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THE ZEROS DISTRIBUTION OF SOLUTIONS OF HIGHER ORDER DIFFERENTIAL EQUATIONS IN AN ANGULAR DOMAIN

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Abstract. In this paper, we investigate the zeros distribution and Borel direction for the solutions of linear homogeneous differential equation

$$f^{(n)}+A_{n-2}(z)f^{(n-2)}+\cdots+A_1(z)f'+A_0(z)f=0\ (n\geq 2)$$

in an angular domain. Especially, we establish a relation between a cluster ray of zeros and Borel direction.

1. Introduction

We use the standard notations from Nevanlinna theory in this paper (see [4, 5, 13]).

The study of the zeros distribution of solutions of a linear differential equation is one of the difficult aspects in the complex oscillation theory of differential equations. However, different authors have obtained some results (see [3, 7, 10, 11, 12, 14]).

In order to state our results, we give some definitions.

Let $g(z)$ be an entire function in the plane and let $\arg z=\theta\in[0,2\pi)$ be a ray. We denote an angular domain and a sectorial domain, for any $\alpha<\beta$, respectively,

$$\Omega(\alpha,\beta)=\{z|\alpha\leq\arg z\leq\beta,|z|>0\},$$
$$\Omega((\alpha,\beta),r)=\{z|z\in\Omega(\alpha,\beta),|z|<r\}.$$

Moreover, we define the sectorial maximus modulus for an entire function by

$$M(\Omega((\alpha,\beta),r),g)=\sup\{|g(z)|:z\in\Omega((\alpha,\beta),r)\}.$$

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The order $\sigma(\Omega(\alpha,\beta),g)$ of $g(z)$ in the angular domain $\Omega(\alpha,\beta)$ is defined by

$$\sigma(\Omega(\alpha,\beta),g)=\overline{\lim_{r\to\infty}}\frac{\log\log M(\Omega((\alpha,\beta),r),g)}{\log r}.$$

The hyper order $\sigma_2(\Omega(\alpha,\beta),g)$ of $g(z)$ in the angular domain $\Omega(\alpha,\beta)$ is defined by

$$\sigma_2(\Omega(\alpha,\beta),g)=\overline{\lim_{r\to\infty}}\frac{\log\log\log M(\Omega((\alpha,\beta),r),g)}{\log r}.$$

Let $n(\Omega((\alpha,\beta),r),g=a)$ be the number of a-points, i.e., roots of the equation $g(z)=a$ in the sectorial domain $\Omega((\alpha,\beta),r)$.

The exponent of convergence of zero sequence of $g(z)-a$ in the angular domain $\Omega(\alpha,\beta)$ is defined by

$$\lambda(\Omega(\alpha,\beta),g=a)=\overline{\lim_{r\to\infty}}\frac{\log n(\Omega((\alpha,\beta),r),g=a)}{\log r}.$$

The hyper order exponent of convergence of zero sequence of $g(z)-a$ in the angular domain $\Omega(\alpha,\beta)$ is defined by

$$\lambda_2(\Omega(\alpha,\beta),g=a)=\overline{\lim_{r\to\infty}}\frac{\log\log n(\Omega((\alpha,\beta),r),g=a)}{\log r}.$$

We also denote, for each $\varepsilon>0$, the exponent and the hyper order exponent of convergence of zero sequence of $g(z)$ in the angular domain $\Omega(\theta-\varepsilon,\theta+\varepsilon)$ by $\lambda_{\theta,\varepsilon}(g)$ and $\lambda_{2,\theta,\varepsilon}(g)$, respectively, i.e.,

$$\lambda_{\theta,\varepsilon}(g)=\overline{\lim_{r\to\infty}}\frac{\log n(\Omega((\theta-\varepsilon,\theta+\varepsilon),r),g=a)}{\log r},$$

and

$$\lambda_{2,\theta,\varepsilon}(g)=\overline{\lim_{r\to\infty}}\frac{\log\log n(\Omega((\theta-\varepsilon,\theta+\varepsilon),r),g=a)}{\log r},$$

and by $\lambda_\theta(g)=\lim_{\varepsilon\to0}\lambda_{\theta,\varepsilon}(g)$ and $\lambda_{2,\theta}(g)=\lim_{\varepsilon\to0}\lambda_{2,\theta,\varepsilon}(g)$, respectively.

Our proofs also require the Nevanlinna characteristic function for an angular domain (see [3, 9, 10]). If $0<\beta-\alpha\le2\pi$ and $k=\frac{\pi}{\beta-\alpha}$ and $g(z)$ is meromorphic in the angular domain $\Omega(\alpha,\beta)$, we denote

$$A_{\alpha,\beta}(r,g)=\frac{k}{\pi}\int_1^r\left(\frac{1}{t^k}-\frac{t^k}{r^{2k}}\right)\left\{\log^+|g(te^{i\alpha})|+\log^+|g(te^{i\beta})|\right\}\frac{dt}{t};$$

$$B_{\alpha,\beta}(r,g)=\frac{2k}{\pi r^k}\int_\alpha^\beta\log^+|g(re^{i\theta})|\sin k(\theta-\alpha)d\theta;$$

$$C_{\alpha,\beta}(r,g)=2\sum_{1<|b_v|<r}\left(\frac{1}{|b_v|^k}-\frac{|b_v|^k}{r^{2k}}\right)\sin k(\beta_v-\alpha);$$

$$D_{\alpha,\beta}(r,g)=A_{\alpha,\beta}(r,g)+B_{\alpha,\beta}(r,g);$$

$$S_{\alpha,\beta}(r,g)=A_{\alpha,\beta}(r,g)+B_{\alpha,\beta}(r,g)+C_{\alpha,\beta}(r,g),$$

where $b_v = |b_v|e^{i\beta_v}$ $(v = 1, 2, \ldots)$ are the poles of $g(z)$ in the angular domain $\Omega(\alpha, \beta)$, counting multiplicities. If we only consider the distinct poles of $g(z)$, we denote the corresponding angular counting function by $\overline{C}_{\alpha,\beta}(r, g)$. $S_{\alpha,\beta}(r, g)$ and $C_{\alpha,\beta}(r, g)$ are called the Nevanlinna's angular characteristic function and the angular counting function, respectively. The sectorial hyper order $\rho_2(\Omega(\alpha, \beta), g)$ of $g(z)$ in the angular domain $\Omega(\alpha, \beta)$ will be defined by

$$\rho_2(\Omega(\alpha, \beta), g) = \overline{\lim_{r\to\infty}} \frac{\log\log S_{\alpha,\beta}(r, g)}{\log r}.$$

In [7], we considered the equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is an entire function with order $\sigma(A) = +\infty$ and the hyper order $\sigma_2(A) = 0$. We have obtained the following results.

Theorem 1.1. *Let $A(z)$ be an entire function with order $\sigma(A) = +\infty$ and the hyper order $\sigma_2(A) = 0$ and let f_1 and f_2 be two linearly independent solutions of* (1.1)*. Set $E = f_1 f_2$. Then $\lambda_{2,\theta}(E) = +\infty$ if and only if*

$$\overline{\lim_{r\to\infty}} \frac{\log\log\log M(\Omega((\theta - \varepsilon, \theta + \varepsilon), r), E)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log\log S_{\theta-\varepsilon,\theta+\varepsilon}(r, E)}{\log r} = +\infty$$

for all $\varepsilon > 0$.

Theorem 1.2. *Let $A(z)$ be an entire function with order $\sigma(A) = +\infty$ and the hyper order $\sigma_2(A) = 0$ and let f_1 and f_2 be two linearly independent solutions of* (1.1)*. Set $E = f_1 f_2$. Suppose that the hyper order exponent of convergence of zero sequence of E is $+\infty$. Then a ray $\arg z = \theta$ from the origin is a Borel direction of E with the hyper order $+\infty$ and $\rho_2(\Omega(\theta - \varepsilon, \theta + \varepsilon), E) = +\infty$ if and only if $\lambda_{2,\theta}(E) = +\infty$.*

2. The zeros distribution in an angular domain

In order to relate our result, we need the followings.

Suppose that $g(z)$ is analytic. Then $g(z)$ has the power series representation

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \quad (0 \le |z| < \infty).$$

Denote maximum item and center index of $g(z)$ by $\mu(r)$ and $\nu(r)$, respectively, i.e.,

$$\mu(r) = \max_{n\ge 0}\{|a_n|r^n\}$$

and

$$\nu(r) = \max\{m : \mu(r) = |a_m|r^m\}.$$

Setting $a = \max_{n\ge0}\{|a_n|\}$, we have

$$|a_n|r^n \le \mu(r) \le ar^{\nu(r)}.$$

Lemma 2.1 ([6, p. 18]). *Suppose that $g(z)$ is analytic. Then for $r < R$ and $\mu(r) > 1$,*

$$M(r,g) \le \mu(r)\{1 + \log M(R,g)\}\frac{2R}{R-r}.$$

On the other hand, under the hypotheses of Lemma 2.1, we have, for all $0 \le r < R$,

$$T(r,g) \le \log M(r,g) \le \frac{R+r}{R-r}T(R,g).$$

Together with Lemma 2.1 in which we set $R = 2r$, we obtain

$$\begin{aligned} T(r,g) &\le \log\mu(r) + \log\log M(2r,g) + O(1) \\ &\le \nu(r)\log r + \log T(4r,g) + O(1). \end{aligned} \tag{2.1}$$

Lemma 2.2 ([8]). *Let $f_1, f_2, \ldots, f_n$ be n linearly independent meromorphic solutions of the equation*

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,\ n \ge 2,$$

with meromorphic coefficients. Then the Wronskian determinant

$$W = W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

satisfying the differential equation $W' + A_{n-1}(z)W = 0$. Specially, if $A_{n-1}(z)$ is an entire function, then for some $c \in \mathbb{C}, W(f_1, f_2, \ldots, f_n) = c\exp(-\varphi)$, where φ is a primitive function of $A_{n-1}(z)$.

Lemma 2.3 ([3]). *Suppose that $g(z)(\not\equiv$ constant$)$ is meromorphic in the plane and that $\Omega(\alpha,\beta)$ is an angular domain, where $0 < \beta - \alpha \le 2\pi$. Then*

(i) *for any complex number $a \ne \infty$,*

$$S_{\alpha,\beta}\left(r, \frac{1}{g-a}\right) = S_{\alpha,\beta}(r,g) + O(1);$$

(ii) *for any $r < R$,*

$$A_{\alpha,\beta}\left(r, \frac{g'}{g}\right) \le K\left\{\left(\frac{R}{r}\right)^k \int_1^R \frac{\log T(t,g)}{t^{1+k}}dt + \log\frac{r}{R-r} + \log\frac{R}{r} + 1\right\},$$

and

$$B_{\alpha,\beta}\left(r, \frac{g'}{g}\right) \le \frac{4k}{r^k}m\left(r, \frac{g'}{g}\right),$$

where $k = \frac{\pi}{\beta-\alpha}$ and K is a positive constant not depending on r and R.

Lemma 2.4 ([7]). *Suppose that* $\Omega(\alpha,\beta)$ *and* $\Omega(\alpha',\beta')$ *are two angular domains such that* $\alpha<\alpha'<\beta'<\beta$ *and that* $g(z)$ *is an entire function with*

$$\overline{\lim_{r\to\infty}}\frac{\log\log\log M(r,g)}{\log r}=\overline{\lim_{r\to\infty}}\frac{\log\log S_{\alpha',\beta'}(r,g)}{\log r}=\sigma_2(\Omega(\alpha',\beta'),g),$$

and

$$\overline{\lim_{r\to\infty}}\frac{\log\log\log\log M(r,g)}{\log r}=0.$$

Then we have for every finite complex a with at most one exception

$$\overline{\lim_{r\to\infty}}\frac{\log\log n(\Omega((\alpha,\beta),r),g=a)}{\log r}=\sigma_2(\Omega(\alpha',\beta'),g).$$

Now we consider the equation

$$f^{(n)}+A_{n-2}(z)f^{(n-2)}+\cdots+A_1(z)f'+A_0(z)f=0,\ n\geq 2, \tag{2.2}$$

where $A_j(z)(j=0,1,\ldots,n-2)$ are entire functions. Whether there has result similar to Theorem 1.1. About this, we have the following result.

Theorem 2.5. *Let* $A_j(z)$ $(j=0,1,\ldots,n-2)$ *be entire functions with order* $\sigma(A_j)=+\infty$ *and the hyper order* $\sigma_2(A_j)=0$ $(j=0,1,2,\ldots,n-2)$, *and let* $f_1,f_2,\ldots,f_n$ *be n linearly independent solutions of* (2.2). *Set* $E=f_1f_2\cdots f_n$. *Then* $\lambda_{2,\theta}(E)=+\infty$ *if and only if*

$$\overline{\lim_{r\to\infty}}\frac{\log\log\log M(\Omega((\theta-\varepsilon,\theta+\varepsilon),r),E)}{\log r}=\overline{\lim_{r\to\infty}}\frac{\log\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,E)}{\log r}=+\infty$$

for all $\varepsilon>0$.

Proof. Suppose that $f(z)$ is a non-trivial solution of (2.2). Then

$$\frac{f^{(n)}}{f}+A_{n-2}(z)\frac{f^{(n-2)}}{f}+\cdots+A_1(z)\frac{f'}{f}+A_0(z)=0. \tag{2.3}$$

We apply Wiman-Valiron theory to (2.3). Hence there exists a set $D_1\subset[0,+\infty)$ of finite logarithmic measure such that if $r\notin D_1$ and z is a point on $|z|=r$ at which $|f(z)|=M(r,f)$, then

$$\left|\frac{f^{(j)}}{f}\right|=\left(\frac{\nu(r)}{z}\right)^j(1+o(1)),\quad j=1,2,\ldots,n, \tag{2.4}$$

where $\nu(r)$ denotes the central index of $f(z)$.

It follows from (2.3) and (2.4) that

$$\begin{aligned}\nu(r)^n(1+o(1))+\nu(r)^{n-2}z^2A_{n-2}(z)(1+o(1))+\cdots\\+\nu(r)z^{n-1}A_1(z)(1+o(1))+z^nA_0(z)=0.\end{aligned} \tag{2.5}$$

Set $\sigma_2=\max_{0\leq j\leq n-2}\{\sigma_2(A_j)\}$. For all arbitrary $\varepsilon>0$, there exists a set $D_2\subset(1,+\infty)$ of finite logarithmic measure such that

$$|A_j(z)|\leq\exp\{\exp\left(r^{\sigma_2+\varepsilon}\right)\},\quad j=0,1,2,\ldots,n-2, \tag{2.6}$$

when $|z|\notin[0,1]\cup D_2$ and $r\to+\infty$.

It follows from (2.5) and (2.6) that

$$\nu(r) \le nr^n \exp\{\exp\left(r^{\sigma_2+\varepsilon}\right)\} \le \exp\{\exp\left(r^{\sigma_2+2\varepsilon}\right)\}. \tag{2.7}$$

Since $f(z)$ is analytic, $f(z)$ satisfies the condition of Lemma 2.1. Thus, (2.1) and (2.7) implies that

$$\overline{\lim_{r\to+\infty}} \frac{\log\log\log T(r,f)}{\log r} \le \sigma_2. \tag{2.8}$$

Now we suppose that $f_1, f_2, \ldots, f_n$ be n linearly independent solutions of (2.2). Set $E = f_1 f_2 \cdots f_n$, and Wronskian determinant

$$W = W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}. \tag{2.9}$$

It follows from Lemma 2.2, without loss of generality, we can set

$$W(f_1, f_2, \ldots, f_n) = 1.$$

From (2.8), we have

$$\overline{\lim_{r\to+\infty}} \frac{\log\log\log T(r,f_j)}{\log r} \le \sigma_2, \quad j = 1, 2, \ldots, n. \tag{2.10}$$

Hence

$$\overline{\lim_{r\to+\infty}} \frac{\log\log\log T(r,E)}{\log r} \le \sigma_2. \tag{2.11}$$

Now dividing (2.9) by E, we have

$$\begin{aligned} \frac{1}{E} = \frac{W}{E} &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{f_1'}{f_1} & \frac{f_2'}{f_2} & \cdots & \frac{f_n'}{f_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{f_1^{(n-1)}}{f_1} & \frac{f_2^{(n-1)}}{f_2} & \cdots & \frac{f_n^{(n-1)}}{f_n} \end{vmatrix} \\ &= \sum_{1\le j_s \ne j_1 \le n} (-1)^{\tau(j_1,j_2,\ldots,j_n)} \cdot 1_{j_1} \cdot \frac{f_{j_2}'}{f_{j_2}} \cdot \frac{f_{j_3}''}{f_{j_3}} \cdots \frac{f_{j_s}^{(s-1)}}{f_{j_s}} \cdots \frac{f_{j_n}^{(n-1)}}{f_{j_n}} \\ &= \sum_{1\le j_s \ne j_1 \le n} (-1)^{\tau(j_1,j_2,\ldots,j_n)} \prod_{s=2}^{n} \frac{f_{j_s}^{(s-1)}}{f_{j_s}}, \end{aligned} \tag{2.12}$$

where 1_{j_1} denotes the number 1 in row 1 and in column j_1 and $\tau(j_1, j_2, \ldots, j_n)$ denotes the inverse order number of $j_1, j_2, \ldots, j_n$, and $j_1, j_2, \ldots, j_n$ is an arrangement of $1, 2, \ldots, n$.

We deduce from (2.10) and Lemma 2.3 (ii) in which we set $R = 2r$ that, for $j = 1, 2, \ldots, n$ and for all sufficiently small $\varepsilon > 0$,

$$\begin{aligned} A_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{f_j'}{f_j}\right) &\leq K \int_1^{2r} \frac{\log T(r, f_j)}{t^{1+k}} dt + O(1) \\ &\leq K \int_1^{2r} \frac{\exp(t^{\sigma_2+\varepsilon})}{t^{1+\frac{\pi}{2\varepsilon}}} dt + O(1) \leq K \exp((2r)^{\sigma_2+\varepsilon}), \end{aligned}$$

where K is a sufficiently large positive constant and the following K is the same but can be different.

Since, for $j = 1, 2, \ldots, n$ and for all sufficiently small $\varepsilon > 0$,

$$m\left(r, \frac{f_j'}{f_j}\right) = O\left(\log T(2r, f_j) + \log r\right) \leq K \exp((2r)^{\sigma_2+\varepsilon}),$$

we deduce from Lemma 2.3 (ii) that, for $j = 1, 2, \ldots, n$ and for all sufficiently small $\varepsilon > 0$,

$$B_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{f_j'}{f_j}\right) \leq K \exp((2r)^{\sigma_2+\varepsilon}).$$

Therefore we have, for all sufficiently small $\varepsilon > 0$,

$$D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{f_j'}{f_j}\right) \leq K \exp((2r)^{\sigma_2+\varepsilon}), \quad j = 1, 2, \ldots, n. \tag{2.13}$$

Similarly, we have, for $j = 1, 2, \ldots, n$ and for all sufficiently small $\varepsilon > 0$,

$$D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{f_j^{(s)}}{f_j}\right) \leq \sum_{l=1}^{s} D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{f_j^{(l)}}{f_j^{(l-1)}}\right) \leq K \exp((2r)^{\sigma_2+\varepsilon}). \tag{2.14}$$

It follows from (2.12) and (2.14) that

$$\begin{aligned} D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{E}\right) &= D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \sum_{1\leq j_s\neq j_1\leq n} (-1)^{\tau(j_1,j_2,\ldots,j_n)} \prod_{s=2}^{n} \frac{f_{j_s}^{(s-1)}}{f_{j_s}}\right) \\ &\leq K \exp((2r)^{\sigma_2+\varepsilon}) \end{aligned}$$

for all sufficiently small $\varepsilon > 0$.

Since, by Lemma 2.3 (i),

$$\begin{aligned} S_{\theta-\varepsilon,\theta+\varepsilon}(r, E) &= S_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{E}) + O(1) \\ &= D_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{E}) + C_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{E}) + O(1), \end{aligned}$$

we have, for all sufficiently small $\varepsilon > 0$,

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r, E) \leq K \left\{C_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{E}) + \exp\left((2r)^{\sigma_2+\varepsilon}\right)\right\}. \tag{2.15}$$

Sufficiency. Now we suppose that $\theta_0 \in \mathbb{R}$ such that for any sufficiently small $\varepsilon > 0$,

$$\begin{aligned}&\overline{\lim_{r\to\infty}}\frac{\log\log\log M(\Omega((\theta_0-\frac{\varepsilon}{3},\theta_0+\frac{\varepsilon}{3}),r),E)}{\log r}\\ =&\overline{\lim_{r\to\infty}}\frac{\log\log S_{\theta_0-\frac{\varepsilon}{3},\theta_0+\frac{\varepsilon}{3}}(r,E)}{\log r}=+\infty.\end{aligned}$$

Together with (2.11) and $\sigma_2(A_j)=0$ $(j=0,1,2,\ldots,n-2)$, E satisfies the conditions of Lemma 2.4. Thus, we can find a finite complex a such that

$$\overline{\lim_{r\to\infty}}\frac{\log\log n(\Omega((\theta_0-\frac{2\varepsilon}{3},\theta_0+\frac{2\varepsilon}{3}),r),E=a)}{\log r}=+\infty. \tag{2.16}$$

For any given $M>\frac{\pi}{2\varepsilon}$, we deduce from (2.16) that there exists a sequence $\{r_n\}$ of real numbers with $r_n\to+\infty$ $(n\to+\infty)$ such that for every $\varepsilon>0$ we have

$$n\left(\Omega\left(\left(\theta_0-\frac{2\varepsilon}{3},\theta_0+\frac{2\varepsilon}{3}\right),r_n\right),E=a\right)\ge\exp\left(r_n^M\right). \tag{2.17}$$

Suppose that $a_v=|a_v|e^{i\alpha_v}$ $(v=1,2,\ldots)$ are the roots of $E=a$, counting multiplicities, in the angular domain $\Omega(\theta_0-\varepsilon,\theta_0+\varepsilon)$. To compute $\sigma_2(\Omega(\theta_0-\varepsilon,\theta_0+\varepsilon),E)$, we first observe that $\theta_0-\frac{2\varepsilon}{3}<\alpha_v<\theta_0+\frac{2\varepsilon}{3}$ implies for $k=\frac{\pi}{2\varepsilon}$ the inequalities

$$k\cdot\frac{\varepsilon}{3}<k(\alpha_v-\theta_0+\varepsilon)<\pi-k\cdot\frac{\varepsilon}{3}.$$

Hence

$$\sin k(\alpha_v-\theta_0+\varepsilon)\ge\sin(k\cdot\frac{\varepsilon}{3})=\sin\frac{\pi}{6}=\frac{1}{2}. \tag{2.18}$$

Moreover, we write a sum below as a *Stieltjes-integral*,

$$\begin{aligned}\sum\left(\frac{1}{|a_v|^k}-\frac{|a_v|^k}{(2r_n)^{2k}}\right)&=\sum\left(\frac{1}{|a_v|^k}\right)-\sum\left(\frac{|a_v|^k}{(2r_n)^{2k}}\right)\\&=\int_1^{r_n}\frac{dn(t)}{t^k}-\frac{1}{(2r_n)^{2k}}\int_1^{r_n}t^k dn(t),\end{aligned}$$

where a short hand notation $n(t)=n(\Omega(\theta_0-\frac{2\varepsilon}{3},\theta_0+\frac{2\varepsilon}{3}),E=a)$ will be used.

Application of Lemma 2.3(i), (2.17), (2.18), the partial integration of the above *Stieltjes-integral* and the definition of $S_{\alpha,\beta}(r,E)$ now results in

$$\begin{aligned}&S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(2r_n,E)\\ &=S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(2r_n,\frac{1}{E-a})+O(1)\\ &\ge C_{\theta_0-\varepsilon,\theta_0+\varepsilon}(2r_n,\frac{1}{E-a})+O(1) \qquad (2.19)\\ &=2\sum_{1<|a_v|<2r_n}\left(\frac{1}{|a_v|^k}-\frac{|a_v|^k}{(2r_n)^{2k}}\right)\sin k(\alpha_v-\theta_0+\varepsilon)+O(1)\end{aligned}$$

$$\geq 2 \sum_{\substack{1<|a_v|<r_n \\ \theta_0-\frac{2\varepsilon}{3}<\alpha_v<\theta_0+\frac{2\varepsilon}{3}}} \left(\frac{1}{|a_v|^k} - \frac{|a_v|^k}{(2r_n)^{2k}}\right) \sin(k \cdot \frac{\varepsilon}{3}) + O(1)$$
$$= 2\left\{\int_1^{r_n} \frac{dn(t)}{t^k} - \frac{1}{(2r_n)^{2k}} \int_1^{r_n} t^k dn(t)\right\} \sin\frac{\pi}{6} + O(1)$$
$$= \frac{n(r_n)}{r_n^k} + k\int_1^{r_n} \frac{n(t)}{t^{1+k}}dt - \frac{r_n^k n(r_n)}{(2r_n)^{2k}} + \frac{k}{(2r_n)^{2k}} \int_1^{r_n} t^{k-1} n(t)dt + O(1)$$
$$\geq \left(1 - \frac{1}{2^{2k}}\right) \frac{n(r_n)}{r_n^k} + O(1) \geq \exp(r_n^{M-\varepsilon}),$$

where $n(t)$ is the numbers of the roots of the equation $E(z) = a$, counting multiplicities, in the sector $\Omega((\theta_0 - \frac{2\varepsilon}{3}, \theta_0 + \frac{2\varepsilon}{3}), t)$. Therefore we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r, E)}{\log r} \geq M - \varepsilon. \tag{2.20}$$

Since ε can be arbitrary small and M be arbitrary large, it follows from (2.15), (2.20) and $\sigma_2(A_j) = 0$ $(j = 0, 1, 2, \ldots, n-2)$ that $\lambda_{2,\theta_0}(E) = +\infty$.

Necessary. Suppose that $\lambda_{2,\theta_0}(E) = +\infty$. Similar to the proof of (2.19), we have

$$S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(2r, E) \geq \left(1 - \frac{1}{2^{2k}}\right) \frac{n(r)}{r^k} + O(1), \tag{2.21}$$

where $n(r) = n(\Omega((\theta_0 - \frac{2\varepsilon}{3}, \theta_0 + \frac{2\varepsilon}{3}), r), E = a), k = \frac{\pi}{2\varepsilon}$.

By the Maximum Modulus Theorem, we have

$$\begin{aligned}
&S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(2r, E) \\
&= A_{\theta_0-\varepsilon,\theta_0+\varepsilon}(2r, E) + B_{\theta_0-\varepsilon,\theta_0+\varepsilon}(2r, E) \\
&= \frac{k}{\pi} \int_1^r \left(\frac{1}{t^k} - \frac{t^k}{r^{2k}}\right) \left\{\log^+ |E(te^{i(\theta_0-\varepsilon)})| + \log^+ |E(te^{i(\theta_0+\varepsilon)})|\right\} \frac{dt}{t} \\
&\quad + \frac{2k}{\pi r^k} \int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon} \log^+ |g(re^{i\theta})| \sin k(\theta - \theta_0 + \varepsilon) d\theta \qquad (2.22)\\
&\leq \log^+ M(\Omega((\theta_0 - \varepsilon, \theta_0 + \varepsilon), r), E) \left\{\frac{2k}{\pi} \int_1^r \frac{1}{t^{k+1}} dt + \frac{2k}{\pi r^k} \int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon} d\theta\right\} \\
&\leq 2\log^+ M(\Omega((\theta_0 - \varepsilon, \theta_0 + \varepsilon), r), E),
\end{aligned}$$

where $k = \frac{\pi}{2\varepsilon}, r \geq 1$.

It follows from (2.21), (2.22) and $\lambda_{2,\theta_0} = +\infty$ that

$$\begin{aligned}
\overline{\lim_{r\to\infty}} \frac{\log\log\log M(\Omega((\theta_0 - \varepsilon, \theta_0 + \varepsilon), r), E)}{\log r} &= \overline{\lim_{r\to\infty}} \frac{\log\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r, E)}{\log r} \\
&= +\infty.
\end{aligned}$$

The proof of Theorem 2.5 is completed. □

3. A relation between a cluster ray of zeros and Borel direction

We begin with some preparations.

Let $g(z)$ be an entire function with the hyper order ρ $(0 < \rho \leq +\infty)$. A ray $L : \arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ is called a cluster ray of zeros of $g = a$ with hyper order $\lambda_{2,\theta_0}(g)$, if for any small positive number ε, we have

$$\overline{\lim_{r\to+\infty}} \frac{\log\log n(\Omega((\theta_0-\varepsilon, \theta_0+\varepsilon), r), g = a)}{\log r} = \lambda_{2,\theta_0}(g).$$

It is similar to the proof of Theorem 3.11 in [13], we have:

Lemma 3.1. *Suppose that $g(z)$ is meromorphic in $\mathbb{C}$ with the hyper order ρ $(0 < \rho \leq +\infty)$. If*

$$B : \quad \arg z = \theta_0, \quad 0 \leq \theta_0 < 2\pi,$$

is a Borel direction of $g(z)$ with the hyper order ρ, then there exists a sequence of filling disks Γ_j of $g(z)$ with the hyper order ρ such that

$$\Gamma_j : |z - z_j| < \varepsilon_j |z_j| \ (j = 1, 2, \ldots), z_j = |z_j| e^{i\theta_0}, \lim_{j\to+\infty} |z_j| = +\infty, \lim_{j\to+\infty} \varepsilon_j = 0,$$

and $g(z)$ takes every complex value in Γ_j at least $n_j \geq \exp\{|z_j|^{\rho_j}\}$ times, except for some values which can be covered in two small disks with radii $\exp\{-n_j\}$ on the Riemann sphere, where $\rho_j \to \rho$ $(j \to +\infty)$.

The above disks of Lemma 3.1 are called filling disks of $g(z)$ with the hyper order ρ.

Lemma 3.2 ([7])**.** *Suppose that $\Omega(\alpha, \beta)$ and $\Omega(\alpha', \beta')$ are two angular domains such that $\alpha < \alpha' < \beta' < \beta$ and that $g(z)$ is an entire function and satisfies the hypotheses of Lemma 2.5. Then $g(z)$ has a Borel direction with hyper order $\sigma_2(\Omega(\alpha', \beta'), g)$ in the angular domain $\Omega(\alpha', \beta')$.*

As for a relation between a cluster ray of zeros and Borel direction, similar to Theorem 1.2, we have the following result.

Theorem 3.3. *Let $A_j(z)$ $(j = 0, 1, \ldots, n-1)$ be entire functions with order $\sigma(A_j) = +\infty$ and the hyper order $\sigma_2(A_j) = 0$ $(j = 0, 1, 2, \ldots, n-2)$, and let $f_1, f_2, \ldots, f_n$ be n linearly independent solutions of* (2.2)*. Set $E = f_1 f_2 \cdots f_n$. Suppose that the hyper order exponent of convergence of zeros sequence of E is $+\infty$. Then a ray $\arg z = \theta$ from the origin is a Borel direction of E with the hyper order $+\infty$ and $\rho_2(\Omega(\theta - \varepsilon, \theta + \varepsilon), E) = +\infty$ if and only if $\lambda_{2,\theta}(E) = +\infty$.*

Proof. It is well know that $\sigma_2(E) = +\infty$ in this case.

Necessary. We suppose that ray $\arg z = \theta_0$ is a Borel direction of E with the hyper order $\rho = +\infty$. We then prove that a ray $\arg z = \theta_0$ is a cluster ray of zeros of E with the hyper order $\rho = +\infty$.

By Lemma 3.1, there exists a sequence of filling discs Γ_j of E with the hyper order $\rho = +\infty$ such that

$$\Gamma_j : |z - z_j| < \varepsilon_j |z_j|, z_j = |z_j| e^{i\theta_0}, \lim_{j\to+\infty} |z_j| = +\infty, \lim_{j\to+\infty} \varepsilon_j = 0, j = 1, 2, \ldots$$

and $g(z)$ takes every complex value in Γ_j at least $n_j \geq \exp\{|z_j|^{\rho_j}\}$ times, except for some values which can be covered in two small disks with radii $\exp\{-n_j\}$ on the Riemann sphere, where $\rho_j \to \rho$ $(j \to +\infty)$.

For any given $\varepsilon > 0$, we have

$$\Gamma_j \subset \Omega_\varepsilon(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$$

for all sufficiently large j.

Noting $E(z)$ is an entire function, ∞ is a Picard exceptional value. Therefore, ∞ lies in one of two small spherical disks in the definition of filling disks.

Let us denote the spherical distance of z_1, z_2 by $|z_1, z_2|$. Therefore, for all sufficiently large j, we can find a point $a_j \in \Gamma_j$ such that

$$|E(a_j), \infty| = \frac{1}{(1 + |E(a_j)|^2)^{\frac{1}{2}}} \leq 2\exp\{-n_j\}.$$

Thus we can find a positive constant c not dependent on j such that

$$|E(a_j)| > c\exp\{n_j\} \geq c\exp\{\exp(|z_j|^{\rho_j})\}$$

for all sufficiently large j.

Noting $|a_j| = (1 + o(1))|z_j|$ and M is arbitrary sufficiently large, we have

$$\overline{\lim_{r\to+\infty}} \frac{\log\log\log M(\Omega(\theta_0 - \varepsilon, \theta_0 + \varepsilon) \cap \overline{D}(0, r), E)}{\log r} = +\infty,$$

where $\overline{D}(0, r) = \{z; |z| \leq r\}$.

Together with the hypotheses of Theorem 3.3, E satisfies the sufficient condition of Theorem 2.5. Thus, we have $\lambda_{2,\theta_0}(E) = +\infty$, i.e., $\arg z = \theta_0$ is a cluster ray of zeros of E with hyper order $+\infty$.

Sufficiency. Suppose that $\arg z = \theta_0$ is a ray such that $\lambda_{2,\theta_0}(E) = +\infty$. By Theorem 2.5 again, we have

$$\begin{aligned}\overline{\lim_{r\to+\infty}} \frac{\log\log\log M(\Omega((\theta_0 - \varepsilon, \theta_0 + \varepsilon), r), E)}{\log r} &= \overline{\lim_{r\to+\infty}} \frac{\log\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r, E)}{\log r} \\ &= +\infty\end{aligned}$$

for all sufficiently small $\varepsilon > 0$.

It follows from (2.11) and $\sigma_2(A_j) = 0$ $(j = 0, 1, 2, \ldots, n-2)$ that E satisfies the conditions of Lemma 3.2. Hence $E(z)$ has a Borel direction with the hyper order $+\infty$ in the angular domain $\Omega(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$. Since ε is arbitrary, the ray $\arg z = \theta_0$ is a Borel direction of E with the hyper order $+\infty$. This concludes the proof of Theorem 3.3. $\square$

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