*Journal of the Korean Data &
Information Science Society*
2010, **21**(1), 173–177

한국데이터정보과학회지

Failure rate of a bivariate exponential distribution

Yeon Woong Hong[1]

[1]School of Business Administration, Dongyang University

Received 14 November 2009, revised 7 January 2010, accepted 12 January 2010

Abstract

It is well known that if the parent distribution has a nonnegative support and has increasing failure rate, then all the order statistics have increasing failure rate (IFR). The result is not necessarily true in the case of bivariate distributions with dependent structures. In this paper we consider a symmetric bivariate exponential distribution and show that, two marginal distributions are IFR and the distributions of the minimum and maximum are constant failure rate and IFR, respectively.

Keywords: Bivariate exponential distribution, failure rate, reliability.

1. Introduction

The distribution of minimum and maximum of two randoms X and Y play an important role in various statistical applications. For example in reliability studies, $\min(X, Y)$ is the observable time of failure if the components are arranged in the series system and $\max(X, Y)$ is the observable if the components are arranged in the parallel system.

In the case of independent and identically distributed random variables from distribution $F(\cdot)$, $\min(X, Y)$ and $\max(X, Y)$ constitute order statistics for a random sample of size 2 from a distribution $F(\cdot)$. In reliability theory literature, it is well known that if the parent distribution has a nonnegative support and has increasing failure rate (IFR), then all the order statistics have IFR, see for example Barlow and Porschan (1981), a monograph by Kamps (1995) and Finkelstein and Esaulova (2005).

Nagaraja and Baggs (1996) have studied the order statistics of bivariate exponential random variables and noted that even if the marginal distribution is IFR, $\min(X, Y)$ and $\max(X, Y)$ do not necessarily have IFR. For example, for Raftery's (1984) bivariate exponential distribution, the marginal distributions are exponential and yet the failure rate of $\min(X, Y)$ is non-monotonic for certain values of parameters.

In this paper we consider a random variable (X, Y) having a Freund's symmetric bivariate exponential distribution. We are interested in the failure rates of X, Y, $\min(X, Y)$ and $\max(X, Y)$. In section 2, we obtain the marginal distributions of X and Y for the bivariate exponential distribution, and in section 3, we show that 1) the failure rate of $X(Y)$ is decreasing failure rate (DFR), constant failure rate (CFR), or IFR according to the value of parameter, 2) the failure rate of $\min(X, Y)$ is CFR, 3) the failure rate of $\max(X, Y)$ is IFR, and 4) the conditional failure rate of $\max(X, Y)$ given $\min(X, Y)$ is CFR.

[1] Professor, School of Business Administration, Dongyang University, Kyungbuk 750-711, Korea.
E-mail: hong@dyu.ac.kr

2. Marginal distribution of $X(Y)$

Consider a random variable (X, Y) having the Freund's (1961) bivariate exponential distribution,

$$f(x,y) = \begin{cases} \alpha\beta^{'} \exp[-\beta^{'} y - (\alpha+\beta-\beta^{'})x], & 0 < x < y, \\ \\ \alpha^{'}\beta \exp[-\alpha^{'} x - (\alpha+\beta-\alpha^{'})y], & 0 < y < x. \end{cases} \tag{2.1}$$

where $\alpha, \alpha', \beta, \beta^{'} > 0$. This distribution arises in the following setting: X and Y are lifetimes of two components assumed to be independent exponentials with parameters α, β, respectively; but, a dependence between X and Y is introduced by taking a failure of either component to change the parameter of the life distribution of the other component; if component 1 fails, the parameter for Y is changed to β'; and if component 2 fails, the parameter for X is changed to α'. In this paper, we assume that $\alpha = \beta = \lambda$ and $\alpha' = \beta' = \theta\lambda$ for $\theta > 0$ in the model (2.1). Then (2.1) can be simplified as the symmetric bivariate exponential distribution

$$f(x,y) = \theta\lambda^2 \exp\left\{-2\lambda \min(x,y) - \theta\lambda\, |x-y|\right\}, \tag{2.2}$$

Weier (1981) and Hong *et. al.* (1995) considered the estimation problems for this model. The identical marginal densities are given by

$$f_X(t) = f_Y(t) = \frac{1-\theta}{2-\theta} \cdot 2\lambda \exp(-2\lambda t) + \frac{1}{2-\theta} \cdot \theta\lambda \exp(-\theta\lambda t) \quad if \theta \neq 2, \tag{2.3}$$

and

$$f_X(t) = f_Y(t) = \frac{1}{2} \cdot 2\lambda \exp(-2\lambda t) + \frac{1}{2} \cdot (2\lambda)^2 t \exp(-2\lambda t) \quad if \theta = 2. \tag{2.4}$$

The marginal density of X is the same as that of Y, but this does not provide any information about the independency.

Remark 1. Note that the marginal density of X is not exponential, but a mixture of two exponential densities when $\theta \neq 2$ and a mixture of exponential and gamma densities when $\theta = 2$.

Now the expectation of $X(Y)$ is obtained as

$$E(X) = E(Y) = \frac{\theta+1}{2\theta\lambda}.$$

The correlation coefficient is given by

$$\rho = \frac{\theta^2 - 1}{\theta^2 + 3}.$$

We can show that $-1/3 < \rho < 1$.

Remark 2. If $\theta = 1$, then $\rho = 0$ and X and Y are independent random variables having exponential distributions

$$f_X(t) = \lambda \exp(-\lambda t) \ and \ f_Y(t) = \lambda \exp(-\lambda t),$$

respectively.

3. Failure rates

3.1. Failure rates of X(Y)

From (2.3) and (2.4), the reliability functions are easily found to be

$$\boldsymbol{R}_X(t) = \boldsymbol{R}_Y(t) = \frac{1-\theta}{2-\theta}\exp(-2\lambda t) + \frac{1}{2-\theta}\exp(-\theta\lambda t) \quad \text{if } \boldsymbol{\theta} \neq 2 \tag{3.1}$$

and

$$\boldsymbol{R}_X(t) = \boldsymbol{R}_Y(t) = (1+\lambda t)\exp(-2\lambda t) \quad \text{if } \boldsymbol{\theta} = 2 \tag{3.2}$$

Now the failure rate function of $X(Y)$ is easily obtained by dividing the marginal density of $X(Y)$ by the corresponding reliability function. That is,

$$h_X(t) = h_Y(t) = \begin{cases} \dfrac{2(1-\theta)\lambda\exp(-2\lambda t) + \theta\lambda\exp(-\theta\lambda t)}{(1-\theta)\exp(-2\lambda t) + \exp(-\theta\lambda t)}, & \text{if } \theta \neq 2, \\ \lambda + \dfrac{\lambda^2}{(1+\lambda t)^2}, & \text{if } \theta = 2. \end{cases} \tag{3.3}$$

This gives

$$\begin{aligned} \frac{dh_X(t)}{dt} &= \frac{df_X(t)/dt \cdot R_X(t) + f_X^2(t)}{R_X(t)^2} \\ &= \begin{cases} \dfrac{(\theta-1)\lambda^2\exp-(\theta+2)\lambda t}{R_X(t)^2} \begin{cases} <0, & \text{if } 0<\theta<1, \\ =0, & \text{if } \theta = 1, \\ >0, & \text{if } \theta > 1 \text{ and } \theta \neq 2, \end{cases} \\ \dfrac{\lambda^2}{(1+\lambda t)^2} > 0, \qquad\qquad \text{if } \theta = 2. \end{cases} \end{aligned}$$

Remark 3. Note that if $0 < \theta < 1$, $h_X(t)(h_Y(t))$ is DFR, if $\theta = 1$, $h_X(t)(h_Y(t))$ is CFR, and if $\theta > 1$, $h_X(t)(h_Y(t))$ is IFR.

3.2. Failure rates of $\min(\mathbf{X},\mathbf{Y})$ and $\max(\mathbf{X},\mathbf{Y})$

Let $U = \min(X,Y)$, $V = \max(X,Y)$, then the joint pdf of $(U,\ V)$ is given by

$$g(u,v) = 2\theta\lambda^2\exp\left\{-(2\lambda-\theta\lambda)u - \theta\lambda v\right\}, 0<u<v.$$

The marginal densitie of U is given by

$$g_U(u) = 2\lambda\exp(-2\lambda u).$$

This gives the failure rate of U as $h_U(t) = 2\lambda$, which is CFR. And the marginal density of V is

$$g_V(v) = \begin{cases} \dfrac{2\theta\lambda}{2-\theta}\left[\exp(-\theta\lambda v) - \exp(-2\lambda v)\right], & \theta \neq 2, \\ (2\lambda)^2 v\exp(-2\lambda v), & \theta = 2. \end{cases}$$

This gives the reliability function of V as

$$
\begin{aligned}
R_V(t) &= \int_t^{\infty} g_V(v)dt \\
&= \begin{cases} \dfrac{2}{2-\theta} \cdot \exp\left(-\theta\lambda t\right) - \dfrac{\theta}{2-\theta} \cdot \exp\left(-2\lambda t\right), & \theta \neq 2, \\ (1+2\lambda t) \cdot \exp\left(-2\lambda t\right), & \theta = 2. \end{cases}
\end{aligned}
$$

Then the failure rate function of V is obtained as

$$
h_V(t) = \begin{cases} 2\theta\lambda \cdot \left(\dfrac{1}{2-\theta\exp\left\{-\lambda(2-\theta)t\right\}} - \dfrac{1}{2exp\left\{\lambda(2-\theta)t\right\} - \theta} \right), & \theta \neq 2, \\ \dfrac{4\lambda^2 t}{1+2\lambda t}, & \theta = 2. \end{cases} \tag{3.4}
$$

This gives

$$
\frac{dh_V(t)}{dt} = \begin{cases} \dfrac{2\theta(\theta-2)^2\lambda^2 \exp\lambda(\theta-2)t}{[2-\theta\exp\lambda(\theta-2)t]^2} > 0, & \text{if } \theta \neq 2, \\ \dfrac{4\lambda^2}{1+2\lambda t} > 0, & \text{if } \theta = 2. \end{cases}
$$

Remark 4. The failure rate function $h_V(t)$ is IFR.
Now we consider the limiting properties of $h_V(t)$. Let $a(t) = \exp\left\{\lambda(\theta-2)t\right\}$, then

$$
\lim_{t\to\infty} a(t) = \begin{cases} \infty, & \text{if } \theta > 2, \\ 0, & \text{if} \theta < 2, \end{cases}
$$

and

$$
\lim_{t\to\infty} h_V(t) = \begin{cases} \lim_{t\to\infty} \dfrac{2\theta\lambda(1-a(t))}{2-\theta a(t)} = \lambda \cdot \min(\theta, 2), & if \theta < 2, \\ \lim_{t\to\infty} \dfrac{4\lambda^2 t}{1+2\lambda t} = 2\lambda, & if \theta = 2. \end{cases}
$$

Remark 5. The failure rate function $h_V(t)$ converges to $\lambda \cdot \min(\theta, 2)$ as $t \to \infty$.

3.3. Conditional failure rate of $\max(\mathbf{X}, \mathbf{Y})$ given $\min(\mathbf{X}, \mathbf{Y})$

The conditional density of V given $U = u$ is obtained as

$$
\begin{aligned}
g(v|u) &= g(u,v)/g_U(u) \\
&= \theta\lambda \exp -\theta\lambda(v-u), \quad v > u.
\end{aligned} \tag{3.5}
$$

This means that the distribution of the minimum has the exponential distribution with failure rate $\theta\lambda$. Thus the conditional reliability function is

$$
R(v|u) = \exp\left\{-\theta\lambda(v-u)\right\}, \quad v > u. \tag{3.6}
$$

Therefore the conditional failure rate function is given by, from (3.5) and (3.6),

$$h(t|U) = \theta\lambda.$$

Hence if the component lifelengths are identical, the conditional failure rate of the system given the first failure is constant. This is not equal to the failure rate of any component of the system.

References

Barlow, R. E. and Prochan, F. (1974). *Statistical theory of reliability and life testing: Probability models*, Holt, Rinehart and Winston, New York.

Cha, Y. J., Hong, Y. W. and Lee, J. M. (1999). Control chart for constant hazard rate. *Journal of the Korean Data & Information Science Society*, **10**, 437-444.

Cho, G. Y. and Lee, O. H. (2002). Control charts for constant failure rate of system. *Journal of the Korean Data & Information Science Society*, **13**, 147-156.

Finkelstein, M. and Esaulova, V. (2005). On the weak IFR aging of bivariate lifetime distributions. *Applied Stochastic Models in Business and Industry*, **21**, 265-272.

Freund, J. E. (1961). A bivariate extension of the exponential distribution. *Journal of the American Statistical Association*, **56**, 971-977.

Hong, Y. W. and Kwon, Y. M. (2002). Estimation of a bivariate exponential distribution with a location parameter. *Journal of the Korean Data & Information Science Society*, **13**, 243-250.

Hong, Y. W., Lee, J. M. and Cha, Y. J. (1995). Reliability estimation for a shared-load system based on Freund model. *Journal of the Korean Data & Information Science Society*, **6**, 1-7.

Kamps, U. (1995). *A concept of generalized order statistics*, Teubner BG Stuttgart.

Nagaraja, H. N. and Baggs, G. E. (1996). Order statistics of bivariate exponential random variables. In *Statistical Theory and Applications*, Springer Verlag, New York.

Raftery, A. E. (1984). A continuous multivariate exponential distribution. *Communications in Statistics-Theory and Methods*, **13**, 947-965.

Weier, D. R. (1981). Bayes estimation for a bivariate survival model based on exponential distributions. *Communications in Statistics-Theory and Methods*, **10**, 1415-1427.