THE EXISTENCE RESULTS OF COUPLED QUASI-SOLUTIONS FOR A CLASS OF OPERATOR EQUATIONS

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ABSTRACT. In this paper, by using the semi-order method, two new existence theorems of coupled quasi-solutions for a class of nonlinear operator equations in Banach spaces are proved under some suitable conditions.

1. Introduction

It is well known that the nonlinear operator theory plays an important role in nonlinear functional analysis, nonlinear programmings, nonlinear differential equations, nonlinear variational inequalities and complementarity problems (see, for example, [2, 1, 6] and the references therein).

In 1987, Guo and Lakshmikantham [7] studied the following nonlinear operator equation:

$$(1.1) \qquad x = A(x, x),$$

where A is a mixed monotone operator in Banach spaces. They obtained some existence results of coupled solution for the nonlinear operator equation (1.1). A generalization of (1.1) involving set-valued operators was introduced and studied by Huang, Tang, and Liu [9].

Recently, Feng and Liu [5] considered the following operator equation:

$$(1.2) \qquad Nx = Ax$$

in complete metric spaces and Banach spaces, respectively. They obtained some existence results of solution for the equation (1.2) by using the technique of partial order theory.

Very recently, He et al. [8] introduced and studied the following nonlinear operator equation:

$$(1.3) \qquad\qquad Nx = A(x, x)$$

in Banach spaces. By using the concept of ϕ concave-($-\psi$) convex operator introduced by Xu and Jia [12], they obtained some new solvability results for the nonlinear operator equation (1.3).

On the other hand, by using the semi-order method, Duan and Li [4] studied the existence of coupled minimal-maximal quasi-solutions for the nonlinear operator equation (1.1) in Banach spaces and proved some new existence results.

Motivated and inspired by the works mentioned above, in this paper, we further study the solvability of coupled quasi-solutions for the nonlinear operator equation (1.3) by using the semi-order method. Under some suitable conditions, we prove two new existence theorems of coupled quasi-solutions for the nonlinear operator equation (1.3) in Banach spaces.

2. Preliminaries

Let E be a real Banach space. A nonempty convex closed set P is called a cone if it satisfies the following conditions:

(i) $x \in P$ and $\lambda \geq 0$ imply that $\lambda x \in P$;
(ii) $x \in P$ and $-x \in P$ imply that $x = \theta$, where θ denotes the zero element of E.

Let $(E, \leq)$ be a partial ordering space induced by the cone P of E, i.e., for any $x, y \in E$, $x \leq y$ if and only if $y - x \in P$.

Now, we define a norm in $E \times E$ as follows:

$$\|(x, y)\|_{E \times E} = \max\{\|x\|, \|y\|\}, \quad \forall x, y \in E.$$

Obviously, $E \times E$ is a Banach space with the norm $\|(\cdot, \cdot)\|$.

Let

$$P_1 = \{(x, y) \in E \times E : x \geq \theta, \ y \leq \theta\},$$

where θ denotes the zero element of E. It is easy to see that P_1 is a cone in $E \times E$ and P_1 defines a partial order in $E \times E$ as follows:

$$(2.1) \qquad (x, y) \leq (u, v) \text{ if and only if } x \leq u \text{ and } y \geq v.$$

Definition 2.1. Let D be a nonempty subset of E. An operator $A : D \times D \to E$ is said to be mixed monotone if, for any $x_i, y_i \in D$ with $i = 1, 2$, $x_1 \leq x_2$ and $y_2 \leq y_1$ imply that

$$A(x_1, y_1) \leq A(x_2, y_2).$$

Definition 2.2. A point $(x^*, y^*) \in D \times D$ is said to be:

(i) a coupled solution of the nonlinear operator equation (1.3) if

$$Nx^* = A(x^*, y^*), \quad Ny^* = A(y^*, x^*);$$

(ii) a coupled minimal-maximal solution of the nonlinear operator equation (1.3) if (x^*, y^*) is a coupled solution of the nonlinear operator equation (1.3) such that, for any coupled solution (u^*, v^*) of (1.3), $x^* \leq u^*$ and $y^* \geq v^*$.

Definition 2.3. Let $L(E)$ be the space of all linear operators on E. $T \in L(E)$ is said to be a linear positive operator if $x \geq \theta$ implies $Tx \geq \theta$.

Definition 2.4. Let $x_0, y_0 \in E$ with $x_0 < y_0$. The set defined by

$$[x_0, y_0] = \{z | x_0 \leq z \leq y_0\}$$

is called an ordered interval in E.

We assume that the following conditions are satisfied:

(H_1) There is $\lambda \in (0, 1]$ such that $(\lambda I + T)^{-1} \in L(E)$ exists and

$$(\lambda I + T)^{-1} x \geq \theta \Longrightarrow x \in P;$$

(H_2) $T : E \to E$ is a bounded linear positive operator.

For any subset D of E, throughout this paper, we denote the weak closure of D, the closed convex hull of D under the norm $\|(\cdot, \cdot)\|_{E \times E}$, and the complement of D by $\overline{D}^w$, $\overline{co}(D)$, and CD, respectively.

Lemma 2.1. *If $T \in L(E)$ and there exists $\lambda \in (0, 1]$ with $(\lambda I + T)^{-1} \in L(E)$, then*

$$(2.2) \qquad (\lambda I + T)^{-1}[\lambda A(x, y) + Tu] = u \Longleftrightarrow A(x, y) = u$$

and

$$(2.3) \qquad (\lambda I + T)^{-1}[\lambda A(y, x) + Tv] = v \Longleftrightarrow A(y, x) = v,$$

where $u = Nx$ and $v = Ny$.

Proof. Obviously,

$$(\lambda I + T)^{-1}[\lambda A(x, y) + Tu] = u$$

if and only if

$$\lambda A(x, y) + Tu = (\lambda I + T)u$$

and $\lambda A(x, y) = \lambda u$ if and only if $A(x, y) = u$. It follows that (2.2) is true. Similarly, we can show that (2.3) holds. This completes the proof. $\qquad \square$

Lemma 2.2 ([3]). *If $T \in L(E)$ satisfies the condition (H_1) for some $\lambda \in (0, 1]$, then $(\lambda I + T)^{-1}$ is a positive linear operator.*

Lemma 2.3. *Let $G : D \times D \to E$ be a mixed monotone operator and N be a nonlinear operator. Let*

$$H(x, y) \overset{\Delta}{=} (G(x, y), G(y, x)), \quad B(x, y) \overset{\Delta}{=} (Nx, Ny), \quad \forall (x, y) \in D \times D.$$

Then the following conclusions hold:

(1) *H is an increasing operator on the partial ordering deduced by P_1;*

(2) $H(x, y) = B(x, y)$ *has a solution* (x^*, y^*) *if and only if* (x^*, y^*) *is a coupled solution of* $Nx = G(x, x)$;

(3) *A minimal solution of* $H(x, y) = B(x, y)$ *is a coupled minimal-maximal solution of* $Nx = G(x, x)$.

Proof. By [16], we know that (1) holds.

Now, we show that (2) is true. In fact, (x^*, y^*) is a solution of $H(x, y) = B(x, y)$ if and only if (x^*, y^*) is a solution of $(Nx, Ny) = (G(x, y), G(y, x))$, i.e.,

$$Nx^* = G(x^*, y^*), \quad Ny^* = G(y^*, x^*).$$

Thus $H(x, y) = B(x, y)$ has a solution (x^*, y^*) if and only if (x^*, y^*) is a coupled solution of $Nx = G(x, x)$.

Next, we prove that (3) holds. Suppose that (u^*, v^*) is a minimal solution of $H(x, y) = B(x, y)$. For any solution (u, v) of $H(x, y) = B(x, y)$, we know that $(u^*, v^*) \leq (u, v)$. It follows from (2.1) that $u^* \leq u$ and $v \leq v^*$. By (2) and Definition 2.2, it is easy to see that (u^*, v^*) is a coupled minimal-maximal solution of $Nx = G(x, x)$. This completes the proof. $\qquad\square$

Lemma 2.4 ([10]). *Let* (E, P) *be a partial ordering Banach space,* D *be a nonempty subset of* E, *and* $y \in E$. *If* $z \leq y$ *(resp.,* $y \leq z$) *for all* $z \in D$, *then* $z \leq y$ *(resp.,* $y \leq z$) *for all* $z \in \overline{co}(D)$.

3. Main results

Theorem 3.1. *Let* E *be a real Banach space and* P *a cone of* E. *Let* $D_0 = [u_0, v_0]$ *be an ordered interval in* E *and* N *be an increasing operator with* $N(D_0) = D_0$. *Suppose that an operator*

$$A : D \overset{\Delta}{=} [(u_0, v_0), (v_0, u_0)] \to E$$

satisfies the conditions (H_1) *and* (H_2). *If*

(i) $Nu_0 \leq A(u_0, v_0)$ *and* $A(v_0, u_0) \leq Nv_0$;

(ii) *For any* $u_0 \leq x_1 \leq x_2 \leq v_0$ *and* $u_0 \leq y_1 \leq y_2 \leq v_0$, *we have*

$$A(x_2, y_1) - A(x_1, y_2) \geq -T(u_2, u_1),$$

   where $u_1 = Nx_1$ *and* $u_2 = Nx_2$;

(iii) *For any* $x_1, x_2 \in D_0$, $Nx_1 \leq Nx_2$ *implies* $x_1 \leq x_2$;

(iv) *Any totally ordered subset in* $G(D)$ *is relatively compact with weak topology, where*

$$G(x, y) \overset{\Delta}{=} (\lambda I + T)^{-1}[\lambda A(x, y) + Tu]$$

   and $u = Nx$ *for all* $(x, y) \in D$.

Then the nonlinear operator equation (1.3) *has a coupled quasi-solution* $(x^*, y^*) \in D$.

Proof. It follows from the condition (H_1) and Lemma 2.2 that $(\lambda I + T)^{-1}$ is a positive operator. By the assumption (H_2) and the conditions (i) and (ii), we have

(I) $G : D \to [u_0, v_0]$ is a mixed monotone operator;

(II) $Nu_0 \leq G(u_0, v_0), G(v_0, u_0) \leq Nv_0$.

Now we show that the following nonlinear operator equation

(3.1) $$B(x, y) = H(x, y)$$

has a solution in D, where

$$H(x, y) \triangleq (G(x, y), G(y, x)), \quad B(x, y) \triangleq (N(x), N(y)).$$

From (I) and Lemma 2.3, it is easy to see that H is an increasing operator. Let

$$M_1 = \{(x, y) \in D : (Nx, Ny) \leq H(x, y)\}, \quad M_2 = \{(y, x) : (x, y) \in M_1\}.$$

By (II), we have $(u_0, v_0) \in M_1$ and so M_1 is nonempty. Suppose that K_1 is a total ordering subset of M_1. Then $K_2 = \{(y, x) | (x, y) \in K_1\}$ is a total ordering subset of M_2. For any fixed $w \in D_0$, let

$$R_1(w) = \{z \in D_0 : w \leq z\}, \quad R_2(w) = \{z \in D_0 : z \leq w\}.$$

Since P is closed and convex, we know that $R_1(w)$ and $R_2(w)$ are both closed and convex. For any $q_1 \in G(K_1)$ and $q_2 \in G(K_2)$, let

$$S_1(q_1) = \overline{co}(G(K_1)) \bigcap R_1(q_1), \quad S_2(q_2) = \overline{co}(G(K_2)) \bigcap R_2(q_2).$$

Since $R_1(q_1)$ and $R_2(q_2)$ are convex and closed, it is easy to see that $S_1(q_1)$ and $S_2(q_2)$ are both convex and closed. Now the mixed monotonicity of G implies that $G(K_i) \, (i = 1, 2)$ are total ordering subsets of $G(D)$. From the condition (iv), we know that $\overline{G(K_i)}^w \, (i = 1, 2)$ are weakly compact sets in $G(D)$. It follows from Krein-Smulian Theorem (see, for example, [2] or [13]) that $\overline{co}(\overline{G(K_i)})^w \, (i = 1, 2)$ are also weakly compact. Since

$$\overline{co}(G(K_i)) \subset \overline{co}(\overline{G(K_i)})^w \, (i = 1, 2),$$

we know that $\overline{co}(G(K_i)) \, (i = 1, 2)$ are weakly compact.

For any $q_i \in G(K_i) \, (i = 1, 2)$, we have $q_i \in S_i(q_i)$ and so $S_i(q_i) \, (i = 1, 2)$ are nonempty. For any $q_1', q_2', \ldots, q_n' \in G(K_1)$ and $q_1'', q_2'', \ldots, q_n'' \in G(K_2)$, without loss of generality, we can suppose that

$$q_1' \leq q_2' \leq \cdots \leq q_n' \quad \text{and} \quad q_1'' \leq q_2'' \leq \cdots \leq q_n''.$$

Thus we have

$$S_1(q_1') \supset S_1(q_2') \supset \cdots \supset S_1(q_n')$$

and

$$S_2(q_1'') \subset S_2(q_2'') \subset \cdots \subset S_2(q_n'').$$

It follows that

(3.2) $$\emptyset \neq S_1(q_n') \subset \bigcap_{i=1}^{n} S_1(q_i'), \quad S_2(q_1'') \subset \bigcap_{i=1}^{n} S_2(q_i'') \neq \emptyset.$$

Now, we prove that $\bigcap_{q_i \in G(K_i)} S_i(q_i)$ $(i = 1, 2)$ are nonempty. In fact, suppose that $\bigcap_{q_i \in G(K_i)} S_i(q_i) = \emptyset$ for $i = 1, 2$. Then we have

$$(3.3) \qquad \overline{co}(G(K_i)) \subset \bigcup_{q_i \in G(K_i)} CS_i(q_i), \quad i = 1, 2.$$

It follows from (3.3) that $\{CS_i(q_i) : q_i \in G(K_i)\}$ is an open covering of $\overline{co}(G(K_i))$ under weakly topology for $i = 1, 2$. Since $\overline{co}(G(K_i))$ is weakly compact, $\overline{co}(G(K_i))$ has a finite sub-covering for $i = 1, 2$. Thus there exist $q_1', q_2', \ldots, q_m' \in G(K_1)$ and $q_1'', q_2'', \ldots, q_l'' \in G(K_2)$ such that

$$(3.4) \qquad \overline{co}(G(K_1)) \subset \bigcup_{i=1}^{m} CS_1(q_i'), \quad \overline{co}(G(K_2)) \subset \bigcup_{i=1}^{l} CS_2(q_i'').$$

Noticing

$$S_1(q_i') \subset \overline{co}(G(K_1)), \quad \forall i = 1, 2, \ldots, m,$$

and

$$S_2(q_i'') \subset \overline{co}(G(K_2)), \quad \forall i = 1, 2, \ldots, l,$$

it follows from (3.4) that

$$\bigcap_{i=1}^{m} S_1(q_i') \subset \overline{co}(G(K_1)) \subset \bigcup_{i=1}^{m} CS_1(q_i'), \quad \bigcap_{i=1}^{l} S_2(q_i'') \subset \overline{co}(G(K_2)) \subset \bigcup_{i=1}^{l} CS_2(q_i'')$$

and so

$$\bigcap_{i=1}^{m} S_1(q_i') = \emptyset, \quad \bigcap_{i=1}^{l} S_2(q_i'') = \emptyset,$$

which contradicts (3.2). Therefore, $\bigcap_{q_i \in G(K_i)} S_i(q_i)$ is nonempty and there exists $q_i^* \in \bigcap_{q_i \in G(K_i)} S_i(q_i)$ such that $q_i^* \in S_i(q_i)$ for all $q_i \in G(K_i)$ with $i = 1, 2$. Thus $q_i^* \in R_i(q_i)$ for all $q_i \in G(K_i)$. By the construction of $R_i(q_i)$, we have

$$(3.5) \qquad q_1 \leq q_1^*, \quad q_2 \geq q_2^*.$$

Since $N(D_0) = D_0$, we know that there exist $w_1, w_2 \in D_0$ such that

$$Nw_1 = q_1^*, \quad Nw_2 = q_2^*.$$

Now, for any $(x, y) \in K_1$, we have $(y, x) \in K_2$. It follows from (3.5) that

$$G(x, y) \leq q_1^* = Nw_1, \quad Nw_2 = q_2^* \leq G(y, x).$$

Since $K_1 \subset M_1$,

$$(Nx, Ny) \leq H(x, y) = (G(x, y), G(y, x))$$

and so

$$Nx \leq G(x, y), \quad G(y, x) \leq Ny.$$

Thus it follows that

$$Nx \leq Nw_1, \quad Nw_2 \leq Ny.$$

From the condition (3), we have

$$x \leq w_1, \quad w_2 \leq y.$$

Now, (2.1) implies that

(3.6) $$(x, y) \leq (w_1, w_2)$$

and so (w_1, w_2) is an upper bound of K_1.

Next, we show that $(w_1, w_2) \in M_1$, that is,

$$B(w_1, w_2) = (Nw_1, Nw_2) \leq H((w_1, w_2)).$$

In fact, for any $(x, y) \in K_1$, we have $(y, x) \in K_2$ and so

$$(x, y) \leq (w_1, w_2), \quad (y, x) \geq (w_2, w_1).$$

Now, the mixed monotonicity of G implies that

$$G(x, y) \leq G(w_1, w_2), \quad G(y, x) \geq G(w_2, w_1).$$

It follows from Lemma 2.4 that, for $Nw_i = q_i^* \in R_i(q_i) \subset \overline{co}(G(K_i)) \, (i = 1, 2)$,

$$Nw_1 \leq G(w_1, w_2), \quad G(w_2, w_1) \leq Nw_2$$

and so

(3.7) $$(Nw_1, Nw_2) \leq (G(w_1, w_2), \quad G(w_2, w_1)) = H(w_1, w_2).$$

Now, (3.7) shows that $(w_1, w_2) \in M_1$. It follows from Zorn's lemma that M_1 contains a maximal element (x^*, y^*).

Finally, we prove that (x^*, y^*) is a solution of the nonlinear operator equation (3.1). We first show the following conclusion holds:

(3.8) $$B(x_1, y_1) \leq B(x_2, y_2), (x_i, y_i) \in D \, (i = 1, 2) \Longrightarrow (x_1, y_1) \leq (x_2, y_2).$$

In fact, it follows from the definition of B that

$$B(x_1, y_1) = (Nx_1, Ny_1) \leq (Nx_2, Ny_2) = B(x_2, y_2)$$

and so

$$Nx_1 \leq Nx_2, \quad Ny_2 \leq Ny_1.$$

By the condition (iii), we have $x_1 \leq x_2$ and $y_2 \leq y_1$ and so $(x_1, y_1) \leq (x_2, y_2)$. Since $(x^*, y^*) \in M_1$,

$$B(x^*, y^*) \leq H(x^*, y^*) = B(B^{-1}H(x^*, y^*))$$

and so (3.8) implies that

$$(x^*, y^*) \leq B^{-1}H(x^*, y^*).$$

By the monotonicity of H, we have

$$B(B^{-1}H(x^*, y^*)) = H(x^*, y^*) \leq H(B^{-1}H(x^*, y^*))$$

and so, $B^{-1}H(x^*, y^*) \in M_1$. We next show that B is an increasing operator. If $(x_1, y_1) \leq (x_2, y_2)$, then $x_1 \leq x_2$ and $y_2 \leq y_1$. By the monotonicity of N, we have

$$Nx_1 \leq Nx_2, \quad Ny_2 \leq Ny_1$$

and so
$$B(x_1, y_1) = (Nx_1, Ny_1) \leq (Nx_2, Ny_2) = B(x_2, y_2).$$

Thus B is an increasing operator. Since (x^*, y^*) is a maximal element of M_1 and B is increasing, we have
$$B(B^{-1}H(x^*, y^*)) = H(x^*, y^*) \leq B(x^*, y^*)$$

and so $H(x^*, y^*) = B(x^*, y^*)$, that is,
$$Nx^* = G(x^*, y^*), \quad Ny^* = G(y^*, x^*).$$

It follows from Lemma 2.1 that
$$Nx^* = A(x^*, y^*), \quad Ny^* = A(y^*, x^*)$$

and so (x^*, y^*) is a coupled quasi-solution of the equation (1.3). This completes the proof. □

Theorem 3.2. *Suppose that all the conditions of Theorem 3.1 are satisfied. Then the nonlinear operator equation* (1.3) *has a coupled minimal and maximal quasi-solution* $(x^*, y^*) \in D$.

Proof. Let $F(H) = \{(x, y) \in D : H(x, y) = B(x, y)\}$. Then Theorem 3.1 implies that $F(H)$ is nonempty. Let
$$S \overset{\triangle}{=} \{[(u, v), (v, u)] : B(u, v) \leq H(u, v), (u, v) \in D, F(H) \subset [(u, v), (v, u)]\},$$

where $[(u, v), (v, u)]$ is an ordered interval in $E \times E$. It is easy to see that $D \in S$ and so $S \neq \emptyset$. We define the ordering " $\leq$ " in S as follows:

$$(3.9) \qquad\qquad I_1, I_2 \in S, \quad I_1 \leq I_2 \Longleftrightarrow I_1 \subset I_2.$$

Now, we show that S has a minimal element. Suppose that
$$\Gamma = \{[(u_\alpha, v_\alpha), (v_\alpha, u_\alpha)] : \alpha \in \Lambda\}$$

is a total ordering subset of S, where Λ is an index set. Let
$$R_1 = \{(u_\alpha, v_\alpha) : \alpha \in \Lambda\}, R_2 = \{(v_\alpha, u_\alpha) : \alpha \in \Lambda\}.$$

Then R_1 and R_2 are total ordering subsets of D. It follows from the mixed monotonicity of G that $G(R_i)\,(i = 1, 2)$ are total ordering subsets of E.

Let $K_1 = R_1$ and $K_2 = R_2$ be the same as in the proof of Theorem 3.1. Then, by the similar proofs for (3.6) and (3.7), we know that there exist $\overline{q_i} \in \overline{co}(G(R_i))\,(i = 1, 2)$ with $N\overline{w_i} = \overline{q_i}$ such that

$$(3.10) \qquad\qquad (u_\alpha, v_\alpha) \leq (\overline{w_1}, \overline{w_2}), \quad \forall \alpha \in \Lambda,$$

and

$$(3.11) \qquad\qquad B(\overline{w_1}, \overline{w_2}) = (N\overline{w_1}, N\overline{w_2}) \leq H(\overline{w_1}, \overline{w_2}).$$

Let $(x, y) \in F(H)$. For any given $(u_\alpha, v_\alpha) \in R_1$, we have $(v_\alpha, u_\alpha) \in R_2$. It follows from the definition of S that

$$(u_\alpha, v_\alpha) \leq (x, y) \leq (v_\alpha, u_\alpha).$$

Now, the mixed monotonicity of G implies that

$$G(u_\alpha, v_\alpha) \leq G(x, y) \leq G(v_\alpha, u_\alpha).$$

Since $\overline{q_i} \in \overline{co}(G(R_i))$, Lemma 2.4 implies that

(3.12) $$N\overline{w_1} \leq G(x, y) \leq N\overline{w_2}.$$

Let $(x, y) \in F(H)$. For any given $(v_\alpha, u_\alpha) \in R_2$, it is easy to see that $(u_\alpha, v_\alpha) \in R_1$. Similarly to the proof of (3.12), we have

$$N\overline{w_1} \leq G(x, y) \leq N\overline{w_2}.$$

Thus it follows that

$$(N\overline{w_1}, N\overline{w_2}) \leq (G(x, y), G(y, x)) = H(x, y) = (Nx, Ny)$$

and

$$(N\overline{w_2}, N\overline{w_1}) \geq (G(x, y), G(y, x)) = H(x, y) = (Nx, Ny).$$

By the condition (3),

(3.13) $$(\overline{w_1}, \overline{w_2}) \leq (x, y) \leq (\overline{w_2}, \overline{w_1}), \quad \forall (x, y) \in F(H).$$

Let

$$I = [(\overline{w_1}, \overline{w_2}), (\overline{w_2}, \overline{w_1})].$$

Then it follows from (3.11) and (3.13) that $I \in S$.

Now, we prove that I is a lower bound of Γ. In fact, for any $\alpha \in \Lambda$, (3.10) implies that

$$(u_\alpha, v_\alpha) \leq (\overline{w_1}, \overline{w_2}), \quad (v_\alpha, u_\alpha) \geq (\overline{w_2}, \overline{w_1}),$$

which show that

$$I \subset [(u_\alpha, v_\alpha), (v_\alpha, u_\alpha)]$$

and so I is a lower bound of Γ in S. It follows from Zorn's lemma that S contains a minimal element I^*. Let $I^* = [(x^*, y^*), (y^*, x^*)]$. Then

$$B(x^*, y^*) \leq H(x^*, y^*) = B(B^{-1}H(x^*, y^*)).$$

From (3.8), we have $(x^*, y^*) \leq B^{-1}H(x^*, y^*)$. By the monotonicity of H,

$$B(B^{-1}H(x^*, y^*)) = H(x^*, y^*) \leq H(B^{-1}H(x^*, y^*)).$$

For any $(x, y) \in F(H)$, the monotonicity of H and the definition of S show that

$$H(x^*, y^*) \leq H(x, y) = B(x, y) \leq H(y^*, x^*)$$

and so

$$B(B^{-1}H(x^*, y^*)) \leq B(B^{-1}H(x, y)) = B(x, y) \leq B(B^{-1}H(y^*, x^*)).$$

It follows from (3.8) that

$$B^{-1}H(x^*, y^*) \leq (x, y) \leq B^{-1}H(y^*, x^*)$$

and so $[B^{-1}H(x^*, y^*), B^{-1}H(y^*, x^*)] \in S$. Therefore, we have

$$(x^*, y^*) \leq B^{-1}H(x^*, y^*), \quad (y^*, x^*) \geq B^{-1}H(y^*, x^*).$$

It follows from (3.9) that

$$I^* = [(x^*, y^*), (y^*, x^*)] \geq [B^{-1}H(x^*, y^*), B^{-1}H(y^*, x^*)].$$

By virtue of the minimality of I^*, we know that $B^{-1}H(x^*, y^*) = (x^*, y^*)$ and so $H(x^*, y^*) = B(x^*, y^*)$. Therefore, (x^*, y^*) is a minimal solution of the nonlinear operator equation $H(x, y) = B(x, y)$. It follows from Lemma 2.3 that (x^*, y^*) is a coupled minimal-maximal solution of the nonlinear operator equation $Nx = G(x, x)$. Now, Lemmas 2.1 and 2.2 imply that (x^*, y^*) is a coupled minimal-maximal quasi-solution of the nonlinear operator equation (1.3). This completes the proof. $\qquad\square$

Remark 3.1. Theorems 3.1 and 3.2 presented in this paper generalize and improve Theorems 2.1 and 2.2 of Duan and Li [4], respectively. Therefore, our results also generalize and improve some corresponding results of Chen [3], Zhou and Yu [16], Liu and Wu [10], Syau [11], Zhang [14], Zhang and Xie [15].

Remark 3.2. We would like to point out Theorems 3.1 and 3.2 do not require any continuity for the operator A.

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