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Cusum of squares test for discretely observed sample from diffusion processes†

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Abstract

In this paper, we consider the change point problem in diffusion processes based on discretely observed sample. Particularly, we consider the change point test for the dispersion parameter when the drift has unknown parameters. In performing a test, we employ the cusum of squares test based on the residuals. It is shown that the test has a limiting distribution of the sup of a Brownian bridge. A simulation result as to the Ornstein-Uhlenbeck process is provided for illustration. It demonstrates the validity of our test.

Keywords: Diffusion process, discretely observed sample, residual based cusum test.

1. Introduction

The diffusion process has long been popular in analyzing random phenomena occurring in various fields such as finance, engineering, physical and medical sciences. As a representative text, we can refer to Karatzas and Shreve (1991) and Shiryayev (1999). Since the application of diffusion processes to real world is various, much attraction has been drawn to statistical inference for diffusion processes and many sophisticated methods have been established by researchers and practitioners. For a general review, we refer to Prakasa Rao (1999) and Kutoyants (2004). According to past experience, time series models are not well fitted to financial time series in many occasions due to structural changes governed by the change of financial policies and critical events. This phenomenon is observed in most financial time series data with high volatility. See, for instance, Lee *et al.* (2004), who empirically verified by using the cusum test that most stock prices of Nikei 225 suffer from serious parameter

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changes when the underlying model of the data is assumed to be a GARCH(1,1) model. With regard to the parameter change test for time series models, we refer to Lee *et al.* (2003), Park and Lee (2006), Park and Lee (2007), Song *et al.* (2007), Choi *et al.* (2006). See also Lee *et al.* (2006) and the articles cited in these papers for continuous time stochastic processes. Recently, Gregorio and Iacus (2008) considered the change point test for the dispersion parameter in diffusion processes based on discretely observed sample. However, they only handled the case that the drift is completely known with no unknown parameters. Thus, in this paper, we consider the change point test for diffusion processes when the drift has unknown parameters.

In Section 2, it is shown that under regularity conditions, the cusum of squares test based on residuals has a limiting distribution of the sup of a Brownian bridge. In Section 3, we carry out a simulation study.

2. Main result

Let us consider the stochastic differential equation:

$$dX_t = a(X_t;\theta)dt + \sigma dW_t, \quad X_0 = x_0, \ t \geq 0, \tag{2.1}$$

where θ is a p-dimensional unknown parameter, a is a known real valued function, and $W = \{W_t; t \geq 0\}$ is a standard Brownian motion. Suppose that X_{t_i}, $t_i = ih_n, i = 1, \ldots, n$, are observed, where $\{h_n\}$ is a sequence of positive real numbers such that $h_n \to 0$ and $nh_n \to \infty$, and one wishes to test the following hypotheses:

$$H_0 : \sigma \text{ is constant over } i = 1, \ldots, n \text{ vs. not } H_0.$$

To task this, we assume that

(A1) There exist constants $C, m > 0$ such that for any θ and x, y,

$$|a(x;\theta) - a(y;\theta)| \leq C|x-y|,$$
$$\sup_{\theta' \in N_\theta} ||\dot{a}(x;\theta')|| \leq C(1+|x|^m),$$

where $\dot{a} = \partial a/\partial\theta$, and N_θ is an open neighborhood of θ.

(A2) Under H_0, $\sup_t E|X_t|^\gamma < \infty$ for all $\gamma > 0$.

(A3) Under H_0, there exists an estimator $\hat{\theta}_n$ of θ, such that $(nh_n)^{1/2}(\hat{\theta}_n - \theta) = O_P(1)$.

(A4) $nh_n^4 \to 0$ as $n \to \infty$.

Sufficient conditions for (A3) can be found in Kessler (1997).

By using the Euler approximation, we can express

$$X_{t_i} - X_{t_{i-1}} \simeq h_n a(X_{t_{i-1}};\theta) + \sigma(W_{t_i} - W_{t_{i-1}}).$$

In view of this, we define the residuals as

$$\hat{\eta}_i = \{X_{t_i} - X_{t_{i-1}} - h_n a(X_{t_{i-1}};\hat{\theta}_n)\}/h_n^{1/2}. \tag{2.2}$$

Then, as in Lee *et al.* (2004), we consider the cusum of squares test based on the residuals and obtain the following result.

Theorem 2.1 Assume that (A1)-(A4) hold. Let

$$T_n = \frac{1}{\sqrt{n}\hat{\tau}_n} \max_{1\le k\le n} \left| \sum_{i=1}^{k} \hat{\eta}_i^2 - \left(\frac{k}{n}\right) \sum_{i=1}^{n} \hat{\eta_i}^2 \right|,$$

where $\hat{\tau}_n^2 = \frac{1}{n}\sum_{i=1}^n \hat{\eta_i}^4 - \left(\frac{1}{n}\sum_{i=1}^n \hat{\eta_i}^2\right)^2$. Then, under H_0,

$$T_n \xrightarrow{d} \sup_{0\le u\le 1} \left|W^0(u)\right|, \; n\to\infty,$$

where W^0 is a Brownian bridge.

Proof. Put $\eta_i = \sigma(W_{t_i} - W_{t_{i-1}})h_n^{-1/2}$, $\Delta_i = \int_{t_{i-1}}^{t_i}\{a(X_s;\theta) - a(X_{t_{i-1}};\theta)\}ds$, and $d_i = a(X_{t_{i-1}};\theta) - a(X_{t_{i-1}};\hat{\theta}_n)$. Then, we can express $\hat{\eta}_i = \eta_i + \Delta_i h_n^{-1/2} + d_i h_n^{1/2}$. Note that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \Delta_i^2 h_n^{-1} = o_P(1), \tag{2.3}$$

where we have used (A4) and the fact that $E\Delta_i^2 \le Ch_n^3$ for some $C > 0$ (cf. Kessler, 1997). Moreover, by (A1)-(A3), for all large n,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} d_i^2 h_n \le C^2 \sum_{i=1}^{n}(1 + |X_{t_{i-1}}|^m)^2 ||\hat{\theta}_n - \theta||^2 h_n = o_P(1). \tag{2.4}$$

By using (2.3) and (2.4), we can eventually get

$$\max_{1\le k\le n} \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \left|\hat{\eta}_i^2 - \eta_i^2\right| = o_P(1). \tag{2.5}$$

Since $\eta_1, \ldots, \eta_n$ are iid $N(0, \sigma^2)$, we have

$$\frac{1}{\sqrt{n}\tau} \max_{1\le k\le n} \left| \sum_{i=1}^{k} \eta_i^2 - \left(\frac{k}{n}\right) \sum_{i=1}^{n} {\eta_i}^2 \right| \xrightarrow{d} \sup_{0\le u\le 1} \left|W^0(u)\right|,$$

where τ^2 is the variance of η_i^2. Consequently, (2.5) implies that $(\hat{\tau}_n/\tau)T_n$ converges to $\sup_{0\le u\le 1}\left|W^0(u)\right|$ in distribution. Since $\hat{\tau}_n^2$ converges to τ^2 in probability, we establish the theorem. □

Remark. The test does not detect the change in the drift parameter as will be seen in our simulation study. Thus, the test is free from any drift changes.

3. Simulation study

In this section, we evaluate the performance of the cusum of squares test through a simulation study. In this study, we consider the Ornstein-Uhlenbeck (O-U) process:

$$dX_t = (\alpha - \mu X_t)dt + \sigma dW_t, \quad X_0 = 0, \; t \ge 0. \tag{3.1}$$

Table 3.1 Empirical sizes of T_n at nominal levels 0.1, 0.05 and 0.01

	level		
n	0.10	0.05	0.01
500	0.087	0.043	0.008
1000	0.098	0.046	0.010
2000	0.107	0.054	0.009

Table 3.2 Empirical powers of T_n at nominal level 0.05 when (α,μ,σ) changes from (0.5, 1.0, 1.0) to (α',μ',σ') at $[0.5n]$

	(α',μ',σ')		
n	(0.5,1.0,1.4)	(0.5,1.0,2.0)	(1.0,2.0,1.0)
500	0.997	1.000	0.040
1000	1.000	1.000	0.050
2000	1.000	1.000	0.055

The empirical sizes and powers are calculated as the rejection number of the null hypothesis 'H_0: No changes occur in σ' out of 1000 repetitions. The empirical quantile value for $\sup_{0\le u\le 1}|W^0(u)|$ can be found in Table 1 of Lee *et al.* (2003), page 784. In each simulation, we generate the sample with n=500, 1000 and 2000, and employ the sampling time length $h = n^{-1/3}$. For the empirical size, we consider the O-U process with α=0.5, μ=1 and $\sigma = 1$. The empirical sizes are calculated at the nominal level 0.01, 0.05 and 0.1, respectively. In order to examine the power, we consider the O-U process with α=0.5 and μ=1, and change σ from 1 to 1.4 and 2 at $[0.5n]$. We also consider the O-U process with σ=1 and change α and μ at $[0.5n]$. Table 3.1 shows that the cusum of squares test has no size distortions. Table 3.2 indicates that the test produces good powers. It is noteworthy that the test does not detect the change in the drift as anticipated. The simulation study enables us to conclude that the cusum of squares test is a proper tool to detect the change of the dispersion parameter in diffusion processes.

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