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Bayesian hypothesis testing for homogeneity of coefficients of variation in k Normal populations[†]

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Abstract

In this paper, we deal with the problem for testing homogeneity of coefficients of variation in several normal distributions. We propose Bayesian hypothesis testing procedures based on the Bayes factor under noninformative prior. The noninformative prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. So we propose the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factor under the reference prior. Simulation study and a real data example are provided.

Keywords: Coefficients of variation, fractional Bayes factor, intrinsic Bayes factor, normal distribution, reference prior.

1. Introduction

The coefficient of variation is a very useful measure of precision and repeatability of data in medical and biological studies. In toxicology, the coefficient of variation is commonly used to measure the precision within and between laboratories, or among replicates for each treatment concentration. And the coefficient of variation is often used to assess the meter-to-meter variability when comparing different types of equipment that perform the same task (Plesch and Klimpel, 2002; Tian, 2005).

We consider that $\mathbf{X}_i = (X_{i1}, \cdots, X_{in_i}), i = 1, \cdots, k$, is a random sample of size n_i from a normal distribution with mean μ_i and variance $\mu_i^2\gamma_i^2$. Here γ_i is the coefficient of variation in i-th population. Then the joint probability density function is

$$\begin{aligned} & f(\mathbf{x}_1, \cdots, \mathbf{x}_k | \mu_1, \cdots, \mu_k, \gamma_1, \cdots, \gamma_k) \\ &= (2\pi)^{-\frac{n}{2}} \prod_{i=1}^{k} \mu_i^{-n_i} \gamma_i^{-n_i} \exp\left\{ -\sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{(x_{ij}-\mu_i)^2}{2\gamma_i^2\mu_i^2} \right\}, \end{aligned} \tag{1.1}$$

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where $n = n_1 + \cdots + n_k$ and $\mu_i > 0, i = 1, \cdots, k$. The present paper focuses on testing homogeneity of coefficients of variation in several normal distributions.

In Bayesian model selection or testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989; 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegelhalter and Smith (1982), O'Hagan (1995), and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the device of imaginary training sample in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so there is no guarantee that the Bayes factor of Spiegelhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper prior. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang *et al.*, 2005; 2006; 2008). An excellent exposition of the objective Bayesian method to model selection is Berger and Pericchi (2001).

For k normal populations, there exist several tests for testing the equality of coefficients of variation. Gupta and Ma (1996), Feltz and Miller (1996) and Fung and Tsang (1998) carried out a simulation study to compare several tests. Fung and Tsang (1998) concluded that the modified Miller (1991) asymptotic test of all tests is a very good test for normal distribution with respect to both sizes and power. Tian (2005) proposed a procedures for interval estimation and hypothesis testing for the common coefficient of variation based on the concepts of generalized confidence intervals and generalized p-values. Verrill and Johnson (2007) provided a likelihood ratio test for the equality of coefficients of variation. However there is a little work on the Bayesian inference about testing the homogeneity of coefficients of variation. Lee *et al.* (2003) provided a Bayesian test procedure for the equality of two coefficients of variation based on fractional Bayes factor.

In this paper, we propose the objective Bayesian hypothesis testing procedures based on the Bayes factors for the homogeneity of coefficients of variation in several normal distributions. The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factors. In Section 3, we provide the Bayesian hypothesis testing procedures based on the fractional Bayes and intrinsic Bayes factors. In Section 4, simulation study and a real data example are given.

2. Intrinsic and fractional Bayes factors

Suppose that hypotheses $H_1, \cdots, H_q$ are under consideration, with the data $\mathbf{x} = (x_1, \cdots, x_n)$ having probability density function $f_i(\mathbf{x}|\boldsymbol{\theta}_i)$ under hypothesis H_i. The parameter vectors $\boldsymbol{\theta}_i$ are unknown. Let $\pi_i(\boldsymbol{\theta}_i)$ be the prior distributions of hypothesis H_i, and let p_i be the prior probability of hypothesis $H_i, i = 1, \cdots, q$. Then the posterior probability that the hypothesis H_i is true is

$$P(H_i|\mathbf{x}) = \left(\sum_{i=1}^{q} \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{2.1}$$

where B_{ji} is the Bayes factor of hypothesis H_j to hypothesis H_i defined by

$$B_{ji} = \frac{\int f_j(\mathbf{x}|\boldsymbol{\theta}_j)\pi_j(\boldsymbol{\theta}_j)d\boldsymbol{\theta}_j}{\int f_i(\mathbf{x}|\boldsymbol{\theta}_i)\pi_i(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})}. \tag{2.2}$$

The B_{ji} interpreted as the comparative support of the data for H_i versus H_i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\boldsymbol{\theta}_i)$ and $\pi_j(\boldsymbol{\theta}_j)$. Often in Bayesian analysis, one can use noninformative priors π_i^N. Common choices are the uniform prior, Jeffreys' prior and the reference prior. The noninformative prior π_i^N is typically improper. Hence the use of noninformative prior π_i^N in (2.2) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor, and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\mathbf{x}(l)$ denote the part of the data to be so used and let $\mathbf{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\mathbf{x}(l)) < \infty, i = 1, \cdots, q. \tag{2.3}$$

In view (2.3), the posteriors $\pi_i^N(\boldsymbol{\theta}_i|\mathbf{x}(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, with the remainder of the data $\mathbf{x}(-l)$ using $\pi_i^N(\boldsymbol{\theta}_i|\mathbf{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int f_j(\mathbf{x}(-l)|\boldsymbol{\theta}_j, \mathbf{x}(l))\pi_j(\boldsymbol{\theta}_j|\mathbf{x}(l))d\boldsymbol{\theta}_j}{\int f_i(\mathbf{x}(-l)|\boldsymbol{\theta}_i, \mathbf{x}(l))\pi_i(\boldsymbol{\theta}_i|\mathbf{x}(l))d\boldsymbol{\theta}_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)), \tag{2.4}$$

where

$$B_{ji}^N = B_{ji}^N(\mathbf{x}) = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})} \text{ and } B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\mathbf{x}$ and training sample $\mathbf{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\mathbf{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \cdot \frac{1}{L} \sum_{l=1}^{L} B_{ij}^N(\mathbf{x}(l)), \tag{2.5}$$

where L is the number of all possible minimal training samples. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \cdot ME[B_{ij}^N(\mathbf{x}(l))], \tag{2.6}$$

where ME indicates the median for all possible training sample Bayes factors. Therefore we can also calculate the posterior probability of H_i using (2.1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (2.5) and (2.6), respectively.

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead o f using part of the data to turn noninformative priors into proper priors, it uses a fraction, b of each likelihood function, $L(\boldsymbol{\theta}_i|\mathbf{x}) = f_i(\mathbf{x}|\boldsymbol{\theta}_i)$, with the remaining $1-b$ faction of the likelihood used for model discrimination. Then the factional Bayes factor (FBF) of hypothesis H_j versus hypothesis H_i is

$$B_{ji}^F = \frac{\int L^b(\boldsymbol{\theta}_j|\mathbf{x})\pi_j^N(\boldsymbol{\theta}_j)d\boldsymbol{\theta}_j}{\int L^b(\boldsymbol{\theta}_i|\mathbf{x})\pi_i^N(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i} = B_{ji}^N \cdot \frac{m_i^b(\mathbf{x})}{m_j^b(\mathbf{x})}. \tag{2.7}$$

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined (O'Hagan (1995; 1997) and the discussion by Berger and Mortera in O'Hagan (1995)).

3. Bayesian hypothesis testing procedures

Consider that we have n_1 observations $X_{11}, \cdots, X_{1n_1}$ from the normal distribution $N(\mu_1, \mu_1^2\tau_1^2)$, n_2 observations $X_{21}, \cdots, X_{2n_2}$ from the $N(\mu_2, \mu_2^2\tau_2^2), \cdots$, and n_k observations $X_{k1}, \cdots, X_{kn_k}$ from the $N(\mu_k, \mu_k^2\tau_k^2)$, and that $\mu_1, \cdots, \mu_k > 0$. And we assume that all of theses observations are statistically independent. We are interest to testing the hypotheses $H_1 : \tau_1 = \cdots = \tau_k \equiv \tau$ versus $H_2 : \tau_1 \neq \cdots \neq \tau_k$ based on the fractional Bayes factor and the intrinsic Bayes factor.

3.1. Bayesian hypothesis testing based on the Fractional Bayes factor

Under the hypothesis H_1, the reference prior for $(\tau, \mu_1, \cdots, \mu_k)$ is

$$\pi_1^N(\tau, \mu_1, \cdots, \mu_k) \propto \tau^{-1}(1+2\tau^2)^{-\frac{1}{2}}\mu_1^{-1}\cdots\mu_k^{-1}. \tag{3.1}$$

This reference prior derived by Kim, Kang and Lee (2008). They showed that the posterior distribution under a general prior including the reference is proper. And the likelihood function is given by

$$L(\tau, \mu_1, \cdots, \mu_k|\mathbf{x}) = (2\pi)^{-\frac{n}{2}}\tau^{-n}\prod_{i=1}^k \mu_i^{-n_i} \exp\left\{-\frac{\sum_{j=1}^{n_i}(x_{ij}-\mu_i)^2}{2\mu_i^2\tau^2}\right\}, \tag{3.2}$$

where $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_k)$, $\mathbf{x}_i = (x_{i1}, \cdots, x_{in_i}), i = 1, \cdots, k$, and $n = \sum_{i=1}^k n_i$. Then from the likelihood (3.2) and the reference prior (3.1), the element of the FBF under H_1, $m_1^b(\mathbf{x})$, is given by

$$\begin{aligned} m_1^b(\mathbf{x}) &= \int_0^\infty \cdots \int_0^\infty \int_0^\infty L^b(\tau, \mu_1, \cdots, \mu_k|\mathbf{x}) \pi_1^N(\tau, \mu_1, \cdots, \mu_k) d\tau d\mu_1 \cdots d\mu_k \\ &= \int_0^\infty \cdots \int_0^\infty (2\pi)^{-\frac{nb}{2}} 2^{\frac{nb}{2}-1} \Gamma\left(\frac{nb+1}{2}\right) \left(\prod_{i=1}^k \mu_i^{-n_i b-1}\right) \\ &\times U\left[\frac{nb+1}{2}, \frac{nb+2}{2}, \sum_{i=1}^k \frac{\sum_{j=1}^{n_i} b(x_{ij}-\mu_i)^2}{\mu_i^2}\right] d\mu_1 \cdots d\mu_k, \end{aligned} \tag{3.3}$$

where $U[a,b,z] = \frac{1}{\Gamma[a]} \int_0^\infty e^{-zt} t^{a-1}(1+t)^{b-a-1} dt$ is the confluent hypergeometric function of the second kind.

Under the hypothesis H_2, the reference prior for $(\tau_1, \mu_1, \cdots, \tau_k, \mu_k)$ is

$$\pi_2^N(\tau_1, \mu_1, \cdots, \tau_k, \mu_k) \propto \prod_{i=1}^k \tau_i^{-1}(1+2\tau_i^2)^{-\frac{1}{2}} \mu_i^{-1}, \tag{3.4}$$

The above reference prior can directly derived from Lee *et al.* (2003). And the likelihood function is given by

$$L(\tau_1, \mu_1, \cdots, \tau_k, \mu_k|\mathbf{x}) = (2\pi)^{-\frac{n}{2}} \prod_{i=1}^k \mu_i^{-n_i} \tau_i^{-n_i} \exp\left\{-\frac{\sum_{j=1}^{n_i}(x_{ij}-\mu_i)^2}{2\mu_i^2\tau_i^2}\right\}. \tag{3.5}$$

Then from the likelihood (3.5) and the reference prior (3.4), the element of the FBF under H_2, $m_2^b(\mathbf{x})$, is given by

$$\begin{aligned} m_2^b(\mathbf{x}) &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\infty L^b(\tau_1, \mu_1, \cdots, \tau_k, \mu_k|\mathbf{x}) \\ &\times \pi_2^N(\tau_1, \mu_1, \cdots, \tau_k, \mu_k) d\tau_1 d\mu_1 \cdots d\tau_k d\mu_k \\ &= \int_0^\infty \cdots \int_0^\infty (2\pi)^{-\frac{nb}{2}} \prod_{i=1}^k 2^{\frac{n_i b}{2}-1} \Gamma\left(\frac{n_i b+1}{2}\right) \mu_i^{-n_i b-1} \\ &\times U\left[\frac{n_i b+1}{2}, \frac{n_i b+2}{2}, \frac{\sum_{j=1}^{n_i} b(x_{ij}-\mu_i)^2}{\mu_i^2}\right] d\mu_1 \cdots d\mu_k. \end{aligned} \tag{3.6}$$

Therefore the element B_{21}^N of the FBF is given by

$$B_{21}^N = \frac{S_2(\mathbf{x}_1, \cdots, \mathbf{x}_k)}{S_1(\mathbf{x}_1, \cdots, \mathbf{x}_k)}, \tag{3.7}$$

where

$$\begin{aligned} S_1(\mathbf{x}_1, \cdots, \mathbf{x}_k) &= \int_0^\infty \cdots \int_0^\infty \Gamma\left(\frac{n+1}{2}\right) \left(\prod_{i=1}^k \mu_i^{-n_i-1}\right) \\ &\times U\left[\frac{n+1}{2}, \frac{n+2}{2}, \sum_{i=1}^k \frac{\sum_{j=1}^{n_i}(x_{ij}-\mu_i)^2}{\mu_i^2}\right] d\mu_1 \cdots d\mu_k \end{aligned}$$

and

$$S_2(\mathbf{x}_1,\cdots,\mathbf{x}_k)=\int_0^\infty\cdots\int_0^\infty 2^{-k+1}\prod_{i=1}^k\Gamma\left(\frac{n_i+1}{2}\right)\mu_i^{-n_i-1}$$
$$\times U\left[\frac{n_i+1}{2},\frac{n_i+2}{2},\frac{\sum_{j=1}^{n_i}(x_{ij}-\mu_i)^2}{\mu_i^2}\right]d\mu_1\cdots d\mu_k.$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\mathbf{x})}{m_2^b(\mathbf{x})}=\frac{S_1(\mathbf{x}_1,\cdots,\mathbf{x}_k;b)}{S_2(\mathbf{x}_1,\cdots,\mathbf{x}_k;b)}, \tag{3.8}$$

where

$$S_1(\mathbf{x}_1,\cdots,\mathbf{x}_k;b)=\int_0^\infty\cdots\int_0^\infty \Gamma\left(\frac{nb+1}{2}\right)\prod_{i=1}^k\mu_i^{-n_ib-1}$$
$$\times U\left[\frac{nb+1}{2},\frac{nb+2}{2},\sum_{i=1}^k\frac{\sum_{j=1}^{n_i}b(x_{ij}-\mu_i)^2}{\mu_i^2}\right]d\mu_1\cdots d\mu_k$$

and

$$S_2(\mathbf{x}_1,\cdots,\mathbf{x}_k;b)=\int_0^\infty\cdots\int_0^\infty 2^{-k+1}\prod_{i=1}^k\Gamma\left(\frac{n_ib+1}{2}\right)\mu_i^{-n_ib-1}$$
$$\times U\left[\frac{n_ib+1}{2},\frac{n_ib+2}{2},\frac{\sum_{j=1}^{n_i}b(x_{ij}-\mu_i)^2}{\mu_i^2}\right]d\mu_1\cdots d\mu_k.$$

Thus from (3.7) and (3.8), the FBF of H_2 versus H_1 is given by

$$B_{21}^F=\frac{S_2(\mathbf{x}_1,\cdots,\mathbf{x}_k)}{S_1(\mathbf{x}_1,\cdots,\mathbf{x}_k)}\cdot\frac{S_1(\mathbf{x}_1,\cdots,\mathbf{x}_k;b)}{S_2(\mathbf{x}_1,\cdots,\mathbf{x}_k;b)}. \tag{3.9}$$

Note that the calculation of the FBF of H_2 versus H_1 requires actually two dimensional integration.

3.2. Bayesian hypothesis testing based on the intrinsic Bayes factor

The emement B_{21}^N of the intrinsic Bayes factor is computed in the fractional Bayes factor. So under the minimal training sample, we only calculate the marginal densities for the hypotheses H_1 and H_2, respectively. The marginal density of $(X_{1j_1},X_{1j_2},\cdots,X_{kl_1},X_{kl_2})$ is finite for all $1\le j_1<j_2\le n_1,\cdots$, and $1\le l_1<l_2\le n_k$ under each hypothesis (Lee *et al.*, 2003; Kim *et al.*, 2008). Thus we conclude that any training sample of size $2k$ is a minimal training sample.

The marginal density $m_1^N(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})$ under H_1 is given by

$$\begin{aligned}
&m_1(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2}) \\
&= \int_0^\infty \cdots \int_0^\infty \int_0^\infty f(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2}|\tau, \mu_1, \cdots, \mu_k)\pi_1^N(\tau, \mu_1, \cdots, \mu_k)d\tau d\mu_1 \cdots d\mu_k \\
&= \int_0^\infty \cdots \int_0^\infty (2\pi)^{-k} 2^{k-1}\Gamma\left(\frac{2k+1}{2}\right)\prod_{i=1}^k \mu_i^{-3} \qquad (3.10) \\
&\times U\left[\frac{2k+1}{2}, k+1, \sum_{i=1}^k \frac{\sum_{j=1}^2 (x_{ij}-\mu_i)^2}{\mu_i^2}\right] d\mu_1 \cdots d\mu_k.
\end{aligned}$$

And the marginal density $m_2^N(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})$ under H_2 is given by

$$\begin{aligned}
&m_2(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2}) \\
&= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\infty f(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2}|\tau_1, \mu_1, \cdots, \tau_k, \mu_k) \\
&\times \pi_2^N(\tau_1, \mu_1, \cdots, \tau_k, \mu_k) d\tau_1 d\mu_1 \cdots d\tau_k d\mu_k \\
&= \int_0^\infty \cdots \int_0^\infty (2\pi)^{-k} \frac{\pi^{\frac{k}{2}}}{2^k} \prod_{i=1}^k \mu_i^{-3} U\left[\frac{3}{2}, 2, \frac{\sum_{j=1}^2 (x_{ij}-\mu_i)^2}{\mu_i^2}\right] d\mu_1 \cdots d\mu_k. \qquad (3.11)
\end{aligned}$$

Therefore the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI} = \frac{S_2(\mathbf{x}_1, \cdots, \mathbf{x}_k)}{S_1(\mathbf{x}_1, \cdots, \mathbf{x}_k)} \cdot \frac{1}{L} \sum_{j_1<j_2} \cdots \sum_{l_1<l_2} \frac{T_1(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})}{T_2(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})}, \qquad (3.12)$$

where L is $\prod_{i=1}^k n_i(n_i-1)/2$,

$$\begin{aligned}
T_1(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2}) &= \int_0^\infty \cdots \int_0^\infty 2^{k-1}\Gamma\left(\frac{2k+1}{2}\right)\prod_{i=1}^k \mu_i^{-3} \\
&\times U\left[\frac{2k+1}{2}, k+1, \sum_{i=1}^k \frac{\sum_{j=1}^2 (x_{ij}-\mu_i)^2}{\mu_i^2}\right] d\mu_1 \cdots d\mu_k
\end{aligned}$$

and

$$T_2(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2}) = \int_0^\infty \cdots \int_0^\infty \frac{\pi^{\frac{k}{2}}}{2^k} \prod_{i=1}^k \mu_i^{-3} U\left[\frac{3}{2}, 2, \frac{\sum_{j=1}^2 (x_{ij}-\mu_i)^2}{\mu_i^2}\right] d\mu_1 \cdots d\mu_k.$$

And also the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{S_2(\mathbf{x}_1, \cdots, \mathbf{x}_k)}{S_1(\mathbf{x}_1, \cdots, \mathbf{x}_k)} \cdot ME\left[\frac{T_1(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})}{T_2(x_{1j_1}, x_{1j_2}, \cdots, x_{kl_1}, x_{kl_2})}\right]. \qquad (3.13)$$

Note that the calculations of the AIBF and MIBF of H_2 versus H_1 requires actually two dimensional integration.

4. Numerical study

In order to assess the Bayesian hypothesis testing procedures, we evaluate the posterior probability for several configurations of $(\mu_1, \tau_1, \cdots, \mu_k, \tau_k)$ amd $(n_1, \cdots, n_k)$. In particular, for fixed $(\mu_1, \tau_1, \cdots, \mu_k, \tau_k)$, we take 200 independent random samples of $X_1, \cdots, X_k$ with sample sizes $n_1, \cdots, n_k$ from the model (1.1). We want to test the hypotheses $H_1 : \tau_1 = \cdots = \tau_k \equiv \tau$ versus $H_2 : \tau_1 \neq \cdots \neq \tau_k$.

The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. Tables 4.1 and 4.2 show the results of the averages and the standard deviations in parentheses of posterior probabilities. From Tables 4.1 and 4.2, the FBF, the AIBF and the MIBF give fairly reasonable answers for all configurations. The MIBF favors the hypothesis H_1, but the FBF favors the hypothesis H_2. And the AIBF is between the MIBF and the FBF.

Table 4.1 The average and the standard deviations in parentheses of Posterior probabilities

μ_1, μ_2, μ_3	τ_1, τ_2, τ_3	n_1, n_2, n_3	$P^F(H_1\|\mathbf{x})$	$P^{AI}(H_1\|\mathbf{x})$	$P^{MI}(H_1\|\mathbf{x})$
1.0,1.0,1.0	0.1,0.1,0.1	5,5,5	0.624 (0.167)	0.733 (0.182)	0.748 (0.175)
		5,5,10	0.660 (0.192)	0.748 (0.188)	0.761 (0.181)
		5,10,10	0.747 (0.179)	0.836 (0.157)	0.846 (0.150)
	0.1,0.1,0.2	5,5,5	0.432 (0.239)	0.530 (0.277)	0.557 (0.270)
		5,5,10	0.406 (0.267)	0.517 (0.294)	0.539 (0.291)
		5,10,10	0.396 (0.299)	0.490 (0.328)	0.511 (0.326)
	0.1,0.1,0.4	5,5,5	0.163 (0.210)	0.207 (0.264)	0.234 (0.275)
		5,5,10	0.071 (0.115)	0.111 (0.161)	0.129 (0.173)
		5,10,10	0.047 (0.114)	0.066 (0.153)	0.074 (0.163)
	0.1,0.5,1.0	5,5,5	0.060 (0.077)	0.095 (0.117)	0.119 (0.136)
		5,5,10	0.027 (0.049)	0.062 (0.098)	0.081 (0.116)
		5,10,10	0.025 (0.044)	0.074 (0.103)	0.094 (0.121)
10.0,10.0,10.0	0.1,0.1,0.1	5,5,5	0.656 (0.165)	0.754 (0.177)	0.759 (0.170)
		5,5,10	0.697 (0.170)	0.781 (0.148)	0.786 (0.142)
		5,10,10	0.728 (0.180)	0.813 (0.162)	0.817 (0.159)
	0.1,0.1,0.2	5,5,5	0.445 (0.238)	0.535 (0.277)	0.553 (0.271)
		5,5,10	0.395 (0.250)	0.500 (0.271)	0.515 (0.268)
		5,10,10	0.333 (0.279)	0.431 (0.315)	0.444 (0.314)
	0.1,0.1,0.4	5,5,5	0.167 (0.203)	0.212 (0.251)	0.233 (0.256)
		5,5,10	0.065 (0.112)	0.099 (0.157)	0.113 (0.166)
		5,10,10	0.029 (0.096)	0.043 (0.122)	0.047 (0.127)
	0.1,0.5,1.0	5,5,5	0.054 (0.085)	0.085 (0.122)	0.106 (0.135)
		5,5,10	0.023 (0.034)	0.055 (0.073)	0.070 (0.087)
		5,10,10	0.024 (0.042)	0.070 (0.098)	0.089 (0.116)

Example 4.1 This example taken from Meier (1953). In this example, four experiments are used to estimate the mean percentage of albumin in the plasma protein of normal human subjects. The summary statistics are given in Table 4.3.

The values of the fractional Bayes factor and the posterior probability of H_1 are 0.218 and 0.821, respectively. The p-value from the asymptotic likelihood ratio test by Verrill and Johnson (2007) is 0.134. Thus the Bayesian and classical testing methods give the same result. Also for different values of s_4^2 with the remaining sampling values fixed, we compute values of the fractional Bayes factor, and the p-values based on the asymptotic and sumulation procedures by Verrill and Johnson (2007). For $s_4^2 = 28.510$, the values of the fractional Bayes factor and the posterior probability of H_1 are 0.564 and 0.639, respectively.

Table 4.2 The average and the standard deviations in parentheses of Posterior probabilities

$\mu_1, \mu_2, \mu_3, \mu_4$	$\tau_1, \tau_2, \tau_3, \tau_4$	n_1, n_2, n_3, n_4	$P^F(H_1\|\mathbf{x})$	$P^{AI}(H_1\|\mathbf{x})$	$P^{MI}(H_1\|\mathbf{x})$
1.0,1.0,1.0,1.0	0.1,0.1,0.1,0.1	4,4,4,4	0.611 (0.196)	0.749 (0.210)	0.784 (0.196)
		4,4,8,8	0.698 (0.222)	0.797 (0.214)	0.827 (0.194)
	0.1,0.1,0.1,0.2	4,4,4,4	0.472 (0.232)	0.595 (0.273)	0.650 (0.257)
		4,4,8,8	0.428 (0.296)	0.550 (0.321)	0.598 (0.308)
	0.1,0.1,0.1,0.4	4,4,4,4	0.190 (0.208)	0.248 (0.273)	0.310 (0.290)
		4,4,8,8	0.059 (0.141)	0.088 (0.193)	0.109 (0.216)
	0.1,0.4,0.7,1.0	4,4,4,4	0.106 (0.118)	0.193 (0.187)	0.265 (0.215)
		4,4,8,8	0.055 (0.087)	0.184 (0.188)	0.256 (0.222)
10.0,10.0,10.0,10.0	0.1,0.1,0.1,0.1	4,4,4,4	0.634 (0.164)	0.763 (0.172)	0.788 (0.161)
		4,4,8,8	0.725 (0.213)	0.829 (0.174)	0.849 (0.157)
	0.1,0.1,0.1,0.2	4,4,4,4	0.490 (0.240)	0.600 (0.274)	0.643 (0.260)
		4,4,8,8	0.415 (0.302)	0.533 (0.329)	0.572 (0.321)
	0.1,0.1,0.1,0.4	4,4,4,4	0.208 (0.226)	0.266 (0.287)	0.317 (0.300)
		4,4,8,8	0.071 (0.138)	0.103 (0.191)	0.124 (0.210)
	0.1,0.4,0.7,1.0	4,4,4,4	0.106 (0.114)	0.194 (0.185)	0.258 (0.214)
		4,4,8,8	0.048 (0.077)	0.167 (0.165)	0.235 (0.201)

And the p-values from the asymptotic likelihood ratio test and the simulation procedure by Verrill and Johnson (2007) is 0.050 and 0.078, respectively. For $s_4^2 = 32.547$, the values of the fractional Bayes factor and the posterior probability of H_1 are 0.914 and 0.522, respectively. And the p-values from the asymptotic likelihood ratio test and the simulation procedure by Verrill and Johnson (2007) is 0.030 and 0.050, respectively. So the fractional Bayes factor and the simulation procedure by Verrill and Johnson (2007) give the similar results. Verrill and Johnson (2007) showed that the simulation procedure works well even for small samples than asymptotic procedure.

Table 4.3 Percentage of albumin in Plasma protein

Experiment	n_i	$\bar{x}_i$	s_i^2
A	12	62.3	12.986
B	15	60.3	7.840
C	7	59.5	33.433
D	16	61.5	18.513

5. Concluding remarks

In the normal distributions, we developed the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factor for the homogeneity of coefficients of variation under the reference priors. From our numerical results, the developed testing procedures give fairly reasonable answers for all parameter configurations. The Bayesian and classical testing methods gave the same result. However the FBF favors the hypothesis H_2 and the MIBF favors the hypothesis H_1. And the AIBF is between the MIBF and the FBF. Therefore from our results of simulation and example, we recommend the use of the AIBF and the FBF than the MIBF in practical application.

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