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REAL HYPERSURFACES OF TYPE B IN COMPLEX TWO-PLANE GRASSMANNIANS RELATED TO THE REEB VECTOR

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Abstract. In this paper we give a new characterization of real hypersurfaces of type B, that is, a tube over a totally geodesic $\mathbb{Q}P^n$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, where $m = 2n$, with the Reeb vector ξ belonging to the distribution $\mathfrak{D}$, where $\mathfrak{D}$ denotes a subdistribution in the tangent space T_xM such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$ for any point $x \in M$ and $\mathfrak{D}^{\perp} = \text{Span}\{\,\xi_1, \xi_2, \xi_3\,\}$.

0. Introduction

The study of real hypersurfaces in non-flat complex space forms or quaternionic space forms is a classical topic in differential geometry. For instance, there have been many investigations for homogeneous hypersurfaces of type A_1, A_2, B, C, D and E in complex projective space $\mathbb{C}P^m$. They are completely classified by Berndt [2], Cecil and Ryan [5], Kimura [7] and Takagi [10]. Here, explicitly, we mention that A_1: geodesic hyperspheres, A_2: a tube around a totally geodesic complex projective spaces $\mathbb{C}P^k$, B: a tube around a complex quadric Q^{m-1} and can be viewed as a tube around a real projective space $\mathbb{R}P^m$, C: a tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ into $\mathbb{C}P^{2k+1}$ for some $k{\geq}2$, D: a tube around the Plücker embedding into $\mathbb{C}P^9$ of the complex Grassmannian manifold $G_2(\mathbb{C}^5)$ of complex 2-planes in $\mathbb{C}^5$ and E: a tube around the half spin embedding into $\mathbb{C}P^{15}$ of the Hermitian symmetric space $SO(10)/U(5)$.

But until now there were only a few characterizations of homogeneous real hypersurfaces of type B, that is, a tube over a real projective space $\mathbb{R}P^m$ in complex projective space $\mathbb{C}P^m$. Among them, Yano and Kon [11] gave a

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characterization for real hypersurfaces of type B in $\mathbb{C}P^m$ in such a way that $A\phi + \phi A = k\phi$, where k is non-zero constant.

Now let us consider a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper Kähler manifold (See Berndt and Suh [3], [4]). So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces M that $[\,\xi\,] = \mathrm{Span}\,\{\,\xi\,\}$ or $\mathfrak{D}^\perp = \mathrm{Span}\,\{\,\xi_1, \xi_2, \xi_3\,\}$ are invariant under the shape operator A of M.

The almost contact structure vector field ξ mentioned above is defined by $\xi = -JN$, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ and it is said to be a *Reeb* vector field. The almost contact three structure vector fields $\{\,\xi_1, \xi_2, \xi_3\,\}$ are defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$, where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$.

By using two invariant structures for the Reeb vector field ξ and the distribution $\mathfrak{D}^\perp = \mathrm{Span}\,\{\,\xi_1, \xi_2, \xi_3\,\}$, Berndt and the second author [3] have proved the following:

Theorem A. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\,\xi\,]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator A of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*, we say that the Reeb vector field ξ on M is Killing. Moreover, the Reeb vector field ξ is said to be *Hopf* if it is invariant by the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the *Hopf foliation* of M is totally geodesic. By the formulas in Section 2 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

In particular, the second author [8] gave a characterization of type B among Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ when the almost contact 3-structure tensors $\{\,\phi_1, \phi_2, \phi_3\,\}$ commute with the shape operator A on the orthogonal complement of the one dimensional distribution $[\,\xi\,]$. Moreover, he also gave another characterization of real hypersurfaces of type B in $G_2(\mathbb{C}^{m+2})$ in terms of contact hypersurface, that is, $A\phi + \phi A = k\phi$, where k is non-zero constant (See [9]).

On the other hand, it can be easily seen that the Reeb vector ξ for real hypersurfaces of type B in Theorem A belongs to the distribution $\mathfrak{D}$ (See [2]). Then naturally we are able to consider a converse problem. It should be an interesting problem to check that whether a real hypersurface of type B, that is, a tube around a totally geodesic $\mathbb{Q}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$, is only a hypersurface with its Reeb vector ξ belonging to the distribution $\mathfrak{D}$.

From such a view point, we affirmatively answer for this problem. In this paper we give a new characterization of real hypersurfaces of type B in $G_2(\mathbb{C}^{m+2})$ as follows:

Main Theorem. *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [1], [3] and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition, that is, a Cartan decomposition

$$\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R},$$

where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with

$(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Let $p \in G_2(\mathbb{C}^{m+2})$ and W a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that W is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some fundamental formulae which will be used in the proof of our main theorem. Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N.

The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). More explicitly, we can define a tensor field ϕ of type $(1,1)$, a vector field ξ and its dual 1-form η on M by $g(\phi X, Y) = g(JX, Y)$ and $\eta(X) = g(\xi, X)$ for any tangent vector fields X and Y on M. Then they satisfy the following

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ \eta(\phi X) = 0 \text{ and } \eta(\xi) = 1 \tag{2.1}$$

for any tangent vector field X.

Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M in such a way that a tensor field ϕ_ν of type $(1,1)$, a vector field ξ_ν and its dual 1-form η_ν on M defined by $g(\phi_\nu X, Y) = g(J_\nu X, Y)$ and $\eta_\nu(X) = g(\xi_\nu, X)$ for any tangent vector fields X and Y on M. Then they also satisfy the following

$$\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu,\ \phi_\nu\xi_\nu = 0,\ \eta_\nu(\phi_\nu X) = 0 \text{ and } \eta_\nu(\xi_\nu) = 1 \tag{2.2}$$

for any vector field X tangent to M and $\nu = 1, 2, 3$.

Using the above expression (1.2) for the curvature tensor $\bar{R}$, the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)X - g(X,Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &+ \sum_{\nu=1}^{3} \{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\} \\ &+ \sum_{\nu=1}^{3} \{g(\phi_\nu \phi Y, Z)\phi_\nu \phi X - g(\phi_\nu \phi X, Z)\phi_\nu \phi Y\} \\ &- \sum_{\nu=1}^{3} \{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\} \\ &- \sum_{\nu=1}^{3} \{\eta(X)g(\phi_\nu\phi Y, Z) - \eta(Y)g(\phi_\nu\phi X, Z)\}\xi_\nu \\ &+ g(AY, Z)AX - g(AX, Z)AY\,, \end{aligned}$$

and

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X =& \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ &+ \sum_{\nu=1}^{3} \{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\ &+ \sum_{\nu=1}^{3} \{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\ &+ \sum_{\nu=1}^{3} \{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu\,, \end{aligned}$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations (See [8] and [9]):

$$\begin{aligned} &\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},\\ &\phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X),\\ &\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\ &\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}. \end{aligned} \tag{2.3}$$

Now let us note that

$$JX = \phi X + \eta(X)N, \quad JN = -\xi\,,$$

and

$$J_\nu X = \phi_\nu X + \eta_\nu(X)N, \quad J_\nu N = -\xi_\nu\,, \quad \nu = 1, 2, 3$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from these and the formulae (1.1) and (2.3) we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.4}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.5}$$

$$\begin{aligned} (\nabla_X\phi_\nu)Y = &-q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX\\ &- g(AX, Y)\xi_\nu. \end{aligned} \tag{2.6}$$

Summing up these formulae, we find the following

$$\begin{aligned} \nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu)\\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu)\\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX\\ &\quad - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX. \end{aligned} \tag{2.7}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1, 2, 3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.8}$$

On the other hand, using the fact, $A\xi = \alpha\xi$, Berndt and the second author gave the following lemma (See [4]):

Lemma 2.1. *If M is a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, then*

$$\begin{aligned} &\alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y) + 2g(\phi X, Y)\\ &= 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\\ &\quad - 2\eta(X)\eta_\nu(\phi Y)\eta_\nu(\xi) + 2\eta(Y)\eta_\nu(\phi X)\eta_\nu(\xi)\} \end{aligned} \tag{2.9}$$

for all vector fields X and Y on M.

3. Proof of Main Theorem

Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. Now let us denote by the distribution $\mathfrak{D}$ the orthogonal complement of the distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\,\xi_1, \xi_2, \xi_3\,\}$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$ for any point $x \in M$.

In order to prove our Main Theorem in the introduction we give a key proposition as follows:

Proposition 3.1. *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}$, then the distribution $\mathfrak{D}$ is invariant under the shape operator A of M, that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. To prove this it suffices to show that $g(A\mathfrak{D}, \xi_\nu) = 0$, $\nu = 1, 2, 3$. In order to do this, we put

$$\mathfrak{D} = [\,\xi\,] \oplus [\,\phi_1\xi, \phi_2\xi, \phi_3\xi\,] \oplus \mathfrak{D}_0,$$

where the distribution $\mathfrak{D}_0$ is an orthogonal complement of $[\,\xi\,] \oplus [\,\phi_1\xi, \phi_2\xi, \phi_3\xi\,]$ in the distribution $\mathfrak{D}$ of the tangent space T_xM, $x{\in}M$, of M in $G_2(\mathbb{C}^{m+2})$. First, from the assumption $\xi{\in}\mathfrak{D}$ we know $g(A\xi, \xi_\nu) = 0$, $\nu = 1, 2, 3$, because we have assumed that M is Hopf. Next we assert the formula $g(A\phi_i\xi, \xi_\nu) = 0$ for $i,\ \nu = 1, 2, 3$. In fact, by using (2.5) and $\xi{\in}\mathfrak{D}$ we have the following:

$$\begin{aligned} g(A\phi_i\xi, \xi_\nu) &= -g(\phi A\xi_\nu, \xi_i) \\ &= -g(\nabla_{\xi_\nu}\xi, \xi_i) \\ &= g(\xi, \nabla_{\xi_\nu}\xi_i) \\ &= g(\xi, q_{i+2}(\xi_\nu)\xi_{i+1} - q_{i+1}(\xi_\nu)\xi_{i+2} + \phi_i A\xi_\nu) \\ &= g(\xi, \phi_i A\xi_\nu) \\ &= -g(A\phi_i\xi, \xi_\nu), \end{aligned}$$

which gives our assertion (See [6], page 1127). Finally, we consider for the case $X \in \mathfrak{D}_0$. From (2.9) in above Lemma 2.1, we have

$$\begin{aligned} &\alpha A\phi X + \alpha\phi AX - 2A\phi AX + 2\phi X \\ &= 2\sum_{\nu=1}^{3}\Big\{ - \eta_\nu(X)\phi_\nu\xi - \eta_\nu(\phi X)\xi_\nu - \eta_\nu(\xi)\phi_\nu X \\ &\qquad + 2\eta(X)\eta_\nu(\xi)\phi_\nu\xi + 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\Big\} \end{aligned}$$

for any tangent vector field $X \in T_xM$, $x{\in}M$. From now on, we show that $g(AX, \xi_\nu) = 0$ for any $X \in \mathfrak{D}_0$. In order to do this, we restrict $X \in T_xM$, $x{\in}M$ to $X \in \mathfrak{D}_0$ unless otherwise stated. Now by taking ϕ into above equation and using the fact $A\xi = \alpha\xi$ we get

$$\alpha\phi A\phi X - \alpha AX - 2\phi A\phi AX - 2X = 0 \tag{3.1}$$

for any $X \in \mathfrak{D}_0$.

Taking inner product in (3.1) with ξ_μ we have

$$\alpha g(\phi A\phi X, \xi_\mu) - \alpha g(AX, \xi_\mu) - 2g(\phi A\phi AX, \xi_\mu) = 0,$$

that is,

$$\alpha g(AX, \xi_\mu) = \alpha g(\phi A\phi X, \xi_\mu) - 2g(\phi A\phi AX, \xi_\mu) \quad \text{for } X \in \mathfrak{D}_0. \tag{3.2}$$

On the other hand, since $g(\phi A\phi X, \xi_\mu) = g(\nabla_{\phi X}\xi, \xi_\mu) = -g(\xi, \nabla_{\phi X}\xi_\mu)$, we have

$$g(\phi A\phi X, \xi_\mu) = -g(\xi, \phi_\mu A\phi X) = -g(\xi_\mu, \phi A\phi X)$$

by virtue of (2.3) and (2.5). Accordingly, we get

$$g(\phi A\phi X, \xi_\mu) = 0$$

for any $X \in \mathfrak{D}_0$.

Next let us show that $g(\phi A\phi AX, \xi_\mu) = 0$ for any $X \in \mathfrak{D}_0$.

In fact, (2.4) and (2.5) give

$$\begin{aligned} g(\phi A\phi AX, \xi_\mu) &= g(\nabla_{\phi AX}\xi, \xi_\mu) = -g(\xi, \nabla_{\phi AX}\xi_\mu) \\ &= -g(\xi, \phi_\mu A\phi AX) = -g(\xi_\mu, \phi A\phi AX), \end{aligned}$$

which gives our assertion. Thus, from (3.2) we know that

$$\alpha g(AX, \xi_\mu) = 0 \quad \text{for any } X \in \mathfrak{D}_0. \tag{3.3}$$

Then we are able to divide two cases as follows:

Case 1. $\alpha \neq 0$

From (3.3) the conclusion is obvious.

Case 2. $\alpha = 0$

From an assumption, $\alpha = 0$, together with (3.1), we have

$$X = -\phi A\phi AX \quad \text{for any } X \in \mathfrak{D}_0.$$

From this, let us apply the shape operator A. Then it follows that

$$AX = -A\phi A\phi AX \quad \text{for any } X \in \mathfrak{D}_0. \tag{3.4}$$

Taking an inner product of (3.4) and ξ_μ, we have

$$g(AX, \xi_\mu) = -g(A\phi A\phi AX, \xi_\mu) \quad \text{for any } X \in \mathfrak{D}_0. \tag{3.5}$$

On the other hand, we know the following

$$g(A\phi A\phi AX, \xi_\mu) = -g(A\phi AX, \phi A\xi_\mu) = -g(A\phi AX, \nabla_{\xi_\mu}\xi).$$

Then it follows that

$$\begin{aligned} g(A\phi A\phi AX, \xi_\mu) &= -g(A\phi AX, \nabla_{\xi_\mu}\xi) \\ &= g((\nabla_{\xi_\mu}A)\phi AX, \xi) + g(A(\nabla_{\xi_\mu}\phi)AX, \xi) \\ &\quad + g(A\phi(\nabla_{\xi_\mu}A)X, \xi) + g(A\phi A(\nabla_{\xi_\mu}X), \xi), \end{aligned}$$

where we have used $g(A\phi AX, \xi) = 0$. From this, together with $A\xi = 0$, it follows that

$$g(A\phi A\phi AX, \xi_\mu) = g((\nabla_{\xi_\mu}A)\phi AX, \xi). \tag{3.6}$$

On the other hand, by using the equation of Codazzi in Section 2, we have the following:

Lemma 3.2. $g((\nabla_{\xi_\mu} A)\phi AX, \xi) = -g(A\xi_\mu, \phi A\phi AX) - 4g(AX, \xi_\mu)$.

Proof. By the Codazzi equation we know

$$\begin{aligned}
(\nabla_{\xi_\mu} A)\phi AX &= (\nabla_{\phi AX} A)\xi_\mu + \eta(\xi_\mu)\phi^2 AX - \eta(\phi AX)\phi\xi_\mu - 2g(\phi\xi_\mu, \phi AX)\xi \\
&\quad + \sum_{\nu=1}^{3} \Big\{\eta_\nu(\xi_\mu)\phi_\nu\phi AX - \eta_\nu(\phi AX)\phi_\nu\xi_\mu - 2g(\phi_\nu\xi_\mu, \phi AX)\xi_\nu\Big\} \\
&\quad + \sum_{\nu=1}^{3} \Big\{\eta_\nu(\phi\xi_\mu)\phi_\nu\phi^2 AX - \eta_\nu(\phi^2 AX)\phi_\nu\phi\xi_\mu\Big\} \\
&\quad + \sum_{\nu=1}^{3} \Big\{\eta(\xi_\mu)\eta_\nu(\phi^2 AX) - \eta(\phi AX)\eta_\nu(\phi\xi_\mu)\Big\}\xi_\nu \\
&= (\nabla_{\phi AX} A)\xi_\mu - 2g(\xi_\mu, AX)\xi + \phi_\mu\phi AX + \sum_{\nu=1}^{3} g(\phi\xi_\nu, AX)\phi_\nu\xi_\mu \\
&\quad + 2\sum_{\nu=1}^{3} g(\phi\phi_\nu\xi_\mu, AX)\xi_\nu + \sum_{\nu=1}^{3} \eta_\nu(AX)\phi_\nu\phi\xi_\mu.
\end{aligned}$$

From this, taking an inner product with ξ and using the fact that $\phi\phi_\mu\xi = -\xi_\mu$, we have

$$\begin{aligned}
g((\nabla_{\xi_\mu} A)\phi AX, \xi) &= g((\nabla_{\phi AX} A)\xi_\mu, \xi) - 2g(\xi_\mu, AX) + g(\phi_\mu\phi AX, \xi) \\
&\quad + \sum_{\nu=1}^{3} g(\phi\xi_\nu, AX)g(\phi_\nu\xi_\mu, \xi) + 2\sum_{\nu=1}^{3} g(\phi\phi_\nu\xi_\mu, AX)g(\xi_\nu, \xi) \\
&\quad + \sum_{\nu=1}^{3} \eta_\nu(AX)g(\phi_\nu\phi\xi_\mu, \xi) \\
&= g((\nabla_{\phi AX} A)\xi_\mu, \xi) - 4g(AX, \xi_\mu).
\end{aligned}$$

On the other hand, since $g(A\xi_\mu, \xi) = g(\xi_\mu, A\xi) = \alpha g(\xi_\mu, \xi)$ and $\alpha = 0$, we have

$$\begin{aligned}
g((\nabla_{\phi AX} A)\xi_\mu, \xi) &= -g(A(\nabla_{\phi AX}\xi_\mu), \xi) - g(A\xi_\mu, \phi A\phi AX) \\
&= -\alpha g(\nabla_{\phi AX}\xi_\mu, \xi) - g(A\xi_\mu, \phi A\phi AX) \\
&= -g(A\xi_\mu, \phi A\phi AX).
\end{aligned}$$

Therefore we have

$$g((\nabla_{\xi_\mu} A)\phi AX, \xi) = -g(A\xi_\mu, \phi A\phi AX) - 4g(AX, \xi_\mu)$$

for any $X \in \mathfrak{D}_0$. □

Consequently, from (3.6), Lemma 3.2, the symmetry of the shape operator A, and together with the fact that $A\xi = 0$, we get

$$g(A\phi A\phi AX, \xi_\mu) = -2g(AX, \xi_\mu). \tag{3.7}$$

From (3.5) and (3.7) for $\alpha = 0$, we have $g(AX, \xi_\mu) = 0$ for any tangent vector field X belongs to the distribution $\mathfrak{D}_0$.

Then summing up all situation mentioned above we conclude that the distribution $\mathfrak{D}$ is invariant under the shape operator of M if the Reeb vector ξ belong to the distribution $\mathfrak{D}$. □

Then by Proposition 3.1 and Theorem A we know that a Hopf real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with the Reeb vector ξ belongs to the distribution $\mathfrak{D}$ is congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ or a tube over a totally geodesic $\mathbb{Q}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+1})$. But in [3] it was known that the Reeb vector ξ of type A in the first case belongs to the distribution $\mathfrak{D}^\perp$. From this we complete the proof of our main theorem in the introduction.

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References

[1] J. Berndt, *Riemannian geometry of complex two-plane Grassmannians*, Rend. Sem. Mat. Univ. Politec. Torino **55** (1997), no. 1, 19–83.
[2] ———, *Real hypersurfaces with constant principal curvatures in complex space forms*, Proceedings of the Tenth International Workshop on Differential Geometry, 1–12, Kyungpook Nat. Univ., Taegu, 2006.
[3] J. Berndt and Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians*, Monatsh. Math. **127** (1999), no. 1, 1–14.
[4] ———, *Real hypersurfaces with isometric Reeb flow in complex two-plane Grassmannians*, Monatsh. Math. **137** (2002), no. 2, 87–98.
[5] T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), no. 2, 481–499.
[6] I. Jeong and Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with Lie ξ-parallel normal Jacobi operator*, J. Korean Math. Soc. **45** (2008), no. 4, 1113–1133.
[7] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), no. 1, 137–149.
[8] Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with commuting shape operator*, Bull. Austral. Math. Soc. **68** (2003), no. 3, 379–393.
[9] ———, *Real hypersurfaces of type B in complex two-plane Grassmannians*, Monatsh. Math. **147** (2006), no. 4, 337–355.
[10] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.
[11] K. Yano and M. Kon, *CR Submanifolds of Kaehlerian and Sasakian Manifolds*, Progress in Mathematics, 30. Birkhauser, Boston, Mass., 1983.

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