

BROWDER'S TYPE STRONG CONVERGENCE THEOREM FOR S -NONEXPANSIVE MAPPINGS

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ABSTRACT. We prove a common fixed point theorem for S -contraction mappings without continuity. Using this result we obtain an approximating curve for S -nonexpansive mappings in a Banach space and prove Browder's type strong convergence theorem for this approximating curve. The demiclosedness principle for S -nonexpansive mappings is also established.

1. Introduction

Let C be a subset of a normed space X . A mapping $T : C \rightarrow X$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. Suppose $S : C \rightarrow X$ is a mapping. Then T is said to be *S -nonexpansive* if

$$\|Tx - Ty\| \leq \|Sx - Sy\|$$

for all $x, y \in C$. The class of S -nonexpansive mappings is a generalization of that of nonexpansive mappings [7].

In 1967, Browder [2] proved the following strong convergence theorem for nonexpansive mappings: Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \rightarrow C$ a nonexpansive self-mapping. Let $u \in C$ and for each $t \in (0, 1)$, let

$$G_t x = tu + (1 - t)Tx, \quad x \in C.$$

Then G_t has a unique fixed point x_t in C and x_t converges strongly to a fixed point of T in C as $t \rightarrow 0$.

On the other hand, Shahzad [8] introduced the notion of R -subweakly commutativity which provides existence of a curve $\{x_\lambda\}$ in C defined by

$$(1.1) \quad x_\lambda = Sx_\lambda = (1 - \lambda)q + \lambda Tx_\lambda \quad \text{for all } \lambda \in (0, 1)$$

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for S -nonexpansive mappings (see Proposition 3.3).

In this paper, an important existence result as a significant improvement of a result of Shahzad [8] for S -contraction mappings without continuity is proved. This result is applied for existence of approximating curve $\{x_\lambda\}$ defined by (1.1) and existence of fixed points of S -nonexpansive mappings without continuity. Also we prove demiclosedness principle for S -nonexpansive mappings and prove strong convergence of curve $\{x_\lambda\}$ in a reflexive Banach space with a weakly continuous duality mapping. Our results are significant improvements of corresponding results of Al-Thagafi [1], Dotson [3], Jungck [4] and Shahzad [8]. One of our results is an extension of celebrated result of Browder [2] from Hilbert space to Banach space for the class of S -nonexpansive mappings.

2. Preliminaries

Let (X, d) be a metric space, C a nonempty subset of X and let $S, T : C \rightarrow C$ be two mappings. Throughout this paper, we adopt the following notations:

$$C_{S,T} = \{u \in C : Su = Tu\}$$

and

$$F(T) = \{u \in C : Tu = u\}.$$

The pair $\{S, T\}$ is said to be R -weakly commuting on C [6] if there exists $R > 0$ such that

$$d(STx, TSx) \leq Rd(Sx, Tx)$$

for all $x \in C$.

Let C be a nonempty subset of a normed space X . The set C is called q -starshaped with $q \in C$ if for all $x \in C$, the segment $[x, p]$ joining x to q is contained in C , that is, $tx + (1-t)q \in C$ for all $x \in C$ and $0 \leq t \leq 1$. Note that if S is q -starshaped for every $q \in C$, then C is convex.

Let C be a nonempty q -starshaped subset of a normed space X . Then a mapping $S : C \rightarrow C$ is said to be q -affine if

$$tSx + (1-t)Sq \in C$$

for all $x \in C$ and $0 \leq t \leq 1$.

Definition 2.1. Let C be a nonempty subset of a normed space X and let $S, T : C \rightarrow C$ be two mappings such that $F(S) \neq \emptyset$. Suppose $q \in F(S)$ and C is q -starshaped. Then S and T are called R -subweakly commuting on C if there exists a real number $R > 0$ such that

$$\|STx - TSx\| \leq Rd(Sx, [Tx, q])$$

for all $x \in C$, where $d(y, D) = \inf\{\|y - z\| : z \in D\}$ for $D \subseteq C$ and $y \in C$.

It is clear from Definition 2.1 that commutativity implies R -subweak commutativity, but the converse is not true in general (see, example in [8]).

Let C be a nonempty subset of a normed space X and $T : C \rightarrow C$ a mapping. When $\{x_n\}$ is a sequence in X , we denote the strong convergence of $\{x_n\}$ to

x by $x_n \rightarrow x$, the weak convergence of $\{x_n\}$ to x by $x_n \rightharpoonup x$ and the weak* convergence of $\{x_n\}$ to x by $x_n \rightharpoonup^* x$. T is said to be *demicontinuous* if $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$, then $Tx_n \rightarrow Tx$. The mapping T is said to be *weakly continuous* if $\{x_n\}$ is a sequence in X such that $x_n \rightharpoonup x$, then $Tx_n \rightarrow Tx$. Note that every continuous mapping is demicontinuous.

Let X be a Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in X is said to be *demiclosed* at a point $y \in R(T)$ if whenever $\{x_n\}$ is a sequence in $D(T)$ which converges weakly to a point $u \in D(T)$ and $\{Tx_n\}$ converges strongly to y , then $Tu = y$.

A Banach space X is said to satisfy *the Opial condition* ([5]) if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

for all $y \in X, y \neq x$.

Let X be a Banach space. Then a mapping $J : X \rightarrow 2^{X^*}$ defined by

$$J(u) = \{j \in X^* : \langle u, j \rangle = \|u\|^2, \|j\| = \|u\|\}$$

is called the normalized duality mapping. Suppose that J is single-valued. Then J is said to be *weakly sequentially continuous* if, for each $\{x_n\}$ in X with $x_n \rightharpoonup x$, $J(x_n) \rightharpoonup^* J(x)$. It is well known that if X admits a weakly sequentially continuous duality mapping, then X satisfies the Opial condition.

3. Auxiliary results

The following lemma is an improvement of Lemma 2.1 of Shahzad [8] in the following ways:

- (i) C is not necessarily closed,
- (ii) T is not necessarily continuous,
- (iii) location of unique common fixed point is given.

Lemma 3.1. *Let (X, d) be a metric space and C a nonempty subset of X . Let $S, T : C \rightarrow C$ be two mappings such that*

- (i) $T(C) \subseteq S(C)$,
- (ii) T is S -contraction, i.e., there exists a constant $k \in (0, 1)$ such that

$$\|Tx - Ty\| \leq k\|Sx - Sy\| \text{ for all } x, y \in C,$$

- (iii) $\{S, T\}$ is R -weakly commuting on C .

Then we have the following:

- (a) $F(S) \cap F(T) \cap T(C)$ is a singleton if $T(C)$ is complete,
- (b) $F(S) \cap F(T) \cap S(C)$ is a singleton if $S(C)$ is complete.

Proof. Pick $x_0 \in C$. Since $T(C) \subseteq S(C)$, we can construct a sequence $\{x_n\}$ in C such that $Sx_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Then

$$d(Sx_{n+1}, Sx_n) = d(Tx_n, Tx_{n-1}) \leq kd(Sx_n, Sx_{n-1}) \text{ for all } n \in \mathbb{N},$$

it follows that $\{Sx_n\}$ is a Cauchy sequence in C .

(a) If $T(C)$ is complete, there exists a point $z \in T(C)$ such that $Tx_n \rightarrow z \in T(C)$. Thus, $Sx_n \rightarrow z$. Since $z \in T(C) \subseteq S(C)$, there exists $u \in C$ such that $z = Su$. By the S -contractivity of T , we have

$$d(Tu, Tx_n) \leq kd(Su, Sx_n).$$

Taking limit as $n \rightarrow \infty$ yields

$$d(Tu, z) \leq kd(z, z) = 0.$$

Thus, $Su = Tu = z$. Since $\{S, T\}$ is R -weakly commuting on C , it follows that $Sz = Tz$. Note that

$$d(Tz, Tx_n) \leq kd(Sz, Sx_n).$$

Letting limit as $n \rightarrow \infty$, we obtain

$$d(Tz, z) \leq kd(Sz, z) = kd(Tz, z).$$

It shows that $Sz = Tz = z$. By the S -contractivity of T , we conclude that $F(S) \cap F(T) \cap T(C) = \{z\}$.

(b) Suppose $S(C)$ is complete. Then $Sx_n \rightarrow z$ for some $z \in S(C)$ and there exist $u \in C$ such that $z = Su$. As part (a) we can show that $Sz = Tz = z$. The S -contractivity of T implies that $F(S) \cap F(T) \cap S(C) = \{z\}$. \square

Before presenting existence results, we establish the demiclosedness principle for S -nonexpansive nonself mappings.

Proposition 3.2 (Demiclosedness principle). *Let X be a Banach space satisfying the Opial condition and C a nonempty weakly closed subset of X . Let $S, T : C \rightarrow X$ be two mappings such that S is weakly continuous and T is S -nonexpansive. Then $S - T$ is demiclosed on C , i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightarrow u \in C$ and $(S - T)x_n \rightarrow y$, then $(S - T)u = y$.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightarrow u \in C$ and $\lim_{n \rightarrow \infty} \|(S - T)x_n - y\| = 0$ for some $y \in X$. We show that $(S - T)u = y$. Suppose, for contradiction, that $(S - T)u \neq y$. Observe that

$$\|Sx_n - Tu - y\| \leq \|Sx_n - Tx_n - y\| + \|Tx_n - Tu\|,$$

which implies that

$$\liminf_{n \rightarrow \infty} \|Sx_n - Tu - y\| \leq \liminf_{n \rightarrow \infty} \|Sx_n - Su\|.$$

By the weak continuity of S , we obtain that $Sx_n \rightarrow Su \in C$. By the Opial condition, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \|Sx_n - Su\| &< \liminf_{n \rightarrow \infty} \|Sx_n - Tu - y\| \\ &\leq \liminf_{n \rightarrow \infty} \|Sx_n - Su\|, \end{aligned}$$

a contradiction. Therefore, $(S - T)u = y$. \square

Proposition 3.3. *Let C be a nonempty subset of a normed space X . Let $S, T : C \rightarrow C$ be two mappings such that*

- (i) *T is S -nonexpansive and S is q -affine with $q \in F(S)$,*
- (ii) *$T(C) \subseteq S(C)$ and C is q -starshaped,*
- (iii) *$\{S, T\}$ is R -subweakly commuting,*
- (iv) *$S(C)$ is complete.*

Then there exists exactly one point $x_\lambda \in S(C)$ such that

$$(3.1) \quad x_\lambda = Sx_\lambda = (1 - \lambda)q + \lambda Tx_\lambda$$

for all $\lambda \in (0, 1)$.

Proof. For each $\lambda \in (0, 1)$, define a mapping T_λ by

$$(3.2) \quad T_\lambda x = (1 - \lambda)q + \lambda Tx$$

for all $x \in C$. Note that each $T_\lambda : C \rightarrow C$ is an S -contraction on C . Indeed,

$$\begin{aligned} \|T_\lambda x - T_\lambda y\| &= \lambda \|Tx - Ty\| \\ &\leq \lambda \|Sx - Sy\| \end{aligned}$$

for all $x, y \in C$. Since $\{S, T\}$ is R -subweakly commuting and S is q -affine, we have

$$\begin{aligned} \|ST_\lambda x - T_\lambda Sx\| &= \|[(1 - \lambda)q + \lambda STx] - [(1 - \lambda)q + \lambda TSx]\| \\ &= \lambda \|TSx - STx\| \\ &\leq \lambda R \|Sx - T_\lambda x\| \end{aligned}$$

for all $x \in C$. Thus, the pair $\{S, T_\lambda\}$ is R -weakly commuting on C .

For $x \in C$, we have $Tx \in T(C) \subseteq S(C)$, i.e., there exists a point $y \in C$ such that $Tx = Sy \in S(C)$. Observe that

$$T_\lambda x = (1 - \lambda)q + \lambda Tx = (1 - \lambda)q + \lambda Sy \in S(C).$$

It follows that $T_\lambda(C) \subseteq S(C)$ for all $\lambda \in (0, 1)$.

For each $\lambda \in (0, 1)$, we conclude that

- (i)' $T_\lambda(C) \subseteq S(C)$,
- (ii)' T_λ is S -contraction,
- (iii)' $S(C)$ is complete,
- (iv)' $\{S, T_\lambda\}$ is R -weakly commuting on C .

Therefore, Lemma 3.1 implies that there exists exactly one point $x_\lambda \in S(C)$ such that $x_\lambda = Sx_\lambda = T_\lambda x_\lambda$. \square

The main results of this section are the following:

Theorem 3.4. *Let C be a nonempty subset of a normed space X . Let $S, T : C \rightarrow C$ be two mappings satisfy the conditions (i) \sim (iii) of Proposition 3.3. Suppose $S(C)$ is compact. Then we have the following:*

- (a) *There exists $y \in S(C)$ such that $Sy = Ty$.*
- (b) *If S or T is demicontinuous, then $y \in F(S) \cap F(T)$.*

Proof. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\lambda_n \rightarrow 1$. By Proposition 3.3, there exists exactly one point $x_{\lambda_n} \in S(C)$ such that

$$x_{\lambda_n} = Sx_{\lambda_n} = (1 - \lambda_n)q + \lambda_n Tx_{\lambda_n} \text{ for all } n \in \mathbb{N}.$$

Set $x_{\lambda_n} := x_n$. By the compactness of $S(C)$, there exists a subsequence $\{\lambda_m\}$ of $\{\lambda_n\}$ such that $\lim_{m \rightarrow \infty} Sx_m = y \in S(C)$. Thus, $y = Su$ for some $u \in C$. The assumption (ii) implies that $\{Tx_m\}$ is bounded. It follows that

$$\|x_m - Tx_m\| \leq (1 - \lambda_m)\|q - Tx_m\| \rightarrow 0 \text{ as } m \rightarrow \infty.$$

This gives that $\lim_{m \rightarrow \infty} Tx_m = y$. By S -nonexpansiveness of T , we have

$$\|Tx_m - Tu\| \leq \|Sx_m - Su\| = \|Sx_m - y\|.$$

Taking limit as $m \rightarrow \infty$ yields $Tu = y$.

(a) Since $\{S, T\}$ is R -weakly commuting, it follows that $Sy = Ty$.

(b) Suppose S is demicontinuous. Since $\lim_{m \rightarrow \infty} x_m = \lim_{m \rightarrow \infty} Sx_m = y$, it follows from the demicontinuity of S that $Sy = y$. Thus, we conclude from $Sy = Ty$ that $y \in F(S) \cap F(T)$. Similarly, we can prove that $y \in F(S) \cap F(T)$ when T is demicontinuous. \square

Example 3.5. Let $X = [0, 1]$ with the usual metric and $C = X$. Define

$$Sx = \begin{cases} 0 & \text{if } x \in [0, 1), \\ 1 & \text{if } x = 1 \end{cases} \quad \text{and } Tx = 0 \text{ for all } x \in C.$$

Then all hypotheses of Theorem 3.4 are satisfied. Note that $0 \in F(S) \cap F(T)$.

Remark 3.6. The mapping S in Theorem 3.4 is not necessarily linear. Therefore, Theorem 3.4 improves Theorem 2.2 of Al-Thagafi [1], Theorem 1 of Dotson [3], Theorem 3.1 of Jungck [4] and Lemma 2.2 of Shahzad [8].

Theorem 3.7. *Let C be a nonempty subset of a Banach space X . Let $S, T : C \rightarrow C$ be two mappings satisfying conditions (i) \sim (iii) of Proposition 3.3. Suppose $S(C)$ is weakly compact. Then $F(S) \cap F(T) \neq \emptyset$ if one of the following conditions holds:*

- (C₁) $I - T$ is demiclosed at zero.
- (C₂) X satisfies the Opial condition.

Proof. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ with $\lambda_n \rightarrow 1$. Since $S(C)$ is weakly compact, it follows that $S(C)$ is norm-closed and hence it is complete. Then from Proposition 3.3, there exists exactly one point x_n such that

$$x_n = Sx_n = (1 - \lambda_n)q + \lambda_n Tx_n \text{ for all } n \in \mathbb{N}.$$

The condition $T(C) \subseteq S(C)$ implies that $T(C)$ is bounded and hence $x_n - Tx_n \rightarrow 0$. By the weak compactness of $S(C)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. The weak continuity of S implies that $v = Sv$.

If $I - T$ is demiclosed at zero, then $v = Tv$ and hence theorem is proved. If we assume that X satisfies the Opial condition, then we can conclude from Proposition 3.2 that $Sv = Tv$. \square

Remark 3.8. The main results of this section can be extended in complete p -normed spaces.

4. Browder's type strong convergence theorem

The following result extends Browder's strong convergence theorem for S -nonexpansive mappings.

Theorem 4.1. *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \rightarrow X^*$. Let C be a weakly closed subset of X , $S : C \rightarrow C$ a affine weakly continuous mapping with $q \in F(S)$ such that C is q -starshaped. Let $T : C \rightarrow C$ be an S -nonexpansive mapping with $C_{S,T} \neq \emptyset$ and let $\{x_\lambda : \lambda \in (0,1)\}$ be the approximating curve in C defined by (3.1). Then $\lim_{\lambda \rightarrow 1} x_\lambda = \tilde{x}$ exists and $\tilde{x} \in F(S) \cap F(T)$.*

Proof. First, we show that $\{x_\lambda\}$ is bounded. Let $p \in C_{S,T}$. Then $S p = T p = u$ for some $u \in C$. From (3.1), we have

$$\begin{aligned} \|x_\lambda - u\| &\leq (1 - \lambda)\|q - u\| + \lambda\|Tx_\lambda - Tp\| \\ &\leq (1 - \lambda)\|q - u\| + \lambda\|Sx_\lambda - Sp\| \\ &= (1 - \lambda)\|q - u\| + \lambda\|x_\lambda - u\|, \end{aligned}$$

which implies that

$$\|x_\lambda - u\| \leq \|q - u\|.$$

Thus, $\{x_\lambda\}$ is bounded. Assume that $\{\lambda_n\}$ is a sequence in $(0,1)$ such that $\lim_{n \rightarrow \infty} \lambda_n = 1$ and $\{x_{\lambda_n}\}$ is bounded. Since X is reflexive, we may assume that $x_{\lambda_n} \rightharpoonup v \in C$. Set $x_{\lambda_n} := x_n$. Again, from (3.1), we have

$$\|x_n - Tx_n\| \leq (1 - \lambda_n)\|q - Tx_n\| \rightarrow 0.$$

On the other hand, we have

$$\begin{aligned} \langle x_\lambda - Tx_\lambda, J(x_\lambda - u) \rangle &= \langle x_\lambda - u + Tp - Tx_\lambda, J(x_\lambda - u) \rangle \\ &= \|x_\lambda - u\|^2 - \langle Tx_\lambda - Tp, J(x_\lambda - p) \rangle \\ (4.1) \qquad \qquad \qquad &\geq \|x_\lambda - u\|^2 - \|Tx_\lambda - Tp\| \|x_\lambda - u\| \\ &\geq \|x_\lambda - u\|^2 - \|Sx_\lambda - Sp\| \|x_\lambda - u\| \\ &= 0. \end{aligned}$$

Since $x_\lambda - Tx_\lambda = \frac{1-\lambda}{\lambda}(q - x_\lambda)$, it follows from (4.1) that

$$(4.2) \qquad \qquad \qquad \langle x_\lambda - q, J(x_\lambda - u) \rangle \leq 0.$$

Since S is weakly continuous, $x_n \rightharpoonup v \in C$ and $Sx_n - Tx_n \rightarrow 0$, we obtain from Proposition 3.2 that $Sv = Tv$. Suppose that $Sv = Tv = w$ for some $w \in C$. Using (4.2), we get

$$(4.3) \qquad \qquad \qquad \langle x_n - q, J(x_n - w) \rangle \leq 0.$$

From (4.3), we have

$$\begin{aligned}
 \|x_n - w\|^2 &= \langle x_n - w, J(x_n - w) \rangle \\
 (4.4) \qquad &= \langle x_n - q, J(x_n - w) \rangle + \langle q - w, J(x_n - w) \rangle \\
 &\leq \langle q - w, J(x_n - w) \rangle.
 \end{aligned}$$

Since J is weakly continuous, it follows from (4.4) that $x_n \rightarrow w$ as $n \rightarrow \infty$. Note that the weak continuity of S implies that $x_n = Sx_n \rightarrow Sw$. By the uniqueness of weak limit of $\{x_n\}$, we have $Sw = w$. Observe that

$$\begin{aligned}
 \|w - Tw\| &\leq \|w - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tw\| \\
 &\leq \|w - x_n\| + \|x_n - Tx_n\| + \|Sx_n - Sw\| \\
 &\leq 2\|w - x_n\| + \|x_n - Tx_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Thus, $w = Sw = Tw$.

Now, it remains to prove that the approximating curve $\{x_\lambda\}$ converges strongly to w . Suppose, for contradiction, that there exists another sequence $\{x_{\lambda_{n'}}\} \subset \{x_\lambda\}$ such that $x_{\lambda_{n'}} \rightarrow w' \neq w$ as $\lambda_{n'} \rightarrow 1$. Then, we have $w' \in F(S) \cap F(T)$. From (4.2), we have

$$(4.5) \qquad \langle x_\lambda - q, J(x_\lambda - p) \rangle \leq 0 \text{ for all } p \in F(S) \cap F(T).$$

Using (4.5), we have

$$\langle w - q, J(w - w') \rangle \leq 0 \text{ and } \langle q - w', J(w - w') \rangle = \langle w' - q, J(w' - w) \rangle \leq 0.$$

Adding these two inequalities, we obtain

$$\|w - w'\|^2 = \langle w - w', J(w - w') \rangle \leq 0.$$

Thus, $w = w'$. Therefore, $\lim_{\lambda \rightarrow 1} x_\lambda$ exists and $\lim_{\lambda \rightarrow 1} x_\lambda = w \in F(S) \cap F(T)$. \square

Remark 4.2. Nonemptiness of $\mathcal{C}_{S,T}$ can be replaced by boundedness of $T(C)$.

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