Journal of the Korean Data & Information Science Society
2010, **21**(1), 195–200

한국데이터정보과학회지

Reliability and ratio in a right truncated Rayleigh distribution†

Jang-Choon Lee[1] · Chang-Soo Lee[2]

[1] Division of Computer Engineering, Taegu Science College
[2] Department of Mobile Engineering, Kyungwoon University
Received 30 November 2009, revised 7 January 2010, accepted 12 January 2010

Abstract

In this paper, we consider estimators and a confidence interval for a reliability in two independent right truncated Rayleigh distributions and consider the density of a ratio in two independent right truncated Rayleigh distributions. And we obtain the density of an estimator for a changing point in the density of a ratio in two independent right truncated Rayleigh distributions.

Keywords: MLE, ratio, reliability, right truncated Rayleigh distribution

1. Introduction

For two independent random variables X and Y, and a real number c, the probability $P(X < cY)$ is as given in Woo (2006): (i) it is a reliability when $c = 1$, (ii) it is a distribution of a ratio $X/(X+Y)$ when $c = t/(1-t)$ for $o < t < 1$.

McCool (1991) and Ali and Woo (2005) studied a inference on a reliability in the Weibull distribution and Levy distribution, respectively. Bowman and Shenton (1998) and Ali *et al.* (2005) studied the distribution of a ratio in a gamma distribution with the unit shape parameter and a power function distribution, respectively. Woo (2006) provided a reliability and a ratio in two independent random variables. Woo (2007) studied a reliability in a half-triangle distribution. A truncated Rayleigh distribution has been widely applied to a reliability of a life time in Saunders (2007). Woo (2008) studied reliability estimations and a density function of a ratio in two independent different variates. Moon and Lee (2009) studied a inference on the reliability $P(X < Y)$ in the gamma case. Moon *et al.* (2009) considered a reliability and a ratio in two exponentiated complementary power function distributions.

In this paper, we consider estimators and a confidence interval for a reliability in two independent right truncated Rayleigh distributions and consider the density of a ratio in two independent right truncated Rayleigh distributions. We obtain the density of an estimator for a changing point in the density of a ratio in two independent right truncated Rayleigh distribution

† This study was supported by the fund of Taegu Science College 2009 College Competency Upgrade Program.
[1] Assistant Professor, Division of Computer Engineering, Taegu Science College, Daegu 702-723, Korea.
[2] Corresponding author: Associated Professor, Department of Mobile Engineering, Kyungwoon University, Gumi 730-850, Korea. E-mail: cslee@ikw.ac.kr

2. Reliability estimation

The density function of a right truncated Rayleigh life time with a truncated point θ was as given in Johnson *et al.* (1994):

$$F'(x;\theta) = f(x;\theta) = 2xe^{-x^2}/(1-e^{-\theta^2}),\ 0 < x < \theta. \tag{2.1}$$

Let $X \sim f(x;\theta_1)$ and $Y \sim f(y;\theta_2)$ be independent life times. Then the reliability is given as:

$$R \equiv P(Y < X) = (1-e^{-\theta_1^2})/(2(1-e^{-\theta_2^2})),\ \text{if } \theta_1 \le \theta_2 \tag{2.2}$$

and

$$R = 1-(1-e^{-\theta_2^2})/(2(1-e^{-\theta_1^2})),\ \text{if } \theta_2 < \theta_1. \tag{2.3}$$

From the equations (2.2) and (2.3), it's clear that $P(Y < X) = 0.5$ when $\theta_1 = \theta_2$.

Here, we consider the estimation of $R = P(Y < X)$ when $\theta_1 \le \theta_2$ because we can consider it in the similar manner when $\theta_1 > \theta_2$.

Assume $X_1, ..., X_m$ are iid life times from $X \sim f(x;\theta_1)$ and $Y_1, ..., Y_n$ are iid life times from $Y \sim f(x;\theta_2)$. And $X_i's$ and $Y_j's$ are two independent random samples.

Let $X_{(m)}$ and $Y_{(n)}$ be the corresponding greatest order statistics. Then the corresponding density functions are given as:

$$f_{X_{(m)}}(x) = \frac{m}{(1-e^{-\theta_1^2})^m}(1-e^{-x^2})^{m-1}\cdot 2xe^{-x^2},\ 0 < x < \theta_1$$

and

$$f_{Y_{(n)}}(y) = \frac{n}{(1-e^{-\theta_2^2})^n}(1-e^{-y^2})^{n-1}\cdot 2ye^{-y^2},\ 0 < y < \theta_2. \tag{2.4}$$

When $\theta_1 \le \theta_2$, a MLE (Maximum Likelihood Estimator) of the reliability is given by :

$$\widehat{R} = \widehat{P(Y \le X)} = (1-e^{-X_{(m)}^2})/(2(1-e^{-Y_{(n)}^2})).$$

From two density functions (2.4), we can obtain the k-moment of a MLE $\widehat{R}$ as follows:

$$E(\widehat{R}^k) = \frac{mn}{(m+k)(n-k)}R^k,\ \text{if } k < n,\quad \theta_1 \le \theta_2. \tag{2.5}$$

From the k-th moment (2.5), the expectation of the a MLE $\widehat{R}$ is given as:

$$E(\widehat{R}) = \frac{mn}{(m+1)(n-1)}R. \tag{2.6}$$

From the expectation (2.6), we define an unbiased estimator $\widehat{R}_U$ of a reliability as follows:

$$\widehat{R}_U = \frac{(m+1)(n-1)}{2mn}\cdot\frac{1-e^{-X_{(m)}^2}}{1-e^{-Y_{(n)}^2}}. \tag{2.7}$$

From the k-th moment (2.5) and an unbiased estimator $\widehat{R}_U$ in (2.7), we obtain mean squared errors (MSEs) of $\widehat{R}$ and $\widehat{R}_U$:

$$MSE(\widehat{R}) = \left[\frac{mn}{(m+2)(n-2)} - \frac{2mn}{(m+1)(n-1)} + 1\right] R^2, n > 2$$

and

$$MSE(\widehat{R_U}) = \left[\frac{(m+1)^2(n-1)^2}{mn(m+2)(n-2)} - 1\right] R^2, n > 2. \tag{2.8}$$

From the results (2.8), we obtain the following.

Fact 1. Assume $\theta_1 \leq \theta_2$ in (2.2). If $n > m \geq 2$, then a MLE $\widehat{R}$ is more efficient in the sense of the MSEs than an unbiased estimator $\widehat{R}_U$, and vice versa if $2 < n \leq m$.

Next, we consider a confidence interval of a reliability. From a quotient distribution in Rohatgi (1976) and formula 3.381(1) in Gradshteyn and Ryzhik (1965), the density function for a MLE $\widehat{R}$ is obtained as follows:

$$f_{\widehat{R}}(x) = \begin{cases} \dfrac{mn}{m+n} R^{-m} \cdot x^{m-1}, & \text{if } 0 < x < R \\ \dfrac{mn}{m+n} R^{n} \cdot x^{-n-1}, & \text{if } x \geq R. \end{cases} \tag{2.9}$$

From the density function (2.9), $Q \equiv \widehat{R}/R$ is a pivotal quantity with the following density:

$$f_Q(x) = \begin{cases} \dfrac{mn}{m+n} x^{m-1}, & \text{if } 0 < x < 1 \\ \dfrac{mn}{m+n} x^{-n-1}, & \text{if } x \geq 1\ . \end{cases} \tag{2.10}$$

For given $0 < p_i < 1$. $i = 1, 2$, from the density function (2.10) of the pivotal quantity Q , we provide a lower limit and an upper limit as follows

$$\int_0^{l(p_1)} f_Q(x)dx = p_1 \text{and} \int_{u(p_2)}^{\infty} f_Q(x)dx = p_2.$$

Hence, a $(1 - p_1 - p_2)100\%$ confidence interval for reliability when $\theta_1 \leq \theta_2$ is

$$(u(p_2)^{-1} \cdot \widehat{R}, l(p_1)^{-1} \cdot \widehat{R}) = \left(\left[p_2 \cdot (1 + \frac{n}{m}) \right]^{1/n} \cdot \widehat{R}, \left[p_1 (1 + \frac{m}{n}) \right]^{-1/m} \cdot \widehat{R} \right). \tag{2.11}$$

Remark 1. When $\theta_1 > \theta_2$, we consider a reliability estimation by the similar manner which we consider a reliability estimation when $\theta_1 \leq \theta_2$ in (2.2).

3. Distribution of ratio X/(X+Y)

Let $X \sim f(x;\theta_1)$ and $Y \sim f(y;\theta_2)$ be independent life times. From a quotient density in Rohatigi (1976) and formula 3.381(1) in Gradshteyn and Ryzhik (1965), we obtain the density of a quotient $W = Y/X$ as follows

$$f_W(w) = \begin{cases} \dfrac{2}{(1-e^{-\theta_1^2})((1-e^{-\theta_2^2})} \cdot \dfrac{w}{(1+w^2)^2} \cdot \gamma(2,(1+w^2)\theta_1^2), & \text{if } 0 < w < \theta_2/\theta_1 \\ \dfrac{2}{(1-e^{-\theta_1^2})(1-e^{-\theta_2^2})} \cdot \dfrac{w}{(1+w^2)^2} \cdot \gamma(2,\theta_2^2(1+w^2)/w^2), & \text{if } w \geq \theta_2/\theta_1 \ , \end{cases} \tag{3.1}$$

where $\gamma(k,x)$ is an incomplete gamma function in Gradshteyn and Ryzhik (1965).

From the density (3.1) we obtain the density of a ratio $V = X/(X+Y)$ as follows

$$f_V(v) = \begin{cases} \dfrac{2}{(1-e^{-\theta_1^2})(1-e^{-\theta_2^2})} \cdot \dfrac{v(1-v)}{(v^2+(1-v)^2)^2} \cdot \gamma\left(2, \theta_2^2 \dfrac{v^2+(1-v)^2}{(1-v)^2}\right), \\ \text{if } 0 < v < \theta_1/(\theta_1+\theta_2) \\ \dfrac{2}{(1-e^{-\theta_1^2})(1-e^{-\theta_2^2})} \cdot \dfrac{v(1-v)}{(v^2+(1-v)^2)^2} \cdot \gamma\left(2, \theta_1^2 \dfrac{v^2+(1-v)^2}{v^2}\right), \\ \text{if } \theta_1/(\theta_1+\theta_2) \leq v < 1. \end{cases} \tag{3.2}$$

It's clear that the density (3.2) is symmetric about 1/2 when $\theta_1 = \theta_2$.

From the density (3.2) of a ratio and formula 8.352(1) in Gradshteyn and Ryzhik (1965), the approximate values of the mean, variance, and coefficient of skewness by integral computations are given in Table 3.1.

Table 3.1 Mean, variance, and coefficient of skewness of the ratio density (3.2)

θ_1	θ_2	mean	variance	skewness
0.5	0.5	0.50000	0.02386	0.00000
0.25	0.75	0.28135	0.01807	1.16405
0.75	0.25	0.71865	0.18072	-1.16696
1.0	1.0	0.50000	0.02734	0.00000
1.25	1.75	0.46811	0.03119	0.14244
1.75	1.25	0.53189	0.03120	-0.14325
1.5	1.5	0.50000	0.03153	0.00000
2.0	2.0	0.50000	0.03430	0.00000
1.5	3.5	0.47817	0.03339	0.08437
3.5	1.5	0.52183	0.03340	-0.08526
3.5	3.5	0.50000	0.03540	0.00000

From Table 3.1, we observe the following:

Fact 2. (a) Mean and variance of ratio having $\theta_1 > \theta_2$ are greater than those of a ratio having $\theta_1 < \theta_2$.

(b) Variance of a ratio are slightly increasing as $\theta_1 = \theta_2 = \theta \leq 3.5$ is increasing.

(c) The density function (3.2) is skewed to the left when $\theta_1 > \theta_2$, but skewed to the right when $\theta_1 < \theta_2$.

Let's consider the density of an estimator of a changing point $\theta \equiv \theta_1/(\theta_1+\theta_2)$ in the density of a ratio $V = X/(X+Y)$.

The MLE of a changing point $\theta \equiv \theta_1/(\theta_1+\theta_2)$ is given by

$$\widehat{\theta} \equiv \widehat{\theta}_1/(\widehat{\theta}_1+\widehat{\theta}_2) = X_{(m)}/(X_{(m)}+Y_{(n)}). \tag{3.3}$$

From the pdfs (probability density functions) (2.4) of $X_{(m)}$ and $Y_{(n)}$ and the formula 3.381(1) in Gradshteyn and Ryzhik (1965), the density function of MLE $\hat{\theta}$ is obtained as follows:

For $\dfrac{\theta_1}{\theta_1+\theta_2} < x < 1$,

$$f_{\widehat{\theta}}(x) = \frac{mn}{(1-e^{-\theta_1^2})^m(1-e^{-\theta_2^2})^n}\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{n-1}{i}\binom{m-1}{j}\cdot$$

$$\frac{2}{(1-x)^2}\cdot\frac{x}{1-x}\frac{\gamma(2,[1+i+(1+j)(\frac{1-x}{x})^2]\theta_1^2)}{[1+i+(1+j)(\frac{x}{1-x})^2]^2}$$

and for $0 < x \le \dfrac{\theta_1}{\theta_1+\theta_2}$,

$$f_{\widehat{\theta}}(x) = \frac{mn}{(1-e^{-\theta_1^2})^m(1-e^{-\theta_2^2})^n}\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{n-1}{i}\binom{m-1}{j}\cdot$$

$$\frac{2}{x^2}\cdot\frac{1-x}{x}\frac{\gamma(2,[1+j+(1+i)(\frac{x}{1-x})^2]\theta_2^2)}{[1+i+(1+j)(\frac{1-x}{x})^2]^2},$$

which is symmetric about 1/2 when $\theta_1 = \theta_2$ and $m = n$.

References

Ali, M. M. and Woo, J. (2005). Inference on reliability $P(T < X)$ in the Levy distribution. *Mathematical and Computing Modelling*, **41**, 965-971.

Ali, M, M., Woo, J. and Nadarajah, S. (2005). On the ration $X/(X+Y)$ for the power function distribution. *Pakistan Journal of Statistics*, **21**, 131-138.

Bowman, K. O. and Shenton, I. R. (1998). Distribution of the ratio of gamma variates. *Communication in Statistics-Simulations*, **27**, 1-19.

Gradshteyn, I. S. and Ryzhik, I. M. (1965). *Tables of Integrals, Series, and Products*, Academic Press, New York.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions* **I**, 2nd Ed., John Wiley & Sons, New York.

McCool, J. I. (1991). Inference on $P(T < X)$ in the Weibull case. *Communications in Statistics-Simulations*, **20**, 129-148.

Moon, Y. G. and Lee, C. S. (2009). Inference on the reliability $P(X < Y)$ in the gamma case. *Journal of the Korean Data & Information Science Society*, **20**, 219-223.

Moon, T. G., Lee, C. S. and Ryu, S. G. (2009). Reliability and ratio in exponentiated complementary power function distribution. *Journal of the Korean Data & Information Sciences Society*, **20**, 955-960.

Rohatgi, V. K. (1976). *An introduction to probability theory and mathematical statistics*, John Wiley & Sons, New York.

Saunders, S. C. (2007). *Reliability, life testing, and prediction of service lives*, Springer, New York.

Woo, J. (2006). Reliability $P(T < X)$, ratio $X/(X+Y)$, and a skewed-symmetric distribution of two independent random variables. *Proceedings of Korean Data & Information Science Society*, 37-42.

Woo, J. (2007). Reliability in a half-triangle distribution and a skew-symmetric distribution. *Journal. of the Korean Data & Information Science Society*, **18**, 543-552.

Woo, J. (2008). Estimating reliability and distribution of ratio in two independent different variates. *Journal of the Korean Data & Information Science Society*, **19**, 967-977.