ so that (14) still holds and $M(x)^* M(x) = I_{2^d}$ for a.e. $x \in \sigma(V_0)$. First we need an obvious observation: If N is an $(n+1) \times 2^d$ matrix such that whose 2^d rows are orthonormal and the remaining rows are 0, then $N^* N = I_{2^d}$. This holds since the matrix obtained by deleting all the 0 rows are $2^d \times 2^d$ unitary matrix and since $N^* N$ is simply the product of the adjoint of the deleted matrix and the deleted matrix.

Suppose $x \in \sigma(V_0)$. Then [7, Lemma 8] shows that $\sigma(V_1) = 2\sigma(V_0) \,(\mathrm{mod}\, 1)$. Hence there exists $k \in \mathbb{Z}^d$ such that $y := 2x + k \in \sigma(V_1)$. Note that if we permute the columns of $M(y/2)$ suitably, then $M(y/2)$ becomes $M(x)$ since $y/2 = x + k/2$. If $y \in \Delta_{2^d}$, then $M(y/2)^* M(y/2) = I_{2^d}$ by Condition (1) of Theorem 2.2 and by the above observation. Hence $M(x)^* M(x) = I_{2^d}$ by the nature of the column permutation. Now suppose that $y \in \Delta_{2^d - l}^{k_1, \ldots, k_l}$. Then $0 = a_{y, q_{k_1}} = \cdots = a_{y, q_{k_l}}$. Hence $k_1, \ldots, k_l$-th columns of $M(y/2)$ have no contribution in (14). Moreover, $M(y/2)^{k_1, \ldots, k_l}$ satisfies Condition (2) of Theorem 2.2. It is easy to see that we can change $k_1, \ldots, k_l$-th columns of $M(y/2)$ so that exactly 2^d rows are orthonormal and the remaining rows are 0. Hence $M(y/2)^* M(y/2) = I_{2^d}$ also by the above observation. Therefore, $M(x)^* M(x) = I_{2^d}$ by the nature of the column permutation. In particular, if $\lim_{x \to 0} \hat{\varphi}(x) = 1$, then $\{D^j T_k \psi_l : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq l \leq n\}$ is a tight frame by [15]. $\qquad\square$

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