Bull. Korean Math. Soc. **47** (2010), No. 3, pp. 541–549
DOI 10.4134/BKMS.2010.47.3.541

# SCALAR CURVATURE OF CONTACT $CR$-SUBMANIFOLDS IN AN ODD-DIMENSIONAL UNIT SPHERE

Hyang Sook Kim* and Jin Suk Pak

Abstract. In this paper we derive an integral formula on an $(n+1)$-dimensional, compact, minimal contact $CR$-submanifold $M$ of $(n-1)$ contact $CR$-dimension immersed in a unit $(2m+1)$-sphere $S^{2m+1}$. Using this integral formula, we give a sufficient condition concerning with the scalar curvature of $M$ in order that such a submanifold $M$ is to be a generalized Clifford torus.

## 1. Introduction

Let $S^{2m+1}$ be a $(2m+1)$-dimensional unit sphere, that is,

$$S^{2m+1} = \{z \in \mathbb{C}^{m+1} \; : \; \|z\| = 1\}.$$

For any point $z \in S^{2m+1}$ we put $\xi = Jz$, where $J$ denotes the almost complex structure of $\mathbb{C}^{m+1}$. We consider the orthogonal projection $\pi : T_z\mathbb{C}^{m+1} \to T_zS^{2m+1}$. Putting $\phi = \pi \circ J$, we can see that the set $(\phi, \xi, \eta, g)$ is a Sasakian structure on $S^{2m+1}$, where $\eta$ is a 1-form dual to $\xi$ and $g$ the standard metric tensor field on $S^{2m+1}$. So $S^{2m+1}$ can be considered as a Sasakian manifold of constant $\phi$-sectional curvature 1, that is, of constant curvature 1 (cf. [1, 2, 12]).

Let $M$ be an $(n+1)$-dimensional submanifold tangent to the structure vector field $\xi$ of $S^{2m+1}$ and denote by $\mathcal{D}_x$ the $\phi$-invariant subspace $\phi T_xM \cap T_xM$ of the tangent space $T_xM$ of $M$ at $x \in M$. Then $\xi$ cannot be contained in $\mathcal{D}_x$ at any point $x \in M$.

When the $\phi$-invariant subspace $\mathcal{D}_x$ has constant dimension for any $x \in M$, $M$ is called a *contact $CR$-submanifold* and the constant is called *contact $CR$-dimension* of $M$ (cf. [5, 6, 9, 10]).

On an $(n+1)$-dimensional contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension, there is a non-zero vector $U$ which is orthogonal to $\xi$ and contained

Received December 4, 2008.
2000 *Mathematics Subject Classification.* 53C40, 53C15.
*Key words and phrases.* Sasakian manifold, odd-dimensional unit sphere, contact $CR$-submanifold, scalar curvature.
* This work was supported by the 2008 Inje University research grant.

in the complementary orthogonal subspace $\mathcal{D}_x^{\perp}$ of $\mathcal{D}_x$ in $T_xM$. In this case $N =: \phi U$ must be normal to $M$ and thus $M$ can be dealt with a contact $CR$-submanifold in the sense of Yano-Kon ([12]).

In this paper we shall study $(n+1)$-dimensional contact $CR$-submanifolds of $(n-1)$ contact $CR$-dimension immersed in $S^{2m+1}$ and prove the following theorem as a Sasakian version corresponding to the results provided in [3] and [7].

**Theorem.** *Let $M$ be an $(n+1)(\geq 3)$-dimensional compact, minimal, contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension in $S^{2m+1}$. If the scalar curvature of $M$ is greater or equal to $n^2-1$, then*

$$M = S^{2t+1}(r_1) \times S^{2s+1}(r_2), \quad t+s = \frac{n+1}{2} - 1,$$

*where $r_1^2 + r_2^2 = 1$.*

*Remark.* The above main theorem was provided in [9] under the condition that the distinguished normal vector field $N$ is parallel with respect to the normal connection $\nabla^{\perp}$. For the complex and the quaternionic analogues corresponding to the above theorem, see [3] and [7], respectively.

Manifolds, submanifolds, geometric objects and mappings we discuss in this paper will be assumed to be connected, differentiable and of class $C^{\infty}$.

## 2. Fundamental properties of contact $CR$-submanifolds

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure $(\phi, \xi, \eta, g)$. Then, by definition, it follows that

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{2.1}$$

for any vector fields $X$, $Y$ tangent to $\overline{M}$.

Let $M$ be a contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension in $\overline{M}$, where $n-1$ must be even. Then, as was already mentioned in §1, the structure vector $\xi$ is always contained in $\mathcal{D}_x^{\perp}$ and $\phi\mathcal{D}_x^{\perp} \subset T_xM^{\perp}$ at any point $x \in M$, where $T_xM^{\perp}$ denotes the normal space of $M$ at $x \in M$. Further, by definition $\dim\mathcal{D}_x^{\perp} = 2$ at any point $x \in M$, and so there exists a unit vector field $U$ contained in $\mathcal{D}^{\perp}$ which is orthogonal to $\xi$. Since $\phi\mathcal{D}_x^{\perp} \subset T_xM^{\perp}$ at any point $x \in M$, $\phi U$ is a unit normal vector field to $M$, which will be denoted by $N$, that is,

$$N := \phi U. \tag{2.2}$$

Moreover, it is clear that $\phi TM \subset TM \oplus \mathrm{Span}\{N\}$. Hence we have, for any tangent vector field $X$ and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\ldots,p}$ $(N_1 := N,\ p := 2m-n)$ of normal vectors to $M$, the following decomposition in tangential and normal components:

$$\phi X = FX + u(X)N, \tag{2.3}$$

$$\phi N_\alpha = PN_\alpha, \quad \alpha = 2, \dots, p. \tag{2.4}$$

It is easily shown that $F$ is a skew-symmetric linear endomorphism acting on $T_xM$. Since the structure vector field $\xi$ is tangent to $M$, (2.1) and (2.3) imply

$$F\xi = 0,\ FU = 0,\ g(U, X) = u(X),\ u(\xi) = g(U, \xi) = 0,\ u(U) = 1. \tag{2.5}$$

Next, applying $\phi$ to (2.3) and using (2.1), (2.3) and (2.5), we also have

$$F^2X = -X + \eta(X)\xi + u(X)U, \quad u(FX) = 0. \tag{2.6}$$

On the other hand, it is clear from (2.1), (2.2) and (2.5) that

$$\phi N = -U, \tag{2.7}$$

which and (2.4) yield the existence of a local orthonormal basis $\{N,\ N_a,\ N_{a^*}\}_{a=1,\dots,q}$ of normal vectors to $M$ such that

$$N_{a^*} := \phi N_a, \quad a = 1, \dots, q := (p-1)/2. \tag{2.8}$$

We denote by $\overline{\nabla}$ and $\nabla$ the Levi-Civita connection on $\overline{M}$ and $M$, respectively, and by $\nabla^\perp$ the normal connection induced from $\overline{\nabla}$ on the normal bundle $TM^\perp$ of $M$. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_XY = \nabla_XY + h(X, Y), \tag{2.9}$$

$$(2.10)_1 \qquad \overline{\nabla}_XN = -AX + \nabla^\perp_XN = -AX + \sum_{a=1}^{q}\{s_a(X)N_a + s_{a^*}(X)N_{a^*}\},$$

$$(2.10)_2 \qquad \overline{\nabla}_XN_a = -A_aX - s_a(X)N + \sum_{b=1}^{q}\{s_{ab}(X)N_b + s_{ab^*}(X)N_{b^*}\},$$

$$(2.10)_3 \qquad \overline{\nabla}_XN_{a^*} = -A_{a^*}X - s_{a^*}(X)N + \sum_{b=1}^{q}\{s_{a^*b}(X)N_b + s_{a^*b^*}(X)N_{b^*}\}$$

for any vector fields $X, Y$ tangent to $M$, where $s$'s are coefficients of the normal connection $\nabla^\perp$. Here $h$ denotes the second fundamental form and $A, A_a, A_{a^*}$ the shape operators corresponding to the normals $N, N_a, N_{a^*}$, respectively. They are related by

$$h(X, Y) = g(AX, Y)N + \sum_{a=1}^{q}\{g(A_aX, Y)N_a + g(A_{a^*}X, Y)N_{a^*}\}. \tag{2.11}$$

From now on we specialize to the case of an ambient Sasakian manifold $\overline{M}$, that is,

$$\overline{\nabla}_X\xi = \phi X, \tag{2.12}$$

$$(\overline{\nabla}_X\phi)Y = -g(X, Y)\xi + \eta(Y)X. \tag{2.13}$$

Since $\xi$ is tangent to $M$, from (2.1), (2.3), (2.7), (2.8), $(2.10)_2$, $(2.10)_3$ and (2.13), we can easily verify that

$$A_aX = -FA_{a^*}X + s_{a^*}(X)U, \quad A_{a^*}X = FA_aX - s_a(X)U, \tag{2.14}$$

$$s_a(X) = -u(A_{a^*}X), \quad s_{a^*}(X) = u(A_aX), \quad a = 1, \dots, q. \tag{2.15}$$

Since $F$ is skew-symmetric, (2.14) implies

$$g((FA_a + A_aF)X, Y) = s_a(X)u(Y) - s_a(Y)u(X), \tag{$2.16_1$}$$

$$g((FA_{a^*} + A_{a^*}F)X, Y) = s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X). \tag{$2.16_2$}$$

On the other hand, since $F\mathcal{D}_x = \mathcal{D}_x$ at each point $x \in M$, we take an orthonormal basis $\{e_i\}_{i=1,\dots,n+1}$ of tangent vectors to $M$ such that

$$e_{l+1} := Fe_1, \dots, e_{2l} := Fe_l, \; e_n := U, \; e_{n+1} := \xi, \tag{2.17}$$

where we have put $l = (n-1)/2$. Replacing $X$ by $Fe_i$ in the first equation of (2.15) and using (2.5), we have

$$s_a(Fe_i) = -g(A_{a^*}Fe_i, U),$$

which together with (2.5) and $(2.16)_2$ yields

$$s_a(Fe_i) = -s_{a^*}(e_i), \quad i = 1, \dots, l.$$

Similarly, replacing $X$ by $Fe_i$ in the second equation of (2.15) and using (2.5) and $(2.16)_1$, we have

$$s_a(Fe_i) = -s_{a^*}(e_i), \quad s_{a^*}(Fe_i) = s_a(e_i), \quad i = 1, \dots, l. \tag{2.18}$$

Differentiating (2.3) and (2.7) covariantly along $M$ and comparing the tangential with normal parts, we have

$$(\nabla_Y F)X = -g(Y, X)\xi + \eta(X)Y - g(AY, X)U + u(X)AY, \tag{2.19}$$

$$(\nabla_Y u)X = g(FAY, X), \tag{2.20}$$

$$\nabla_X U = FAX \tag{2.21}$$

with the aid of (2.3), (2.8), (2.9), $(2.10)_1$, (2.11) and (2.13). On the other hand, since $\xi$ is tangent to $M$, from (2.9) and (2.12), it follows that

$$\phi X = \overline{\nabla}_X \xi = \nabla_X \xi + h(X, \xi),$$

which together with (2.3) and (2.11) gives

$$\nabla_X \xi = FX, \tag{2.22}$$

$$g(A\xi, X) = u(X), \quad \text{i.e.,} \quad A\xi = U, \tag{2.23}$$

$$A_a\xi = 0, \quad A_{a^*}\xi = 0, \quad a = 1, \dots, q. \tag{2.24}$$

If the ambient manifold $\overline{M}$ is a $(2m+1)$-dimensional unit sphere $S^{2m+1}$ as a Sasakian manifold of constant curvature 1, then its curvature tensor $\overline{R}$ satisfies

$$\overline{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y$$

for any vector fields $X, Y, Z$ tangent to $\overline{M}$. Therefore, by means of the equation of Gauss, we can easily see that the Ricci tensor $\mathrm{Ric}(Y, Z)$ has the form

$$\begin{aligned}\mathrm{Ric}(Y,Z) = {} & ng(Y,Z) + (\mathrm{tr}A)g(AY,Z) - g(A^2Y,Z) \\ & + \sum_{a=1}^{q}\{(\mathrm{tr}A_a)g(A_aY,Z) + (\mathrm{tr}A_{a^*})g(A_{a^*}Y,Z) \\ & - g(A_a^2Y,Z) - g(A_{a^*}^2Y,Z)\}\end{aligned} \tag{2.25}$$

and consequently the scalar curvature $\rho$ is given by

$$\begin{aligned}\rho = {} & n(n+1) + (\mathrm{tr}A)^2 - \mathrm{tr}A^2 \\ & + \sum_{a=1}^{q}\{(\mathrm{tr}A_a)^2 + (\mathrm{tr}A_{a^*})^2 - \mathrm{tr}A_a^2 - \mathrm{tr}A_{a^*}^2\}.\end{aligned} \tag{2.26}$$

Moreover, from the equation of Codazzi, we also have

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = {} & \sum_{a=1}^{q}\{s_a(X)A_aY - s_a(Y)A_aX \\ & + s_{a^*}(X)A_{a^*}Y - s_{a^*}(Y)A_{a^*}X\}\end{aligned} \tag{2.27}$$

for any vector fields $X, Y$ tangent to $M$ (cf. [1, 2, 12]).

## 3. An integral formula on the compact contact $CR$-submanifold

Let $M$ be an $(n+1)$-dimensional contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension immersed in a $(2m+1)$-dimensional unit sphere $S^{2m+1}$.

We now put

$$T := \nabla_U U + (\mathrm{div}U)U$$

and take the same orthonormal basis $\{e_i\}_{i=1,\dots,n+1}$ of tangent vectors to $M$ as given in (2.17). Then it follows from (2.21) that

$$T = FAU \tag{3.1}$$

since $\mathrm{div}U = \sum_{i=1}^{n+1} g(e_i, \nabla_{e_i}U) = \mathrm{tr}(FA) = 0$.

From now on, for later use we shall compute $\mathrm{div}T = \sum_{i=1}^{n+1} g(e_i, \nabla_{e_i}T)$ (for a general formula of $\mathrm{div}T$, see [11]).

Differentiating (3.1) covariantly and using (2.5), (2.19), (2.21) and (2.23), we have

$$\begin{aligned}\nabla_X T = {} & -g(X,AU)\xi + X - g(A^2U,X)U + u(AU)AX \\ & + FAFAX + F(\nabla_X A)U,\end{aligned} \tag{3.2}$$

from which, taking account of (2.5), (2.6) and (2.23), it follows that

$$
\begin{aligned}
\operatorname{div}T = n - u(A^2U) + (\operatorname{tr}A)u(AU) + \sum_{i=1}^{n+1} g(FAFAe_i, e_i) \\
- \sum_{i=1}^{l} g((\nabla_{e_i}A)Fe_i - (\nabla_{Fe_i}A)e_i, U).
\end{aligned}
\tag{3.3}
$$

On the other hand, using (2.5), (2.6), (2.15), (2.18) and (2.27), we can easily obtain that

$$
\begin{aligned}
&\sum_{i=1}^{l} g((\nabla_{e_i}A)Fe_i - (\nabla_{Fe_i}A)e_i, U) \\
&= \sum_{i=1}^{l}\sum_{a=1}^{q}\{s_a(e_i)^2 + s_a(Fe_i)^2 + s_{a^*}(e_i)^2 + s_{a^*}(Fe_i)^2\}
\end{aligned}
\tag{3.4}
$$

because of $2l = n - 1$. Inserting (3.4) back into (3.3), the equation (3.3) turns out to be

$$
\begin{aligned}
\operatorname{div}T = n + (\operatorname{tr}A)u(AU) + \sum_{i=1}^{n+1} g(FAFAe_i, e_i) - u(A^2U) \\
- \sum_{i=1}^{l}\sum_{a=1}^{q}\{s_a(e_i)^2 + s_a(Fe_i)^2 + s_{a^*}(e_i)^2 + s_{a^*}(Fe_i)^2\}.
\end{aligned}
\tag{3.5}
$$

On the other hand, using (2.5), (2.6) and (2.23), we can easily verify that

$$
\sum_{i=1}^{n+1} g(FAFAe_i, e_i) = \frac{1}{2}\|FA - AF\|^2 - \operatorname{tr}A^2 + u(A^2U) + 1,
$$

which together with (3.5) implies

$$
\begin{aligned}
\operatorname{div}T = n + 1 + \frac{1}{2}\|FA - AF\|^2 + (\operatorname{tr}A)u(AU) - \operatorname{tr}A^2 \\
- \sum_{i=1}^{l}\sum_{a=1}^{q}\{s_a(e_i)^2 + s_a(Fe_i)^2 + s_{a^*}(e_i)^2 + s_{a^*}(Fe_i)^2\}.
\end{aligned}
\tag{3.6}
$$

Moreover, combining (2.26) with (3.6), we have

$$
\begin{aligned}
\operatorname{div}T = &\frac{1}{2}\|FA - AF\|^2 + (\operatorname{tr}A)u(AU) - (\operatorname{tr}A)^2 \\
&+ \rho - (n^2 - 1) - \sum_{a=1}^{q}\{(\operatorname{tr}A_a)^2 + (\operatorname{tr}A_{a^*})^2\} + \sum_{a=1}^{q}(\operatorname{tr}A_a^2 + \operatorname{tr}A_{a^*}^2) \\
&- \sum_{i=1}^{l}\sum_{a=1}^{q}\{s_a(e_i)^2 + s_a(Fe_i)^2 + s_{a^*}(e_i)^2 + s_{a^*}(Fe_i)^2\}.
\end{aligned}
$$

Thus we have:

**Lemma 3.1.** *Let $M$ be an $(n+1)$-dimensional compact contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension immersed in $S^{2m+1}$. Then the following equality is valid*:

$$\begin{aligned}\int_M [\frac{1}{2}\|FA - AF\|^2 + \rho - (n^2-1) + (\mathrm{tr}A)u(AU) - (\mathrm{tr}A)^2 \\ - \sum_{a=1}^{q}\{(\mathrm{tr}A_a)^2 + (\mathrm{tr}A_{a^*})^2\} + \sum_{a=1}^{q}(\mathrm{tr}A_a^2 + \mathrm{tr}A_{a^*}^2) \\ - \sum_{i=1}^{l}\sum_{a=1}^{q}\{s_a(e_i)^2 + s_a(Fe_i)^2 + s_{a^*}(e_i)^2 + s_{a^*}(Fe_i)^2\}] * 1 = 0.\end{aligned} \tag{3.7}$$

## 4. The proof of main theorem

In order to prove the main theorem stated in §1, we prepare:

**Lemma 4.1.** *Let $M$ be an $(n+1)(\geq 3)$-dimensional compact, minimal, contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension in $S^{2m+1}$. If the scalar curvature of $M$ is greater or equal to $n^2-1$, then*

$$FA - AF = 0 \tag{4.1}$$

*and the distinguished normal vector field $N$ is parallel with respect to the normal connection $\nabla^\perp$. Moreover, we have*

$$A_a = 0, \quad A_{a^*} = 0, \quad a = 1, \ldots, q. \tag{4.2}$$

*Proof.* We first notice that (2.15) and (2.24) yield

$$\sum_{i=1}^{l}\{s_a(e_i)^2 + s_a(Fe_i)^2\} = u(A_{a^*}^2U) - u(A_{a^*}U)^2,$$

$$\sum_{i=1}^{l}\{s_{a^*}(e_i)^2 + s_{a^*}(Fe_i)^2\} = u(A_a^2U) - u(A_aU)^2.$$

Inserting these equations back into (3.7) and taking account of (2.24), we have

$$\begin{aligned}\int_M [\frac{1}{2}\|FA - AF\|^2 + \rho - (n^2-1) + (\mathrm{tr}A)u(AU) - (\mathrm{tr}A)^2 \\ - \sum_{a=1}^{q}\{(\mathrm{tr}A_a)^2 + (\mathrm{tr}A_{a^*})^2\} + \sum_{a=1}^{q}\{u(A_aU)^2 + u(A_{a^*}U)^2\} \\ + \sum_{a=1}^{q}\sum_{i=1}^{l}\{g(A_a^2e_i, e_i) + g(A_a^2Fe_i, Fe_i) + g(A_{a^*}^2e_i, e_i) \\ + g(A_{a^*}^2Fe_i, Fe_i)\}] * 1 = 0.\end{aligned} \tag{4.3}$$

If $\rho$ is greater or equal to $n^2-1$, our assumptions yield (4.1) and

$$\begin{aligned} &A_a e_i = A_a F e_i = 0, \quad A_{a^*} e_i = A_{a^*} F e_i = 0, \\ &u(A_a U) = 0, \quad u(A_{a^*} U) = 0, \quad a = 1, \dots, q, \quad i = 1, \dots, l, \end{aligned} \tag{4.4}$$

which and (2.15) imply

$$\begin{aligned} &s_a(e_i) = s_a(Fe_i) = 0, \quad s_{a^*}(e_i) = s_{a^*}(Fe_i) = 0, \\ &s_a(U) = 0, \quad s_{a^*}(U) = 0, \quad a = 1, \dots, q, \quad i = 1, \dots, l. \end{aligned}$$

Since $s_a(\xi) = s_{a^*}(\xi) = 0$ because of (2.24), we have $s_a = s_{a^*} = 0$ ($a = 1, \dots, q$) which means that the distinguished normal vector field $N$ is parallel with respect to the normal connection by means of $(2.10)_1$. Also it is clear from (2.15) that $A_a U = A_{a^*} U = 0$, which combined with (2.24) and (4.4) implies (4.2). □

*Proof of main theorem.* By means of Lemma 4.1, for the submanifold $M$ given in the main theorem, we can easily see that its first normal space is contained in Span$\{N\}$ which is invariant under parallel translation with respect to the normal connection. Thus we may apply Erbacher's reduction theorem ([4]) and so we can see that there exists an $(n+2)$-dimensional totally geodesic unit sphere $S^{n+2}$ such that $M \subset S^{n+2}$. Here we note that $(n+2)$ is odd. Moreover, since the tangent space $T_x S^{n+2}$ of the totally geodesic submanifold $S^{n+2}$ at $x \in M$ is $T_x M \oplus \mathrm{Span}\{N\}$, $S^{n+2}$ is an invariant submanifold of $S^{2m+1}$ with respect to $\phi$ (for definition, see [1, 12]) because of (2.2) and (2.3). Hence the submanifold $M$ can be regarded as a real hypersurface of $S^{n+2}$ which is a totally geodesic invariant submanifold of $S^{2m+1}$.

Tentatively we denote $S^{n+2}$ by $M'$, and by $i_1$ the immersion of $M$ into $M'$ and $i_2$ the totally geodesic immersion of $M'$ onto $S^{2m+1}$. Then, from the Gauss formula (2.9), it follows that

$$\nabla'_{i_1 X} i_1 Y = i_1 \nabla_X Y + h'(X,Y) = i_1 \nabla_X Y + g(A'X, Y)N', \tag{4.5}$$

where $h'$ is the second fundamental form of $M$ in $M'$ and $A'$ is the corresponding shape operator to a unit normal vector field $N'$ to $M$ in $M'$. Since $i = i_2 \circ i_1$, making use of (4.5), we have

$$\begin{aligned} \overline{\nabla}_{(i_2 \circ i_1)X}(i_2 \circ i_1)Y &= i_2(\nabla'_{i_1 X} i_1 Y) \\ &= i_2(i_1 \nabla_X Y + g(A'X, Y)N'), \end{aligned} \tag{4.6}$$

because $M'$ is totally geodesic in $S^{2m+1}$. Comparing (2.9) with (4.6), we easily see that

$$N = i_2 N', \quad A = A'. \tag{4.7}$$

Since $M'$ is an invariant submanifold of $S^{2m+1}$, for any $X' \in TM'$,

$$\phi i_2 X' = i_2 \phi' X' \tag{4.8}$$

is valid, where $\phi'$ is the induced Sasakian structure of $M' = S^{n+2}$. Thus it follows from (2.3), (4.7) and (4.8) that

$$\begin{aligned}\phi i X &= \phi(i_2 \circ i_1)X = i_2\phi' i_1 X = i_2(i_1 F'X + u'(X)N') \\ &= iF'X + u'(X)i_2N' = iF'X + u'(X)N.\end{aligned}$$

Comparing this equation with (2.3), we have $F = F'$ and $u = u'$. By means of Lemma 4.1, it is clear that $M$ is a real hypersurface of $S^{n+2}$ which satisfies $F'A' = A'F'$. Thus, applying a theorem due to Kon ([8]), we may complete the proof of our main theorem. □

## References


[1] A. Bejancu, *Geometry of CR-Submanifolds*, D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, Tokyo, 1886.

[2] B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker Inc., New York, 1973.

[3] M. Djorić and M. Okumura, *The scalar curvature of CR submanifolds of maximal CR dimension of complex projective space*, Monatsh. Math. **154** (2008), no. 1, 11–17.

[4] J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differential Geometry **5** (1971), 333–340.

[5] H. S. Kim and J. S. Pak, *Certain contact CR-submanifolds of an odd-dimensional unit sphere*, Bull. Korean Math. Soc. **44** (2007), no. 1, 109–116.

[6] ———, *Certain class of contact CR-submanifolds of an odd-dimensional unit sphere*, to appear in Taiwanese Math. J.

[7] ———, *Scalar curvature of QR-submanifolds with maximal QR-dimension in a quaternionic projective space*, submitted.

[8] M. Kon, *On hypersurfaces immersed in* $S^{2n+1}$, Ann. Fac. Sci. Univ. Nat. Zaïre (Kinshasa) Sect. Math.-Phys. **4** (1978), no. 1, 1–24.

[9] J.-H. Kwon and J. S. Pak, *On some contact CR-submanifolds of an odd-dimensional unit sphere*, Soochow J. Math. **26** (2000), no. 4, 427–439.

[10] J. S. Pak, J.-H. Kwon, H. S. Kim, and Y.-M. Kim, *Contact CR-submanifolds of an odd-dimensional unit sphere*, Geom. Dedicata **114** (2005), 1–11.

[11] K. Yano, *Integral Formulas in Riemannian Geometry*, Pure and Applied Mathematics, No. 1 Marcel Dekker, Inc., New York 1970.

[12] K. Yano and M. Kon, *CR Submanifolds of Kaehlerian and Sasakian Manifolds*, Progress in Mathematics, 30. Birkhauser, Boston, Mass., 1983.



Hyang Sook Kim
Department of Computational Mathematics
School of Computer Aided Science
Institute of Basic Science
Inje University
Kimhae 621-749, Korea
*E-mail address*: mathkim@inje.ac.kr

Jin Suk Pak
Department of Mathematics Education
Kyungpook National University
Daegu 702-701, Korea
*E-mail address*: jspak@knu.ac.kr