Bull. Korean Math. Soc. **47** (2010), No. 3, pp. 491–502
DOI 10.4134/BKMS.2010.47.3.491

STABILITY OF A FUNCTIONAL EQUATION DERIVING FROM QUARTIC AND ADDITIVE FUNCTIONS

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Abstract. In this paper, we obtain the general solution and the generalized Hyers-Ulam Rassias stability of the functional equation

$$f(2x+y)+f(2x-y)=4(f(x+y)+f(x-y))-\frac{3}{7}(f(2y)-2f(y))+2f(2x)-8f(x).$$

1. Introduction

The stability problem of functional equations originated from a question of Ulam [28] in 1940, concerning the stability of group homomorphisms. Let $(G_1,\cdot)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$, such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x\cdot y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [13] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f:E\to E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\|\le\delta$$

for all $x,y\in E$, and for some $\delta>0$. Then there exists a unique additive mapping $T:E\to E'$ such that

$$\|f(x)-T(x)\|\le\delta$$

for all $x\in E$. Moreover if $f(tx)$ is continuous in t for each fixed $x\in E$, then T is linear. Finally in 1978, Th. M. Rassias [25] proved the following theorem.

Received October 12, 2008.
2000 *Mathematics Subject Classification.* 39B82, 39B52.
Key words and phrases. Hyers-Ulam-Rassias stability.

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Theorem 1.1. *Let $f : E \to E'$ be a mapping from a norm vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$, then inequality (1.1) *holds for all $x, y \neq 0$, and* (1.2) *for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous for each fixed $x \in E$, then T is linear.*

In 1991, Z. Gajda [9] answered the question for the case $p > 1$, which was rased by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [1, 3], [5-15], [22-24]).

In [19], W.-G. Park and J. H. Bae, considered the following functional equation:

$$f(2x+y) + f(2x-y) = 4(f(x+y) + f(x-y)) + 24f(x) - 6f(y). \tag{1.3}$$

In fact they proved that a function f between real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique symmetric multi-additive function $B : X \times X \times X \times X \to Y$ such that $f(x) = B(x, x, x, x)$ for all x (see [2, 4], [16-21], [26, 27]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.3), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

We deal with the next functional equation deriving from quartic and additive functions:

(1.4)

$$f(2x+y)+f(2x-y) = 4(f(x+y)+f(x-y))-\frac{3}{7}(f(2y)-2f(y))+2f(2x)-8f(x).$$

It is easy to see that the function $f(x) = ax^4 + bx$ is a solution of the functional equation (1.4). In the present paper we investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.4).

2. General solution

In this section we establish the general solution of functional equation (1.4).

Theorem 2.1. *Let X,Y be vector spaces, and let $f : X \to Y$ be a function satisfies* (1.4). *Then the following assertions hold.*

a) *If f is even function, then f is quartic.*

b) *If f is odd function, then f is additive.*

Proof. a) Putting $x = y = 0$ in (1.4), we get $f(0) = 0$. Setting $x = 0$ in (1.4), by evenness of f, we obtain

$$f(2y) = 16f(y) \tag{2.1}$$

for all $y \in X$. Hence (1.4) can be written as

$$f(2x+y)+f(2x-y)=4(f(x+y)+f(x-y))+24f(x)-6f(y) \tag{2.2}$$

for all $x, y \in X$. This means that f is a quartic function.

b) Setting $x = y = 0$ in (1.4) to obtain $f(0) = 0$. Putting $x = 0$ in (1.4), then by oddness of f, we have

$$f(2y)=2f(y) \tag{2.3}$$

for all $y \in X$. We obtain from (1.4) and (2.3) that

$$f(2x+y)+f(2x-y)=4(f(x+y)+f(x-y))-4f(x) \tag{2.4}$$

for all $x, y \in X$. Replacing y by $-2y$ in (2.4), it follows that

$$f(2x-2y)+f(2x+2y)=4(f(x-2y)+f(x+2y))-4f(x). \tag{2.5}$$

Combining (2.3) and (2.5) to obtain

$$f(x-y)+f(x+y)=2(f(x-2y)+f(x+2y))-2f(x). \tag{2.6}$$

Interchange x and y in (2.6) to get the relation

$$f(x+y)+f(x-y)=2(f(y-2x)+f(y+2x))-2f(y). \tag{2.7}$$

Replacing y by $-y$ in (2.7), and using the oddness of f to get

$$f(x-y)-f(x+y)=2(f(2x-y)-f(2x+y))+2f(y). \tag{2.8}$$

From (2.4) and (2.8), we obtain

$$4f(2x+y)=9f(x+y)+7f(x-y)-8f(x)+2f(y). \tag{2.9}$$

Replacing $x+y$ by y in (2.9) it follows that

$$7f(2x-y)=4f(x+y)+2f(x-y)-9f(y)+8f(x). \tag{2.10}$$

By using (2.9) and (2.10), we lead to

$$f(2x+y)+f(2x-y)=\frac{79}{28}f(x+y)+\frac{57}{28}f(x-y)-\frac{6}{7}f(x)-\frac{11}{14}f(y). \tag{2.11}$$

We get from (2.4) and (2.11) that

$$3f(x+y)+5f(x-y))=8f(x)-28f(y). \tag{2.12}$$

Replacing x by $2x$ in (2.4) it follows that

$$f(4x+y)+f(4x-y)=16(f(x+y)+f(x-y))-24f(x). \tag{2.13}$$

Setting $2x+y$ instead of y in (2.4), we arrive at

$$f(4x+y)-f(y)=4(f(3x-y)+f(x-y))-4f(x). \tag{2.14}$$

Replacing y by $-y$ in (2.14), and using oddness of f to get

$$f(4x-y)+f(y)=4(f(3x+y)+f(x+y))-4f(x). \tag{2.15}$$

Adding (2.14) to (2.15) to get the relation

$$f(4x+y)+f(4x-y)=4(f(3x+y)+f(3x-y))-4(f(x+y)+f(x-y))-8f(x). \tag{2.16}$$

Replacing y by$x+y$ in (2.4) to obtain

$$f(3x+y)+f(x-y)=4(f(2x+y)-f(y))-4f(x). \tag{2.17}$$

Replacing y by $-y$ in (2.17), and using the oddness of f, we lead to

$$f(3x-y)+f(x+y)=4(f(2x-y)+f(y))-4f(x). \tag{2.18}$$

Combining (2.17) and (2.18) to obtain

$$f(3x+y)+f(3x-y)=15(f(x+y)+f(x-y))-24f(x). \tag{2.19}$$

Using (2.16) and (2.19) to get

$$f(4x+y)+f(4x-y)=56(f(x+y)+f(x-y))-104f(x). \tag{2.20}$$

Combining (2.13) and (2.20), we arrive at

$$f(x+y)+f(x-y)=2f(x). \tag{2.21}$$

Hence by using (2.12) and (2.21) it is easy to see that f is additive. This completed the proof of theorem. $\square$

Theorem 2.2. *Let X,Y be vector spaces, and let $f : X \to Y$ be a function. Then f satisfies* (1.4) *if and only if there exist a unique symmetric multi-additive function $B : X \times X \times X \times X \to Y$ and a unique additive function $A : X \to Y$ such that $f(x) = B(x, x, x, x) + A(x)$ for all $x \in X$.*

Proof. Suppose f satisfies (1.4). We decompose f into the even part and odd part by setting

$$f_e(x)=\frac{1}{2}(f(x)+f(-x)), \quad f_o(x)=\frac{1}{2}(f(x)-f(-x))$$

for all $x \in X$. By (1.4), we have

$$\begin{aligned}
&f_e(2x+y)+f_e(2x-y)\\
=&\frac{1}{2}[f(2x+y)+f(-2x-y)+f(2x-y)+f(-2x+y)]\\
=&\frac{1}{2}[f(2x+y)+f(2x-y)]+\frac{1}{2}[f(-2x+(-y))+f(-2x-(-y))]\\
=&\frac{1}{2}[4(f(x+y)+f(x-y))-\frac{3}{7}(f(2y)-2f(y))+2f(2x)-8f(x)]\\
&+\frac{1}{2}[4(f(-x-y)+f(-x-(-y)))-\frac{3}{7}(f(-2y)-2f(-y))+2f(-2x)-8f(-x)]\\
=&4[\frac{1}{2}(f(x+y)+f(-x-y))+\frac{1}{2}(f(-x+y)+f(x-y))]\\
&-\frac{3}{7}[\frac{1}{2}(f(2y)+f(-2y))-(f(y)-f(-y))]\\
&+2[\frac{1}{2}(f(2x)+f(-2x))]-8[\frac{1}{2}(f(x)+f(-x))]\\
=&4(f_e(x+y)+f_e(x-y))-\frac{3}{7}(f_e(2y)-2f_e(y))+2f_e(2x)-8f_e(x)
\end{aligned}$$

for all $x, y \in X$. This means that f_e holds in (1.4). Similarly we can show that f_o satisfies (1.4). By above theorem, f_e and f_o are quartic and additive respectively. Thus there exists a unique symmetric multi-additive function $B : X \times X \times X \times X \to Y$ such that $f_e(x) = B(x, x, x, x)$ for all $x \in X$. Put $A(x) := f_o(x)$ for all $x \in X$. It follows that $f(x) = B(x) + A(x)$ for all $x \in X$. The proof of the converse is trivially. $\square$

3. Stability

Throughout this section, X and Y will be a real normed space and a real Banach space, respectively. Let $f : X \to Y$ be a function then we define $D_f : X \times X \to Y$ by

$$\begin{aligned} D_f(x, y) = 7[f(2x + y) + f(2x - y)] - 28[f(x + y) + f(x - y)] \\ + 3[f(2y) - 2f(y)] - 14[f(2x) - 4f(x)] \end{aligned}$$

for all $x, y \in X$.

Theorem 3.1. *Let $\psi : X \times X \to [0, \infty)$ be a function satisfies $\sum_{i=0}^{\infty} \frac{\psi(0,2^i x)}{16^i} < \infty$ for all $x \in X$, and $\lim \frac{\psi(2^n x, 2^n y)}{16^n} = 0$ for all $x, y \in X$. If $f : X \to Y$ is an even function such that $f(0) = 0$, and that*

$$\|D_f(x, y)\| \leq \psi(x, y) \tag{3.1}$$

for all $x, y \in X$, then there exists a unique quartic function $Q : X \to Y$ satisfying (1.4) *and*

$$\|f(x) - Q(x)\| \leq \frac{1}{48} \sum_{i=0}^{\infty} \frac{\psi(0, 2^i x)}{16^i} \tag{3.2}$$

for all $x \in X$.

Proof. Putting $x = 0$ in (3.1), then we have

$$\|3f(2y) - 48f(y)\| \leq \psi(0, y). \tag{3.3}$$

Replacing y by x in (3.3) and then dividing by 48 to obtain

$$\| \frac{f(2x)}{16} - f(x) \| \leq \frac{1}{48} \psi(0, x) \tag{3.4}$$

for all $x \in X$. Replacing x by $2x$ in (3.4) to get

$$\| \frac{f(4x)}{16} - f(2x) \| \leq \frac{1}{48} \psi(0, 2x). \tag{3.5}$$

Combine (3.4) and (3.5) by use of the triangle inequality to get

$$\| \frac{f(4x)}{16^2} - f(x) \| \leq \frac{1}{48} \left(\frac{\psi(0, 2x)}{16} + \psi(0, x) \right). \tag{3.6}$$

By induction on $n \in \mathbb{N}$, we can show that

$$\| \frac{f(2^n x)}{16^n} - f(x) \| \le \frac{1}{48} \sum_{i=0}^{n-1} \frac{\psi(0, 2^i x)}{16^i}. \tag{3.7}$$

Dividing (3.7) by 16^m and replacing x by $2^m x$ to get

$$\begin{aligned}\left\| \frac{f(2^{m+n}x)}{16^{m+n}} - \frac{f(2^m x)}{16^m} \right\| &= \frac{1}{16^m} \| f(2^n 2^m x) - f(2^m x) \| \\ &\le \frac{1}{48 \times 16^m} \sum_{i=0}^{n-1} \frac{\psi(0, 2^i x)}{16^i} \\ &\le \frac{1}{48} \sum_{i=0}^{\infty} \frac{\psi(0, 2^i 2^m x)}{16^{m+i}}\end{aligned}$$

for all $x \in X$. This shows that $\{\frac{f(2^n x)}{16^n}\}$ is a Cauchy sequence in Y, by taking the $\lim m \to \infty$. Since Y is a Banach space, then the sequence $\{\frac{f(2^n x)}{16^n}\}$ converges. We define $Q : X \to Y$ by $Q(x) := \lim_n \frac{f(2^n x)}{16^n}$ for all $x \in X$. Since f is even function, then Q is even. On the other hand we have

$$\begin{aligned}\|D_Q(x, y)\| &= \lim_n \frac{1}{16^n} \|D_f(2^n x, 2^n y)\| \\ &\le \lim_n \frac{\psi(2^n x, 2^n y)}{16^n} = 0\end{aligned}$$

for all $x, y \in X$. Hence by Theorem 2.1, Q is a quartic function. To shows that Q is unique, suppose that there exists another quartic function $\acute{Q} : X \to Y$ which satisfies (1.4) and (3.2). We have $Q(2^n x) = 16^n Q(x)$ and $\acute{Q}(2^n x) = 16^n \acute{Q}(x)$ for all $x \in X$. It follows that

$$\begin{aligned}\| \acute{Q}(x) - Q(x) \| &= \frac{1}{16^n} \| \acute{Q}(2^n x) - Q(2^n x) \| \\ &\le \frac{1}{16^n} [\| \acute{Q}(2^n x) - f(2^n x) \| + \| f(2^n x) - Q(2^n x) \|] \\ &\le \frac{1}{24} \sum_{i=0}^{\infty} \frac{\psi(0, 2^{n+i} x)}{16^{n+i}}\end{aligned}$$

for all $x \in X$. By taking $n \to \infty$ in this inequality we have $\acute{Q}(x) = Q(x)$. □

Theorem 3.2. *Let $\psi : X \times X \to [0, \infty)$ be a function satisfies*

$$\sum_{i=0}^{\infty} 16^i \psi(0, 2^{-i-1} x) < \infty$$

for all $x \in X$, and $\lim 16^n \psi(2^{-n} x, 2^{-n} y) = 0$ for all $x, y \in X$. Suppose that an even function $f : X \to Y$ satisfies $f(0) = 0$, and (3.1). Then the limit

$Q(x) := \lim_n 16^n f(2^{-n}x)$ exists for all $x \in X$ and $Q : X \to Y$ is a unique quartic function satisfies (1.4) *and*

$$\|f(x) - Q(x)\| \leq \frac{1}{3}\sum_{i=0}^{\infty} 16^i \psi(0, 2^{-i-1}x) \tag{3.8}$$

for all $x \in X$.

Proof. By putting $x = 0$ in (3.1), we get

$$\|3f(2y) - 48f(y)\| \leq \psi(0, y). \tag{3.9}$$

Replacing y by $\frac{x}{2}$ in (3.9) and result dividing by 3 to get

$$\| 16f(2^{-1}x) - f(x) \| \leq \frac{1}{3}\psi(0, 2^{-1}x) \tag{3.10}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (3.10) it follows that

$$\| 16f(4^{-1}x) - f(2^{-1}x) \| \leq \frac{1}{3}\psi(0, 2^{-2}x). \tag{3.11}$$

Combining (3.10) and (3.11) by use of the triangle inequality to obtain

$$\| 16^2 f(4^{-1}x) - f(x) \| \leq \frac{1}{3}\left(\frac{\psi(0, 2^{-2}x)}{16} + \psi(0, 2^{-1}x)\right). \tag{3.12}$$

By induction on $n \in \mathbb{N}$, we have

$$\| 16^n f(2^{-n}x) - f(x) \| \leq \frac{1}{3}\sum_{i=0}^{n-1} 16^i \psi(0, 2^{-i-1}x). \tag{3.13}$$

Multiplying (3.13) by 16^m and replacing x by $2^{-m}x$ to obtain

$$\begin{aligned} \| 16^{m+n} f(2^{-m-n}x) - 16^m f(2^{-m}x) \| &= 16^m \| f(2^{-n}2^{-m}x) - f(2^{-m}x) \| \\ &\leq \frac{16^m}{3}\sum_{i=0}^{n-1} 16^i \psi(0, 2^{-i-1}x) \\ &\leq \frac{1}{3}\sum_{i=0}^{\infty} 16^{m+i} \psi(0, 2^{-i-1}2^{-m}x) \end{aligned}$$

for all $x \in X$. By taking the $\lim_{m\to\infty}$, it follows that $\{16^n f(2^{-n}x)\}$ is a Cauchy sequence in Y. Since Y is a Banach space, then the sequence $\{16^n f(2^{-n}x)\}$ converges. Now we define $Q : X \to Y$ by

$$Q(x) := \lim_n 16^n f(2^{-n}x)$$

for all $x \in X$. The rest of proof is similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let $\psi : X \times X \to [0, \infty)$ be a function such that*

$$\sum \frac{\psi(0, 2^i x)}{2^i} < \infty \tag{3.14}$$

and

$$\lim_n \frac{\psi(2^n x, 2^n y)}{2^n} = 0 \tag{3.15}$$

for all $x, y \in X$. *If* $f : X \to Y$ *is an odd function such that*

$$\|D_f(x, y)\| \leq \psi(x, y) \tag{3.16}$$

for all $x, y \in X$. *Then there exists a unique additive function* $A : X \to Y$ *satisfies* (1.4) *and*

$$\|f(x) - A(x)\| \leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{\psi(0, 2^i x)}{2^i}$$

for all $x \in X$.

Proof. Setting $x = 0$ in (3.16) to get

$$\|f(2y) - 2f(y)\| \leq \psi(o, y). \tag{3.17}$$

Replacing y by x in (3.17) and result dividing by 2, then we have

$$\left\|\frac{f(2x)}{2} - f(x)\right\| \leq \frac{1}{2}\psi(0, x). \tag{3.18}$$

Replacing x by $2x$ in (3.18) to obtain

$$\left\|\frac{f(4x)}{2} - f(2x)\right\| \leq \frac{1}{2}\psi(0, 2x). \tag{3.19}$$

Combine (3.18) and (3.19) by use of the triangle inequality to get

$$\left\|\frac{f(4x)}{4} - f(x)\right\| \leq \frac{1}{2}(\psi(0, x) + \frac{1}{2}\psi(0, 2x)). \tag{3.20}$$

Now we use iterative methods and induction on n to prove our next relation.

$$\left\|\frac{f(2^n x)}{2^n} - f(x)\right\| \leq \frac{1}{2} \sum_{i=0}^{n-1} \frac{\psi(0, 2^i x)}{2^i}. \tag{3.21}$$

Dividing (3.21) by 2^m and then substituting x by $2^m x$, we get

$$\begin{aligned} \left\|\frac{f(2^{m+n}x)}{2^{m+n}} - \frac{f(2^m x)}{2^m}\right\| &= \frac{1}{2^m}\left\|\frac{f(2^n 2^m x)}{2^n} - f(2^m x)\right\| \\ &\leq \frac{1}{2^{m+1}} \sum_{i=0}^{n-1} \frac{\psi(0, 2^i 2^m x)}{2^i} \\ &\leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{\psi(0, 2^{i+m} x)}{2^{m+i}}. \end{aligned} \tag{3.22}$$

Taking $m \to \infty$ in (3.22), then the right hand side of the inequality tends to zero. Since Y is a Banach space, then $A(x) = \lim_n \frac{f(2^n x)}{2^n}$ exits for all $x \in X$. The oddness of f implies that A is odd. On the other hand by (3.15) we have

$$D_A(x,y) = \lim_n \frac{1}{2^n}\|D_f(2^n x, 2^n y)\| \leq \lim_n \frac{\psi(2^n x, 2^n y)}{2^n} = 0.$$

Hence by Theorem 1.2, A is additive function. The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.4. *Let $\psi : X \times X \to [0,\infty)$ be a function satisfies*

$$\sum_{i=0}^{\infty} 2^i \psi(0, 2^{-i-1}x) < \infty$$

for all $x \in X$ and $\lim 2^n \psi(2^{-n}x, 2^{-n}y) = 0$ for all $x, y \in X$. Suppose that an odd function $f : X \to Y$ satisfies (3.1)*. Then the limit $A(x) := \lim_n 2^n f(2^{-n}x)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive function satisfying* (1.4)*, and*

$$\|f(x) - A(x)\| \leq \sum_{i=0}^{\infty} 2^i \psi(0, 2^{-i-1}x)$$

for all $x \in X$.

Proof. It is similar to the proof of Theorem 3.3. □

Theorem 3.5. *Let $\psi : X \times X \to Y$ be a function such that*

$$\sum_{i=o}^{\infty} \frac{\psi(0, 2^i x)}{2^i} \leq \infty \quad \textit{and} \quad \lim_n \frac{\psi(2^n x, 2^n x)}{2^n} = 0$$

for all $x \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$\|D_f(x,y)\| \leq \psi(x,y)$$

for all $x, y \in X$, and $f(0) = 0$. Then there exist a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1.4) *and*

$$\| f(x) - Q(x) - A(x) \| \leq \frac{1}{48}\left[\sum_{i=0}^{\infty}\left(\frac{\psi(0,2^i x) + \psi(0,-2^i x)}{2 \times 16^i} + \frac{12(\psi(0,2^i x) + \psi(0,-2^i x))}{2^i}\right)\right] \tag{3.23}$$

for all $x, y \in X$.

Proof. We have

$$\|D_{f_e}(x,y)\| \leq \frac{1}{2}[\psi(x,y) + \psi(-x,-y)]$$

for all $x, y \in X$. Since $f_e(0) = 0$ and f_e is and even function, then by Theorem 3.1, there exists a unique quartic function $Q : x \to Y$ satisfying

$$\| f_e(x) - Q(x) \| \leq \frac{1}{48} \sum_{i=0}^{\infty} \frac{\psi(0, 2^i x) + \psi(0, -2^i x)}{2 \times 16^i} \tag{3.24}$$

for all $x \in X$. On the other hand f_0 is odd function and

$$\|D_{f_0}(x, y)\| \leq \frac{1}{2}[\psi(x, y) + \psi(-x, -y)]$$

for all $x, y \in X$. Then by Theorem 3.3, there exists a unique additive function $A : X \to Y$ such that

$$\| f_0(x) - A(x) \| \leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{\psi(0, 2^i x) + \psi(0, -2^i x)}{2 \times 2^i} \tag{3.25}$$

for all $x \in X$. Combining (3.24) and (3.25) to obtain (3.23). This completes the proof of theorem. □

By Theorem 3.5, we are going to investigate the Hyers-Ulam-Rassias stability problem for functional equation (1.4).

Corollary 3.6. *Let $\theta \geq 0$, $P < 1$. Suppose $f : X \to Y$ satisfies the inequality*

$$\|D_f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ and $f(0) = 0$. Then there exists a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1.4)*, and*

$$\| f(x) - Q(x) - A(x) \| \leq \frac{\theta}{48} \|x\|^p \left(\frac{16}{16 - 2^p} + \frac{96}{1 - 2^{p-1}} \right)$$

for all $x \in X$.

By Corollary 3.6, we solve the following Hyers-Ulam stability problem for functional equation (1.4).

Corollary 3.7. *Let ϵ be a positive real number, and let $f : X \to Y$ be a function satisfies*

$$\|D_f(x, y)\| \leq \epsilon$$

for all $x, y \in X$. Then there exist a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1.4)*, and*

$$\| f(x) - Q(x) - A(x) \| \leq \frac{362}{45}\ \epsilon$$

for all $x \in X$.

By applying Theorems 3.2 and 3.4, we have the following theorem.

Theorem 3.8. *Let $\psi : X \times X \to Y$ be a function such that*

$$\sum_{i=o}^{\infty} 16^i \psi(0, 2^{-i-1}x) \leq \infty \quad and \quad \lim_n 16^n \psi(2^n x, 2^n x) = 0$$

for all $x \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$\|D_f(x, y)\| \leq \psi(x, y)$$

for all $x, y \in X$ and $f(0) = 0$. Then there exist a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1.4), and

$$\| f(x) - Q(x) - A(x) \| \leq \sum_{i=0}^{\infty} \left[\left(\frac{16^i}{3} + 2^i \right) \left(\frac{\psi(0, 2^{-i-1}x) + \psi(0, -2^{-i-1}x)}{2} \right) \right]$$

for all $x, y \in X$.

Corollary 3.9. *Let $\theta \geq 0$, $P > 4$. Suppose $f : X \to Y$ satisfies the inequality*

$$\|D_f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, and $f(0) = 0$. Then there exist a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1.4), and

$$\| f(x) - Q(x) - A(x) \| \leq \frac{\theta}{3 \times 2^p} \|x\|^p \left(\frac{1}{1 - 2^{4-p}} + \frac{1}{1 - 2^{1-p}} \right)$$

for all $x \in X$.

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