Bull. Korean Math. Soc. **47** (2010), No. 3, pp. 527–540
DOI 10.4134/BKMS.2010.47.3.527

A WEIGHTED COMPOSITION OPERATOR ON THE LOGARITHMIC BLOCH SPACE

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Abstract. We characterize the boundedness and compactness of the weighted composition operator on the logarithmic Bloch space $\mathcal{LB} = \{f \in H(D) : \sup_D (1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)| < +\infty\}$ and the little logarithmic Bloch space $\mathcal{LB}_0$. The results generalize the known corresponding results on the composition operator and the pointwise multiplier on the logarithmic Bloch space $\mathcal{LB}$ and the little logarithmic Bloch space $\mathcal{LB}_0$.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and $H(D)$ denote the set of all analytic functions on D. For $f \in H(D)$, let

$$\|f\|_{\mathcal{LB}} = \sup \left\{ (1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)|f'(z)| : z \in D \right\}.$$

As in [10, 12], the logarithmic Bloch space $\mathcal{LB}$ consists of all $f \in H(D)$ satisfying $\|f\|_{\mathcal{LB}} < +\infty$ and the little logarithmic Bloch space $\mathcal{LB}_0$ consists of all $f \in H(D)$ satisfying $\lim_{|z|\to 1^-}(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)| = 0$. It is known that with the norm

$$\|f\|_L = |f(0)| + \|f\|_{\mathcal{LB}},$$

$\mathcal{LB}$ is a Banach space and $\mathcal{LB}_0$ is a closed subspace of $\mathcal{LB}$.

An analytic map $\varphi : D \to D$ induces the composition operator C_φ on $H(D)$, defined by

$$C_\varphi f = f \circ \varphi$$

for f analytic on D. It is interesting to provide a function theoretic characterization when φ induces a bounded or compact composition operator on various function spaces. The boundedness and compactness of C_φ on the classical Bloch space $\mathcal{B}$ were described by Madigan and Matheson in [4]. On the

Received December 3, 2008.
2000 *Mathematics Subject Classification.* 47B38, 30D05, 30H05.
Key words and phrases. logarithmic Bloch space, weighted composition operator, boundedness, compactness.
The research was supported by NNSF of China (Grant No. 10771130), NSF of Fujian Province (Grant No. 2009J01004), and a Key Project of Fujian Provincial Universities–Information Technology Research Based on Mathematics.

logarithmic Bloch space $\mathcal{LB}$, this operator is studied by Yoneda in [12]. On the other various function spaces, one may see in [3, 7, 8, 11, 13].

In this paper we study the weighted composition operator uC_φ, which can be regarded as a generalization of a multiplication operator and a composition operator.

For a fixed analytic function u on D and an analytic self-map $\varphi : D \to D$, define a weighted composition operator uC_φ as follows:

$$uC_\varphi f = uf \circ \varphi, \quad f \in H(D).$$

This operator may be firstly studied on the Bloch space and the little Bloch space in [6]. In [5], Ohno, Stroethoff, and Zhao got the characterization on φ and u for the weighted composition operator is bounded or compact between the α-Bloch spaces. Especially, for $\varphi(z) = z$, this operator is a pointwise multiplier operator induced by u. The pointwise multiplier operator was studied on the Bloch spaces [1], on the α-Bloch spaces [14], on the logarithmic Bloch [10], to mention only a few related works.

Here we will consider the boundedness and the compactness of the weighted composition operator uC_φ on the logarithmic Bloch space $\mathcal{LB}$ and the logarithmic little Bloch space $\mathcal{LB}_0$. In what follows C will stand for positive constants not depending on the functions being considered, but whose value may change from line to line.

2. Auxiliary results

In order to prove the main results of this paper, we need some auxiliary results. The first four lemmas may be found in [10]. For the purpose of reference, we give them here.

Lemma 2.1. *If $f \in \mathcal{LB}$, then*

(i) $|f(z)| \le (2 + \ln(\ln \frac{2}{1-|z|}))\|f\|_L$;
(ii) $|f(z)| \le 2\ln(\ln \frac{2}{1-|z|})\|f\|_L, \quad$ where $\quad |z| \ge r_* = 1 - \frac{2}{e^{e^2}}$.

Lemma 2.2. *If $f \in \mathcal{LB}_0$, then* $\lim_{|z|\to 1^-} \frac{|f(z)|}{\ln(\ln \frac{2}{1-|z|})} = 0$.

Lemma 2.3. *Let* $f(z) = \frac{(1-|z|)\ln \frac{2}{1-|z|}}{|1-z|\ln \frac{4}{|1-z|}}$, $z \in D$. *Then* $|f(z)| < 2$.

Lemma 2.4. *Let* $0 \le t \le 1$, $f(z) = \frac{(1-|z|)\ln \frac{2}{1-|z|}}{(1-|tz|)\ln \frac{2}{1-|tz|}}$, $z \in D$. *Then* $|f(z)| < 2$.

Lemma 2.5. *Suppose $f \in \mathcal{LB}$. Then $\|f_t\|_L \le 4\|f\|_L$,$0 < t < 1$, where $f_t(z) = f(tz)$.*

The result is easily proved by lemma 2.4.

Using the same idea of [9], we obtain the following result.

Lemma 2.6. *Let $f \in H(D)$. Then*

$$\|f\|_{\mathcal{LB}} \approx \sup_{a\in D} \int_D |f'(z)|(1-|z|^2)^{-1} \ln\left(\frac{2}{1-|z|}\right)(1-|\varphi_a(z)|^2)^2\, dA(z),$$

where $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ is the Möbius transformation of D, $dA(z)$ denotes the Lebesgue area measure on D, and $\approx$ means the equivalence of two quantities, that is, the quotient of the left side and the right side lies between two positive constants unless both are zero.

Proof. Noting that

$$(1-|z|^2)|\varphi_a'(z)| = 1-|\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}$$

and

$$z \in E\left(a, \frac{1}{2}\right) \triangleq \left\{z \in D : |\varphi_a(z)| < \frac{1}{2}\right\} \Longrightarrow 1-|z|^2 \approx 1-|a|^2$$

by p. 61 in [15], we obtain that

$$\begin{aligned}
|f'(a)| &= |f'(\varphi_a(0))| \\
&\le \frac{4}{\pi} \int_{|z|<\frac{1}{2}} |f'(\varphi_a(z))|\, dA(z) \\
&= \frac{4}{\pi} \int_{E(a,\frac{1}{2})} |f'(z)||\varphi_a'(z)|^2\, dA(z) \\
&\le \frac{4}{\pi(1-|a|^2)} \int_{E(a,\frac{1}{2})} |f'(z)|(1-|z|^2)^{-1}(1-|\varphi_a(z)|^2)^2\, dA(z) \\
&\le \frac{4}{\pi(1-|a|^2)\ln(\frac{2}{1-|a|})} \int_{E(a,\frac{1}{2})} |f'(z)|(1-|z|^2)^{-1} \ln\left(\frac{2}{1-|z|}\right)(1-|\varphi_a(z)|^2)^2\, dA(z).
\end{aligned}$$

Hence

$$\begin{aligned}
&(1-|a|^2)\ln\left(\frac{2}{1-|a|}\right)|f'(a)| \\
&\le \frac{4}{\pi}\sup_{a\in D} \int_D |f'(z)|(1-|z|^2)^{-1} \ln\left(\frac{2}{1-|z|}\right)(1-|\varphi_a(z)|^2)^2\, dA(z),
\end{aligned}$$

i.e.,

$$\|f\|_{\mathcal{LB}} \le \frac{4}{\pi}\sup_{a\in D} \int_D |f'(z)|(1-|z|^2)^{-1} \ln\left(\frac{2}{1-|z|}\right)(1-|\varphi_a(z)|^2)^2\, dA(z).$$

Conversely, by Lemma 4.2.2 of [15], we obtain that

$$\begin{aligned}
&\sup_{a\in D} \int_D |f'(z)|(1-|z|^2)^{-1} \ln\left(\frac{2}{1-|z|}\right)(1-|\varphi_a(z)|^2)^2\, dA(z) \\
&\le \|f\|_{\mathcal{LB}} \sup_{a\in D} \int_D \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}\, dA(z) \le C\|f\|_{\mathcal{LB}}.
\end{aligned}$$

Hence

$$\|f\|_{\mathcal{LB}} \approx \sup_{a\in D}\int_D |f'(z)|(1-|z|^2)^{-1}\ln\left(\frac{2}{1-|z|}\right)(1-|\varphi_a(z)|^2)^2\,dA(z).$$

□

Lemma 2.7. *Suppose $uC_\varphi : \mathcal{LB}_0 \to \mathcal{LB}_0$ is a bounded operator. Then $uC_\varphi : \mathcal{LB} \to \mathcal{LB}$ is a bounded operator.*

Proof. Suppose uC_φ is bounded in $\mathcal{LB}_0$. It is clear that for any $f \in \mathcal{LB}$, we have $f_t \in \mathcal{LB}_0$ for every $0 < t < 1$. According to Lemma 2.5, we obtain that

$$\|uC_\varphi(f_t)\|_L \le \|uC_\varphi\|\|f_t\|_L \le 4\|uC_\varphi\|\|f\|_L < +\infty.$$

For the simple, we write $\omega(|z|) = (1-|z|^2)^{-1}\ln(\frac{2}{1-|z|})(1-|\varphi_a(z)|^2)^2 > 0$. By Lemma 2.6 and Fatou's lemma, we obtain that

$$\begin{aligned}
&\|uC_\varphi f\|_L\\
&= |u(0)f(\varphi(0))| + \|uC_\varphi f\|_{\mathcal{LB}}\\
&\le |u(0)f(\varphi(0))| + C\sup_{a\in D}\int_D |u(z)f'(\varphi(z))\varphi'(z) + u'(z)f(\varphi(z))|\omega(|z|)\,dA(z)\\
&= |u(0)f(\varphi(0))| + C\sup_{a\in D}\int_D \lim_{t\to 1^-}|u(z)f'(t\varphi(z))t\varphi'(z) + u'(z)f(t\varphi(z))|\omega(|z|)\,dA(z)\\
&\le \lim_{t\to 1^-}|u(0)f(t\varphi(0))| + C\sup_{a\in D}\liminf_{t\to 1^-}\int_D |(uC_\varphi(f_t))'(z)|\omega(|z|)\,dA(z)\\
&\le \lim_{t\to 1^-}|u(0)f_t(\varphi(0))| + C\liminf_{t\to 1^-}\|uC_\varphi f_t\|_{\mathcal{LB}}\\
&\le C\|uC_\varphi(f_t)\|_L\\
&\le C\|uC_\varphi\|\|f\|_L < +\infty.
\end{aligned}$$

Hence $uC_\varphi : \mathcal{LB} \to \mathcal{LB}$ is a bounded operator. □

3. Boundedness of uC_φ

In this section we characterize bounded weighted composition operators on the logarithmic Bloch space $\mathcal{LB}$ and the little logarithmic Bloch space $\mathcal{LB}_0$.

Theorem 3.1. *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Then uC_φ is bounded on the logarithmic Bloch space $\mathcal{LB}$ if and only if the following are satisfied*:

$$\sup_{z\in D}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)\ln\left(\ln\frac{2}{1-|\varphi(z)|}\right)|u'(z)| < +\infty; \tag{1}$$

$$\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)u(z)| < +\infty. \tag{2}$$

Proof. Suppose uC_φ is bounded on the logarithmic Bloch space $\mathcal{LB}$. Then we can easily obtain the following results by taking $f(z) = 1$ and $f(z) = z$ in $\mathcal{LB}$ respectively:

$$u \in \mathcal{LB}; \tag{3}$$

$$K = \sup_{z\in D}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)|\varphi'(z)u(z)| < +\infty. \tag{4}$$

Fix $w \in D$, we take the test function

$$f_w(z) = 2\ln\ln\frac{4}{1-\overline{\varphi(w)}z} - \frac{1}{\ln\ln\frac{4}{1-|\varphi(w)|^2}}\left(\ln\ln\frac{4}{1-\overline{\varphi(w)}z}\right)^2 \tag{5}$$

for $z \in D$. Then

$$f_w'(z) = \frac{2\overline{\varphi(w)}}{(1-\overline{\varphi(w)}z)\ln\frac{4}{1-\overline{\varphi(w)}z}} - 2\ln\ln\frac{4}{1-\overline{\varphi(w)}z}\,\frac{\overline{\varphi(w)}}{(1-\overline{\varphi(w)}z)\ln\frac{4}{1-\overline{\varphi(w)}z}}\,\frac{1}{\ln\ln\frac{4}{1-|\varphi(w)|^2}}.$$

By Lemmas 2.3 and 2.4 we obtain that $f_w \in \mathcal{LB}$ and $\|f_w\|_L \le 16$ with a directly calculation. Since $f_w'(\varphi(w)) = 0$ and $f_w(\varphi(w)) = \ln\ln\frac{4}{1-|\varphi(w)|^2}$, it follows that

$$\begin{aligned}
&(1-|w|^2)\ln\left(\frac{2}{1-|w|}\right)|u'(w)f_w(\varphi(w))| \\
&= (1-|w|^2)\ln\frac{2}{1-|w|}|(uC_\varphi f_w)'(w)| \\
&\le \|uC_\varphi f_w\|_{\mathcal{LB}} \le \|uC_\varphi\|\|f_w\|_L \le 16\|uC_\varphi\| < +\infty.
\end{aligned}$$

We have

$$\sup_{w\in D}(1-|w|^2)\ln\left(\frac{2}{1-|w|}\right)\left|\ln\left(\ln\frac{4}{1-|\varphi(w)|^2}\right)\right||u'(w)| \le 16\|uC_\varphi\| < +\infty.$$

So

$$\sup_{z\in D}(1-|z|^2)\ln\frac{2}{1-|z|}\left|\ln\left(\ln\frac{2}{1-|\varphi(z)|}\right)\right||u'(z)| < +\infty.$$

Hence (1) holds.

Next, fix $w \in D$ with $w \ne 0$, let

$$f_w(z) = \int_0^z\left(1-\frac{\overline{w}^2}{|w|^2}z^2\right)^{-1}\left(\ln\frac{4}{1-\frac{\overline{w}^2}{|w|^2}z^2}\right)^{-1}dz. \tag{6}$$

By Lemma 2.3, we have

$$\sup_{z_1\in D}(1-|z_1|^2)\left(\ln\frac{2}{1-|z_1|^2}\right)|1-z_1^2|^{-1}\left|\ln\frac{4}{1-z_1^2}\right|^{-1} < 2 < +\infty,$$

applying $z_1 = \frac{\overline{w}}{|w|}z$, we obtain that

$$\sup_{z\in D}(1-|z|^2)\left(\ln\frac{2}{1-|z|^2}\right)\left|1-\frac{\overline{w}^2}{|w|^2}z^2\right|\left|\ln\frac{4}{1-\frac{\overline{w}^2}{|w|^2}z^2}\right|^{-1} < 2 < +\infty.$$

Hence $f_w \in \mathcal{LB}$ and $\|f_w\|_L < 4$ with $w \neq 0$. Then for $w \neq 0$ we obtain that

$$\begin{aligned} \|uC_\varphi(f_w)\|_{\mathcal{LB}} &\leq \|uC_\varphi(f_w)\|_L \\ &\leq \|uC_\varphi\|\|f_w\|_L \\ &= \|uC_\varphi\|\|f_w\|_{\mathcal{LB}} \\ &= C < +\infty. \end{aligned} \tag{7}$$

So for $\forall z \in D$ with $\varphi(z) \neq 0$, applying $w = \varphi(z)$ to (7), we have that

$$\begin{aligned} &(1-|z|^2)\ln\frac{2}{1-|z|}|u(z)f_w'(\varphi(z))\varphi'(z)| \\ \leq\ &\|uC_\varphi(f_w)\|_{\mathcal{LB}} + \sup_{z\in D}(1-|z|^2)\ln\frac{2}{1-|z|}|u'(z)||f_w(\varphi(z))| \\ \leq\ &C + 2\sup_{z\in D}(1-|z|^2)\ln\frac{2}{1-|z|}\left(2+\ln\left(\ln\frac{2}{1-|\varphi(z)|}\right)\right)|u'(z)| < +\infty, \end{aligned}$$

where we use Lemma 2.1. So,

$$\begin{aligned} &\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|u(z)\varphi'(z)| \\ \leq\ &\sup_{z\in D}\frac{2(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{4}{1-|\varphi(z)|^2}}|u(z)\varphi'(z)| < +\infty. \end{aligned}$$

For $\forall z \in D$ with $\varphi(z) = 0$, by (4), we have

$$\begin{aligned} &\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|u(z)\varphi'(z)| \\ =\ &\sup_{z\in D}\frac{1}{\ln 2}(1-|z|^2)\ln\frac{2}{1-|z|}|u(z)\varphi'(z)| < +\infty. \end{aligned}$$

Hence (2) holds.

Conversely, suppose that (1) and (2) hold. For $f \in \mathcal{LB}$, by Lemma 2.1, we have the following inequality:

$$\begin{aligned} &\|uC_\varphi f\|_{\mathcal{LB}} \\ \leq\ &\sup_{z\in D}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)|u'(z)f(\varphi(z))| \\ &+\sup_{z\in D}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)|u(z)||f'(\varphi(z))\varphi'(z)| \\ \leq\ &\sup_{z\in D}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)|u'(z)|\left(2+\ln\ln\frac{2}{1-|\varphi(z)|}\right)\|f\|_L \\ &+\sup_{z\in D}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)\frac{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|f'(\varphi(z))||\varphi'(z)u(z)| \end{aligned}$$

$$\leq C\|f\|_L + \|f\|_{\mathcal{LB}} \sup_{z\in D} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}} |\varphi'(z)u(z)|$$

$$\leq C\|f\|_L$$

and

$$|u(0)f(\varphi(0))| \leq |u(0)(2+\ln\left(\ln\frac{2}{1-|\varphi(0)|}\right))|\|f\|_L.$$

This shows that uC_φ is bounded. This completes the proof of Theorem 3.1. $\square$

Theorem 3.2. *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Then uC_φ is bounded on the little logarithmic Bloch space $\mathcal{LB}_0$ if and only if $u \in \mathcal{LB}_0$,* (1) *and* (2) *hold, and*

$$\lim_{|z|\to 1^-} (1-|z|^2)\ln\frac{2}{1-|z|}|\varphi'(z)u(z)| = 0.$$

Proof. Suppose that uC_φ is bounded on the little logarithmic Bloch space $\mathcal{LB}_0$. Then $u = uC_\varphi 1 \in \mathcal{LB}_0$. Also $u\varphi = uC_\varphi z \in \mathcal{LB}_0$, thus

$$(1-|z|^2)\ln\frac{2}{1-|z|}|u'(z)\varphi(z)+u(z)\varphi'(z)| \longrightarrow 0 \quad (|z|\to 1^-).$$

Since $|\varphi| \leq 1$ and $u \in \mathcal{LB}_0$, we have $\lim_{|z|\to 1^-}(1-|z|^2)\ln\frac{2}{1-|z|}|\varphi'(z)u(z)| = 0$. On the other hand, by Lemma 2.7 and Theorem 3.1, we obtain that (1) and (2) hold.

Conversely, let

$$M_1 = \sup_{z\in D}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)\left|\ln\left(\ln\frac{2}{1-|\varphi(z)|}\right)\right||u'(z)| < +\infty;$$

$$M_2 = \sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)u(z)| < +\infty.$$

For $\forall f \in \mathcal{LB}_0$, we have both $(1-|z|^2)\ln\frac{2}{1-|z|}|f'(z)| \to 0$ and $\frac{|f(z)|}{\ln(\ln\frac{2}{1-|z|})} \to 0$ as $|z| \to 1^-$ by Lemma 2.2. Given $\epsilon > 0$ there is $0 < \delta < 1$ such that $(1-|z|^2)\ln\frac{2}{1-|z|}|f'(z)| < \frac{\epsilon}{2M_2}$ and $\frac{|f(z)|}{\ln(\ln\frac{2}{1-|z|})} < \frac{\epsilon}{2M_1}$ for all z with $\delta < |z| < 1$.

If $|\varphi(z)| > \delta$, it follows that

$$(1-|z|^2)\ln\frac{2}{1-|z|}|(uC_\varphi(f))'(z)|$$

$$\leq (1-|z|^2)\ln\frac{2}{1-|z|}|u'(z)||f(\varphi(z))| + (1-|z|^2)\ln\frac{2}{1-|z|}|f'(\varphi(z))\varphi'(z)||u(z)|$$

$$\leq (1-|z|^2)\ln\frac{2}{1-|z|}|u'(z)|\ln(\ln\frac{2}{1-|\varphi(z)|})\frac{|f(\varphi(z))|}{\ln(\ln\frac{2}{1-|\varphi(z)|})}$$

$$+ (1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}|f'(\varphi(z))|\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|u(z)\varphi'(z)|$$

$$< M_1 \frac{\epsilon}{2M_1} + M_2 \frac{\epsilon}{2M_2} = \epsilon.$$

We know that there exists a constant M_3 such that $|f(z)| \le M_3$ and $|f'(z)| \le M_3$ for all $|z| \le \delta$.

If $|\varphi(z)| \le \delta$, it follows that

$$\begin{aligned}
&(1-|z|^2) \ln \frac{2}{1-|z|} |(uC_\varphi(f))'(z)| \\
\le\ & (1-|z|^2) \ln \frac{2}{1-|z|} |u'(z)||f(\varphi(z))| + (1-|z|^2) \ln \frac{2}{1-|z|} |f'(\varphi(z))\varphi'(z)||u(z)| \\
\le\ & M_3 (1-|z|^2) \ln \frac{2}{1-|z|} |u'(z)| + M_3 (1-|z|^2) \ln \frac{2}{1-|z|} |u(z)\varphi'(z)|.
\end{aligned}$$

Thus we conclude that $(1-|z|^2)\ln \frac{2}{1-|z|} |(uC_\varphi(f))'(z)| \to 0$ as $|z| \to 1^-$. Hence $uC_\varphi f \in \mathcal{LB}_0$ for all $f \in \mathcal{LB}_0$. On the other hand, uC_φ is bounded on $\mathcal{LB}$ by Theorem 3.1. Hence uC_φ is a bounded operator on the little logarithmic Bloch space $\mathcal{LB}_0$. □

Corollary 3.1. *Let φ be an analytic self-map of D. Then C_φ is a bounded operator on $\mathcal{LB}_0$ if and only if $\varphi \in \mathcal{LB}_0$ and*

$$\sup_{z\in D} \frac{(1-|z|^2)\ln \frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln \frac{2}{1-|\varphi(z)|}} |\varphi'(z)| < +\infty.$$

In the formulation of remark, we use the notation M_u on $H(D)$ defined by $M_u f = uf$ for $f \in H(D)$. Let H^∞ be the algebra of bounded analytic functions in D.

Remark 3.1. From Theorem 3.1, we see that the composition operator $C_\varphi : \mathcal{LB} \to \mathcal{LB}$ is bounded if and only if

$$\sup_{z\in D} \frac{(1-|z|^2)\ln \frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln \frac{2}{1-|\varphi(z)|}} |\varphi'(z)| < +\infty.$$

This fact is proved in Theorem 1 of [12].

Remark 3.2. From Theorem 3.1 and Theorem 3.2, we see that: the pointwise multiplier $M_u : \mathcal{LB}(\text{or } \mathcal{LB}_0) \longrightarrow \mathcal{LB}(\text{or } \mathcal{LB}_0)$ is a bounded operator if and only if $u \in H^\infty$ and

$$\sup_{z\in D} (1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)\left|\ln\left(\ln\frac{2}{1-|z|}\right)\right| |u'(z)| < +\infty.$$

This fact is proved in Theorem 2.4 of [10].

4. Compactness of uC_φ

Lemma 4.1. *Suppose that uC_φ is a bounded operator on $\mathcal{LB}$. Then uC_φ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{LB}$ which converges to 0 uniformly on compact subsets of D, we have $\|uC_\varphi(f_n)\|_L \to 0$ as $n \to \infty$.*

The proof is similar to that of Proposition 3.11 in [2]. The details are omitted.

Theorem 4.1. *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Suppose that uC_φ is bounded on the logarithmic Bloch space $\mathcal{LB}$. Then uC_φ is compact if and only if the following are satisfied:*

(i) $\lim\limits_{|\varphi(z)|\to 1^-} (1-|z|^2)\ln \dfrac{2}{1-|z|} \ln\left(\ln \dfrac{2}{1-|\varphi(z)|}\right)|u'(z)| = 0$;

(ii) $\lim\limits_{|\varphi(z)|\to 1^-} \dfrac{(1-|z|^2)\ln \frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln \frac{2}{1-|\varphi(z)|}}|\varphi'(z)u(z)| = 0.$

Proof. Suppose that uC_φ is compact on the logarithmic Bloch space $\mathcal{LB}$. Let $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. We take the test functions

$$f_n(z) = \frac{3}{a_n}\left(\ln\ln \frac{4}{1-\overline{\varphi(z_n)}z}\right)^2 - \frac{2}{a_n^2}\left(\ln\ln \frac{4}{1-\overline{\varphi(z_n)}z}\right)^3,$$

where $a_n = \ln\ln \frac{4}{1-|\varphi(z_n)|^2}$. Clearly $f_n(z) \to 0$ uniformly on compact subsets of D. By Lemmas 2.3 and 2.4, we obtain that $\sup_n \|f_n\|_L < \infty$. Then $\{f_n\}$ is a bounded sequence in $\mathcal{LB}$ which converges to 0 uniformly on compact subsets of D. Note that $f_n'(\varphi(z_n)) \equiv 0$ and $f_n(\varphi(z_n)) = a_n$, it follows that

$$\begin{aligned}
\|uC_\varphi f_n\|_L &\geq \|uC_\varphi f_n\|_{\mathcal{LB}} \\
&\geq (1-|z_n|^2)\ln\left(\frac{2}{1-|z_n|}\right)|u'(z_n)f_n(\varphi(z_n))+u(z_n)f_n'(\varphi(z_n))\varphi'(z_n)| \\
&= (1-|z_n|^2)\ln\left(\frac{2}{1-|z_n|}\right)|u'(z_n)|\left|\ln\ln \frac{4}{1-|\varphi(z_n)|^2}\right| \\
&\geq (1-|z_n|^2)\ln\left(\frac{2}{1-|z_n|}\right)|u'(z_n)|\left|\ln\ln \frac{2}{1-|\varphi(z_n)|}\right|.
\end{aligned}$$

Then (i) holds by Lemma 4.1.

Next assume that (ii) fails. Then there exist a subsequence $\{z_n\} \subset D$ and an $\epsilon_0 > 0$ such that $|\varphi(z_n)| \to 1 (n \to \infty)$ and

$$\frac{(1-|z_n|^2)\ln \frac{2}{1-|z_n|}}{(1-|\varphi(z_n)|^2)\ln \frac{2}{1-|\varphi(z_n)|}}|\varphi'(z_n)u(z_n)| \geq \epsilon_0.$$

Let $\varphi(z_n) = r_n e^{i\theta_n}$, we take

$$g_n(z) = \int_0^z \left(\frac{r_n}{1-e^{-i\theta_n}r_n w} - \frac{r_n^2}{1-r_n^2 e^{-i\theta_n}w}\right)\left(\ln \frac{4}{1-r_n^2 e^{-i\theta_n}w}\right)^{-1} dw,$$

so

$$g_n'(z) = \left(\frac{r_n}{1 - e^{-i\theta_n} r_n z} - \frac{r_n^2}{1 - r_n^2 e^{-i\theta_n} z} \right) \left(\ln \frac{4}{1 - r_n^2 e^{-i\theta_n} z} \right)^{-1}.$$

One may obtain that $|g_n(z)| \leq \frac{1-r_n}{(1-|z|)^2} (\ln \frac{4}{1-|z|})^{-1}$ by a directly calculation and $\|g_n\|_L \leq 8$ by Lemmas 2.3 and 2.4. Then $\{g_n\}$ is a bounded sequence in $\mathcal{LB}$ which converges to 0 uniformly on compact subsets of D.

On the other hand, for enough large n, by (i) and Lemma 2.1, it follows that

$$\begin{aligned}
&\|uC_\varphi(g_n)\|_L \\
&\geq (1 - |z_n|^2) \ln \frac{2}{1 - |z_n|} |g_n'(\varphi(z_n))| |\varphi'(z_n) u(z_n)| \\
&\quad - (1 - |z_n|^2) \ln \frac{2}{1 - |z_n|} |g_n(\varphi(z_n))| |u'(z_n)| \\
&\geq (1 - |z_n|^2) \ln \frac{2}{1 - |z_n|} \left(\frac{r_n}{1 - r_n^2} - \frac{r_n^2}{1 - r_n^3} \right) \left(\ln \frac{4}{1 - r_n^3} \right)^{-1} |\varphi'(z_n) u(z_n)| \\
&\quad - 2(1 - |z_n|^2) \ln \frac{2}{1 - |z_n|} \ln \left(\ln \frac{2}{1 - |\varphi(z_n)|} \right) \|g_n\|_L |u'(z_n)| \\
&\geq \frac{(1 - |z_n|^2) \ln \frac{2}{1-|z_n|}}{6(1 - |\varphi(z_n)|^2) \ln \frac{2}{1-|\varphi(z_n)|}} |\varphi'(z_n)| \\
&\quad - 16(1 - |z_n|^2) \ln \frac{2}{1 - |z_n|} \ln \left(\ln \frac{2}{1 - |\varphi(z_n)|} \right) |u'(z_n)| \geq \frac{\epsilon_0}{6} \quad (n \to \infty).
\end{aligned}$$

This contradicts the compactness of uC_φ by Lemma 4.1. The proof of the necessary is completed.

Conversely, suppose that (i) and (ii) hold. Let $\{f_n\}$ be a bounded sequence in $\mathcal{LB}$ which converges to 0 uniformly on compact subsets of D. Let $M = \sup_n \|f_n\|_L < +\infty$. We only prove $\lim\limits_{n\to\infty} \|uC_\varphi(f_n)\|_L = 0$ by Lemma 4.1. This amounts to showing that both

$$\sup_{w \in D} (1 - |w|^2) \ln \frac{2}{1 - |w|} |u(w) f_n'(\varphi(w)) \varphi'(w)| \to 0$$

and

$$\sup_{w \in D} (1 - |w|^2) \ln \frac{2}{1 - |w|} |u'(w) f_n(\varphi(w))| \to 0.$$

If $|\varphi(w)| \leq r < 1$, by (4), then

$$(1 - |w|^2) \ln \frac{2}{1 - |w|} |u(w) f_n'(\varphi(w)) \varphi'(w)| \leq K \max_{|z| \leq r} |f_n'(z)|.$$

If $|\varphi(w)| > r$, then

$$(1 - |w|^2) \ln \frac{2}{1 - |w|} |u(w) f_n'(\varphi(w)) \varphi'(w)|$$

$$= (1-|\varphi(w)|^2)\ln\left(\frac{2}{1-|\varphi(w)|}\right)|f_n'(\varphi(w))| \times \frac{(1-|w|^2)\ln\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)\ln\frac{2}{1-|\varphi(w)|}}|\varphi'(w)u(w)|$$

$$\le M\frac{(1-|w|^2)\ln\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)\ln\frac{2}{1-|\varphi(w)|}}|\varphi'(w)u(w)|.$$

Thus

$$\sup_{w\in D}(1-|w|^2)\ln\frac{2}{1-|w|}|u(w)f_n'(\varphi(w))\varphi'(w)|$$

$$\le K\max_{|w|\le r}|f_n'(w)| + \sup_{|\varphi(w)|>r} M\frac{(1-|w|^2)\ln\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)\ln\frac{2}{1-|\varphi(w)|}}|\varphi'(w)u(w)|.$$

First letting n tend to infinity and subsequently r increase to 1, one obtains that

$$\sup_{w\in D}(1-|w|^2)\ln\frac{2}{1-|w|}|u(w)f_n'(\varphi(w))\varphi'(w)| \longrightarrow 0$$

as $n\to\infty$. The other statement is proved similarly.

If $|\varphi(w)|\le r<1$, by (3), then

$$(1-|w|^2)\ln\frac{2}{1-|w|}|u'(w)f_n(\varphi(w))| \le \|u\|_L \max_{|z|\le r}|f_n(z)|.$$

If $|\varphi(w)|>r$, we may suppose that $|r|>r_*$, by Lemma 2.1, then

$$(1-|w|^2)\ln\frac{2}{1-|w|}|u'(w)f_n(\varphi(w))|$$

$$\le 2M(1-|w|^2)\ln\left(\frac{2}{1-|w|}\right)\ln\left(\ln\frac{2}{1-|\varphi(w)|}\right)|u'(w)|.$$

Thus

$$\sup_{w\in D}(1-|w|^2)\ln\frac{2}{1-|w|}|u'(w)f_n(\varphi(w))|$$

$$\le \|u\|_L\max_{|w|\le r}|f_n(w)| + 2M\sup_{|\varphi(w)|>r}(1-|w|^2)\ln\left(\frac{2}{1-|w|}\right)\ln\ln\frac{2}{1-|\varphi(w)|}|u'(w)|,$$

which also implies that

$$\sup_{w\in D}(1-|w|^2)\ln\left(\frac{2}{1-|w|}\right)|u'(w)f_n(\varphi(w))| \longrightarrow 0$$

as $n\to\infty$. This completes the proof of Theorem 3.1. □

In order to prove the compactness of uC_φ on the little logarithmic Bloch space $\mathcal{LB}_0$, we require the following lemma.

Lemma 4.2. *Let $U\subset\mathcal{LB}_0$. Then U is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in U}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)|f'(z)| = 0.$$

The proof is similar to that of Lemma 1 in [4], we omit it.

Theorem 4.2. *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Then uC_φ is compact on the little logarithmic Bloch space $\mathcal{LB}_0$ if and only if the following are satisfied*:

(i) $\lim_{|z|\to 1^-}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)\ln\left(\ln\frac{2}{1-|\varphi(z)|}\right)|u'(z)|=0;$

(ii) $\lim_{|z|\to 1^-}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)u(z)|=0.$

Proof. Assume (i) and (ii) hold. By Theorem 3.2, we know that uC_φ is bounded on the little logarithmic Bloch space $\mathcal{LB}_0$. From (i), we can show that

$$\lim_{|z|\to 1^-}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)|u'(z)|=0. \tag{8}$$

Suppose that $f\in\mathcal{LB}_0$ with $\|f\|_L\le 1$. We obtain that

$$\begin{aligned}&(1-|z|^2)\ln\frac{2}{1-|z|}|(uC_\varphi f)'(z)|\\ &\le (1-|z|^2)\ln\frac{2}{1-|z|}|u'(z)f(\varphi(z))|+(1-|z|^2)\ln\frac{2}{1-|z|}|u(z)||f'(\varphi(z))\varphi'(z)|\\ &\le (1-|z|^2)\ln\frac{2}{1-|z|}|u'(z)|\left(2+\ln\ln\frac{2}{1-|\varphi(z)|}\right)+\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)u(z)|,\end{aligned}$$

thus

$$\begin{aligned}&\sup\left\{|(1-|z|^2)\ln\frac{2}{1-|z|}(uC_\varphi f)'(z)| : f\in\mathcal{LB}_0, \|f\|_L\le 1\right\}\\ &\le (1-|z|^2)\ln\frac{2}{1-|z|}|u'(z)|\left(2+\ln\ln\frac{2}{1-|\varphi(z)|}\right)+\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)u(z)|,\end{aligned}$$

and it follows that

$$\lim_{|z|\to 1^-}\sup\left\{\left|(1-|z|^2)\ln\frac{2}{1-|z|}(uC_\varphi f)'(z)\right| \ : \ f\in\mathcal{LB}_0, \|f\|_L\le 1\right\}=0,$$

hence uC_φ is compact on $\mathcal{LB}_0$ by Lemma 3.2.

Conversely, suppose that uC_φ is compact on $\mathcal{LB}_0$. By Lemma 3.2 we have

$$\lim_{|z|\to 1^-}\sup\left\{\left|(1-|z|^2)\ln\frac{2}{1-|z|}(uC_\varphi f)'(z)\right| \ : \ f\in\mathcal{LB}_0, \|f\|_L\le M\right\}=0 \tag{9}$$

for some $M>0$. Note that the proof of Theorem 3.1 and the fact that the functions given in (5) are in $\mathcal{LB}_0$ and have norms bounded independently of w, we obtain that

$$\lim_{|w|\to 1^-}(1-|w|^2)\ln\frac{2}{1-|w|}\ln\left(\ln\frac{2}{1-|\varphi(w)|}\right)|u'(w)|=0. \tag{10}$$

Similarly, note that the functions given in (6) are in $\mathcal{LB}_0$ and have norms bounded independently of w, we obtain that

$$\begin{aligned}&\lim_{|z|\to 1^-}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|u(z)\varphi'(z)|\\ &\le 2\lim_{|z|\to 1^-}(1-|z|^2)\ln\frac{2}{1-|z|}|(uC_\varphi f_w)'(z)|\\ &\quad+2\lim_{|z|\to 1^-}(1-|z|^2)\ln\frac{2}{1-|z|}\left(2+\ln\left(\ln\frac{2}{1-|\varphi(z)|}\right)\right)|u'(z)|\end{aligned}$$

for $\varphi(z)\neq 0$. So by (9) and (10) it follows that

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|u(z)\varphi'(z)|=0$$

for $\varphi(z)\neq 0$. However, if $\varphi(z)=0$, by taking the constant function and $f(z)=z$ in (9) respectively, we easily have

$$\lim_{|z|\to 1^-}(1-|z|^2)\ln\left(\frac{2}{1-|z|}\right)|u(z)\varphi'(z)|=0.$$

This completes the proof of Theorem 3.2. □

Corollary 4.1. *Let φ be an analytic self-map of D. Then C_φ is a compact operator on $\mathcal{LB}_0$ if and only if*

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)|=0.$$

Corollary 4.2. *Let $u\in H(D)$. Then the pointwise multiplier $M_u:\mathcal{LB}$(or $\mathcal{LB}_0$) $\longrightarrow\mathcal{LB}$(or $\mathcal{LB}_0$) is a compact operator if and only if $u\equiv 0$.*

Remark 4.1. From Theorem 4.1, we see that the composition operator C_φ : $\mathcal{LB}\to\mathcal{LB}$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1^-}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)|=0.$$

This fact is proved in Theorem 2 of [12].

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