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# EXISTENCE OF HOMOCLINIC ORBITS FOR LIÉNARD TYPE SYSTEMS

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ABSTRACT. We investigate the existence of homoclinic orbits of the following systems of Liénard type: $a(x)x' = h(y) - F(x)$, $y' = -a(x)g(x)$, where $h(y) = m|y|^{p-2}y$ with $m > 0$ and $p > 1$ and $a$, $F$, $g$ are continuous functions such that $a(x) > 0$ for all $x \in \mathbb{R}$ and $F(0) = g(0) = 0$ and $xg(x) > 0$ for $x \neq 0$. By a series of time and coordinates transformations of the above system, we obtain sufficient conditions for the positive orbits of the above system starting at the points on the curve $h(y) = F(x)$ with $x > 0$ to approach the origin through only the first quadrant. The method of this paper is new and the results of this paper cover some early results on this topic.

## 1. INTRODUCTION

Consider the existence of homoclinic orbits of the following generalized Liénard system:

$$\begin{cases} a(x)x' = h(y) - F(x), \\ y' = -a(x)g(x), \end{cases} \tag{1}$$

where $x' = \frac{dx}{dt}$ and $y' = \frac{dy}{dt}$. Throughout this paper, we assume that $h(y) = m|y|^{p-2}y$ with constants $m > 0$ and $p > 1$, and $a$, $F$, $g$ are continuous functions on $\mathbb{R}$ such that $a(x) > 0$ for all $x \in \mathbb{R}$ and $F(0) = g(0) = 0$ and $xg(x) > 0$ for all $x \neq 0$. Moreover, we assume that smoothness conditions for the existence and uniqueness of solutions of the initial value problem of (1) are satisfied.

If $a(x) = 1$, $p = 2$ and $m = 1$, then system (1) reduces to the so-called Liénard system:

$$\begin{cases} x' = y - F(x), \\ y' = -g(x), \end{cases} \tag{2}$$

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which is equivalent to the following second order equation

$$x'' + f(x)x' + g(x) = 0, \tag{3}$$

where $F \in C^1$ and $f(x) := F'(x)$.

As for the system (1), it is easy to see that the origin is the unique equilibrium point of (1). An orbit of (1) is said to be a *homoclinic orbit* if its $\alpha-$ and $\omega-$limit sets are the origin. The investigation of the existence of homoclinic orbits of a dynamical system plays an important role in the study of various nonlinear dynamical systems such as Lagrangian systems, Hamiltonian systems, Lorenz systems and Schrödinger systems. Many results have been achieved on this topic. For example, see the results in [1-10] and the references therein. The existence of a homoclinic orbit implies that the zero solution is not stable and the origin is not a center. Moreover, the existence of a homoclinic orbit has a close relationship with the stability of the zero solution, the global attractivity of the origin and oscillation of the solutions and so on(see [3, 8–10]).

We say that system (1) has *property* $Z_1^+$( resp.,$Z_3^+$) if there exists a point $P = (x_0, y_0)$ with $x_0 > 0$ (resp.,$x_0 < 0$) and $h(y_0) = F(x_0)$ such that the positive orbit of (1) starting at $P$ approaches the origin through only the first (resp., third) quadrant. Similarly, we say that system (1) has *property* $Z_2^-$(resp., $Z_4^-$) if there exists a point $Q = (x_1, y_1)$ with $x_1 < 0$ (resp.,$x_1 > 0$) and $h(y_1) = F(x_1)$ such that the negative orbit of (1) starting at $Q$ approaches the origin through only the second (resp., fourth) quadrant. If system (1) has both property $Z_1^+$ and property $Z_2^-$, a homoclinic orbit exists in the upper half-plane. Similarly, if the system (1) has both property $Z_3^+$ and property $Z_4^-$, a homoclinic orbit exists in the lower half-plane.

Sugie et al [8] have obtained some implicit and explicit conditions for the system (1) (with $a(x) = 1$) to have the property $Z_1^+$. Recently, Aghajani et al [9] also have obtained some sufficient conditions for system (1) to have property $Z_1^+$.

In this paper, by a series of coordinates and time transformations, we rewrite the system (1) in the so-called generalized polar coordinates and then by using the information obtained from the second equation of the transformed system, we obtain the same results achieved in [8] and [9]. The method used in this paper is new and the constants obtained are very sharp.

## 2. Coordinates Transformations and Lemmas

First, by using coordinates and time transformations, we change system (1) into the so-called canonical form.

Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $q = \frac{p}{p-1}$ and $q > 1$.

**Lemma 1.** *Define a time transformation $T_1$ by*

$$T_1 : \frac{ds}{dt} = \frac{1}{a(x)}.$$

*Then, under the transformation $T_1$, the system (1) is changed into the following form:*

$$\begin{cases} \dot{x} = h(y) - F(x), \\ \dot{y} = -a^2(x)g(x), \end{cases} \tag{4}$$

*where $\dot{x} = \frac{dx}{ds}$, $\dot{y} = \frac{dy}{ds}$.*

**Lemma 2.** *Define coordinates and time transformation $T_2$ by*

$$T_2 : u = (qG(x))^{\frac{1}{q}}\, sgn(x), \quad y = y, \quad \frac{d\tau}{ds} = \frac{a^2(x)|g(x)|}{(qG(x))^{\frac{1}{p}}}, \tag{5}$$

*where $G(x) = \int_0^x a^2(s)g(s)ds$. Then under the transformation $T_2$, the system (4) is changed into the following canonical form:*

$$\begin{cases} \frac{du}{d\tau} = h(y) - F^*(u), \\ \frac{dy}{d\tau} = -|u|^{q-2}u, \end{cases} \tag{6}$$

*where*

$$F^*(u) = F\left(G^{-1}\left(\frac{|u|^q}{q}\right)\right).$$

Now, we introduce the so-called *generalized sine* and *conine* functions.

Let $(x, y) = (S(t), C(t))$ be the solution of the initial value problem

$$x' = h(y), \quad y' = -|x|^{q-2}x, \qquad (x(0), y(0)) = (0, 1)). \tag{7}$$

Then it follows from (7) that $(S(t), C(t))$ satisfies the following identity:

$$\frac{(p-1)|S(t)|^q}{m} + |C(t)|^p \equiv 1. \tag{8}$$

Note that if $m = 1$ and $p = 2 (= q)$, then the functions $S(t)$ and $C(t)$ reduce to the usual *sine* and *cosine* functions $\sin t$ and $\cos t$ respectively.

**Lemma 3.** *Introduce the generalized polar coordinates transformation $T_3 : (r, \theta) \in (0, +\infty) \times \mathbb{R} \to (u, y) \in \mathbb{R}^2$ by:*

$$u = r^{\frac{p}{2}} S(\theta), \quad y = r^{\frac{q}{2}} C(\theta). \tag{9}$$

*Then, by using $p+q=pq$, (7) and (8), and after some calculations, system (6) is changed into the following form:*

$$(10)\qquad \begin{aligned} \frac{dr}{rd\tau} &= -\frac{2F(x)r^{\frac{q}{2}}|S(\theta)|^{q-2}S(\theta)}{mqr^{\frac{p+q}{2}}}, \\ \frac{d\theta}{d\tau} &= 1-\frac{F(x)r^{\frac{q}{2}}C(\theta)}{mr^{\frac{p+q}{2}}}. \end{aligned}$$

Now note that by using (5), (8) and (9), we can obtain the following relation

$$(11)\qquad r^{\frac{p+q}{2}} = r^{\frac{pq}{2}} = \frac{(p-1)|u|^q}{m} + |y|^p = \frac{pG(x)}{m} + |y|^p,$$

and by using (5), (9) and (11), we can put system (10) back into $(x,\, y)$ coordinates:

$$(12)\qquad \begin{aligned} \frac{dr}{rd\tau} &= -\frac{2F(x)(qG(x))^{\frac{1}{p}}\,sgn x}{q[pG(x)+m|y|^p]}, \\ \frac{d\theta}{d\tau} &= 1-\frac{F(x)y}{pG(x)+m|y|^p}. \end{aligned}$$

From the first equation of (12), we get $r(\tau) = r(0)e^{\int_0^\tau A(s)ds}$, where $A(s)$ equals the right side of the first equation of (12) with $x$ and $y$ replaced by $x(s)$ and $y(s)$. Hence, for $r(0) > 0$, we see that for finite $\tau > 0$, $0 < r(\tau) < +\infty$, which again implies that $r(\tau)$ has no blow up in finite time. From the second equality of (12), we shall derive some important information of our results. For this we need the following lemmas:

**Lemma 4.** *If $A$, $B$ are nonnegative numbers and $p > 1$, $q = \frac{p}{p-1}$, then*

$$\frac{A^p}{p} + \frac{B^q}{q} \geq AB,$$

*where the equality holds if and only if $A = B^{q-1}$.*

**Lemma 5.** *Let*

$$y_0(x) = \left(\frac{qG(x)}{m}\right)^{\frac{1}{p}},$$

*and for any fixed $x$, define*

$$H(y) = \frac{F(x)y}{pG(x)+m|y|^p}.$$

*If $F(x) \geq 0$, then*

$$\max_{y\in\mathbb{R}} H(y) = H(y_0(x)) = \frac{F(x)}{m^{\frac{1}{p}}p\,(qG(x))^{\frac{1}{q}}}.$$

Note that the function $y_0(x) > 0 \forall x \neq 0$ and is independent of the function $F(x)$ and hence the function $H(y)$ always takes its maximum value on $y_0(x)$. Lemma 5 follows immediately by letting $A^p = pm|y|^p$ and $B^q = qpG(x)$ in Lemma 4. The proofs of Lemma 1 – Lemma 4 are also straightforward, so we omit them.

## 3. Main Results and Proofs

Now, we can state the main results of this paper. For simplicity, we discuss only the property $Z_1^+$ of system (1). Other properties can be formulated in analogous ways. Since the property $Z_1^+$ is preserved under the time transformation, we need only to consider system (1) and the second equation of the system (12).

**Theorem 1.** *Let $\lambda = m^{1/p}pq^{1/q} > 0$ and suppose that there exists a $\delta > 0$ such that for $x \in [0, \delta)$, $F(x) \geq \lambda\,(G(x))^{\frac{1}{q}}$. Then the system (1) has property $Z_1^+$.*

*Proof.* We prove Theorem 1 by contradiction. Assume that there exists a point $P_0 = (x_0, y_0)$ on the curve $h(y) = F(x)$ and $x_0 \in (0, \delta)$ such that the positive orbit of system (1) starting from $P_0$ does not approach the origin through the region $D = \{(x, y) : 0 < x < \delta \text{ and } 0 < h(y) < F(x)\}$. Then it rotates in a clockwise direction about the origin. For this reason and by the assumption $F(x) \geq \lambda\,(G(x))^{\frac{1}{q}}$ it is easy to verify that $F(x) > h(y_0(x)) = m^{\frac{1}{p}}q^{\frac{1}{q}}\,(G(x))^{\frac{1}{q}}$ in $D$ and the positive orbit crosses first the the curve $y = y_0(x)$ at some point $P_1 = (x_1, y_1)$ and then crosses the $x-$axis at some point $P_2 = (x_2, 0) = (x_2, y_2)$ with $0 < x_2 < x_1 < x_0$(since $a(x)x' = h(y) - F(x) < 0$ in $D$) and $0 < y_1 = y_0(x_1) < y_0 = h^{-1}(F(x_0))$ in $D$. Let us consider the curve $y(x)$ joining $P_1$ and $P_2$. For $x \in (x_2, x_1)$ it satisfies $0 < y(x) < y_0(x)$ and equation

$$\frac{dx}{dy} = \frac{F(x) - h(y)}{a^2(x)g(x)}.$$

Since $F(x) \geq \lambda\,(G(x))^{\frac{1}{q}} = m^{\frac{1}{p}}q^{\frac{1}{q}}p\,(G(x))^{\frac{1}{q}}$ and $h(y(x)) < h(y_0(x)) = m^{\frac{1}{p}}q^{\frac{1}{q}}\,(G(x))^{\frac{1}{q}}$ we get for $x \in (x_2, x_1)$,

$$\begin{aligned}\frac{dx}{dy} &= \frac{F(x)-h(y(x))}{a^2(x)g(x)} \\ &> \frac{m^{\frac{1}{p}}q^{\frac{1}{q}}p(G(x))^{\frac{1}{q}} - m^{\frac{1}{p}}q^{\frac{1}{q}}(G(x))^{\frac{1}{q}}}{a^2(x)g(x)} \\ &= \frac{m^{\frac{1}{p}}q^{\frac{1}{q}}(p-1)(G(x))^{\frac{1}{q}}}{a^2(x)g(x)},\end{aligned}$$

and hence,

$$\frac{dy}{dx} < \frac{a^2(x)g(x)}{m^{\frac{1}{p}}(p-1)q^{\frac{1}{q}}\,(G(x))^{\frac{1}{q}}}, \quad x_2 < x < x_1. \tag{13}$$

Integrating inequality (13) over $[x_2, x_1]$ and using $y(x_2) = 0$, we obtain

$$y(x_1) - y(x_2) = y(x_1) = y_0(x_1) < \frac{q^{\frac{1}{p}}}{m^{\frac{1}{p}}}\left[(G(x_1))^{\frac{1}{p}} - (G(x_2))^{\frac{1}{p}}\right] = y_0(x_1) - y_0(x_2),$$

which is a contradiction. Here, we have used the following facts

$$y(x_1) = y_0(x_1) = \left(\frac{qG(x_1)}{m}\right)^{\frac{1}{p}} \quad \text{and} \quad y_0(x_2) = \left(\frac{qG(x_2)}{m}\right)^{\frac{1}{p}} > 0.$$

□

**Theorem 2.** *Assume that there exists constants $\delta > 0$ and $\epsilon \in (-\infty, 1)$ such that for $x \in [0, \delta)$,*

$$0 \leq F(x) \leq \epsilon\lambda\,(G(x))^{\frac{1}{q}}.$$

*Then system (1) fails to have property $Z_1^+$.*

*Proof.* Since $0 \leq F(x) \leq \epsilon\lambda\,(G(x))^{\frac{1}{q}}$, we get from the second equation of (12) and Lemma 5 that

$$\begin{aligned}\tfrac{d\theta}{d\tau} &\geq 1 - \max_{y\in\mathbb{R}} \tfrac{F(x)y}{pG(x)+m|y|^p} \\ &= 1 - H(y_0(x)) \\ &= 1 - \tfrac{F(x)}{m^{\frac{1}{p}}p(qG(x))^{\frac{1}{q}}} \\ &\geq 1 - \epsilon = \text{constant} > 0.\end{aligned}$$

This implies that $\theta(\tau) \to +\infty$ as $\tau \to +\infty$ and so for all $P_0 = (x_0, y_0)$ with $x_0 \in (0, \delta)$ and $h(y_0) = F(x_0)$, the positive orbit of $P_0$ rotates around the origin clockwise, and it crosses the positive $x-$axis in finite time $\tau_0 > 0$. This implies that system (1) fails to have property $Z_1^+$. □

**Remark 1.** We see from Theorem 1 and Theorem 2 that the value $\lambda$ is sharp and also that the sign of $\frac{d\theta}{d\tau}$ changes from positive to negative when $y = y_0(x)$ and $F(x)$ changes from $F(x) < \lambda\,(G(x))^{\frac{1}{q}}$ to $F(x) > \lambda\,(G(x))^{\frac{1}{q}}$. Moreover, $\frac{d\theta}{d\tau} = 0$ if $y = y_0(x)$ and $F(x) = \lambda\,(G(x))^{\frac{1}{q}}$. In this case, we say that system (1) has property $X^+$ in the right half-plane (see [3, 10] for details). Besides, it is assumed $p \geq 2$ in [3] with a slightly different form of $h(y)$, but our results hold for $p > 1$.

**Remark 2.** In Theorem 2, if we further assume that there exists a constant $\epsilon_0 > 0$ such that for all $x \in \mathbb{R}$,

$$\frac{d\tau}{dt} = \frac{d\tau}{ds}\frac{ds}{dt} = \frac{1}{a(x)}\frac{a^2(x)|g(x)|}{(qG(x))^{\frac{1}{p}}} = \frac{a(x)|g(x)|}{(qG(x))^{\frac{1}{p}}} \geq \epsilon_0 > 0,$$

then this implies that $\tau(t) \to +\infty$ as $t \to +\infty$. And from the second equation of (12), we also get

$$\frac{d\theta}{dt} = \frac{d\theta}{d\tau}\frac{d\tau}{dt} \geq (1-\epsilon)\epsilon_0 = \text{constant} > 0,$$

which implies that all solutions of system (1) rotate around the origin clockwise and cross the $x$−axis and $y$−axis infinitely many times. Therefore, all solutions of system (1) are oscillatory.

**Example 1.** Consider the system

$$x' = \sqrt{1+x^2}\left[my/\sqrt{|y|} - F(x)\right], \quad y' = -\frac{2x}{\sqrt{1+x^2}}, \tag{14}$$

where $m$ is a positive constant. Then $a(x) = \frac{1}{\sqrt{1+x^2}}$, $p = \frac{3}{2}$, $q = 3$, $g(x) = 2x$, and $G(x) = \int_0^x \frac{2sds}{1+s^2} = \ln(1+x^2)$, and $\lambda = \frac{m^{2/3}3^{4/3}}{2}$.

If $F(x) \geq \lambda\left[\ln(1+x^2)\right]^{\frac{1}{3}}$ for $x > 0$ sufficiently small, then by the Theorem 1, system (14) has property $Z_1^+$. If $0 \leq F(x) \leq \epsilon\lambda\left[\ln(1+x^2)\right]^{\frac{1}{3}}$ for some constant $\epsilon \in (-\infty, 1)$ and $x > 0$ sufficiently small, then by Theorem 2, system (14) fails to have property $Z_1^+$.

**Example 2.** Consider the system

$$x' = |y|y - c|x|^{k-1}x, \quad y' = -3|x|x, \tag{15}$$

where $c$ and $k$ are positive constants. Then $a(x) = m = 1$, $p = 3$, $q = \frac{3}{2}$, $g(x) = 3|x|x$, $G(x) = |x|^3$ and $\lambda = 3\left(\frac{3}{2}\right)^{\frac{2}{3}}$. In this case, system (15) has property $Z_1^+$ provided one of the following two conditions is satisfied:

(*i*) $0 < k < 2$;

(*ii*) $k = 2$, $c \geq \lambda = 3\left(\frac{3}{2}\right)^{\frac{2}{3}}$.

**Example 3.** Consider the system (1) with $a(x) = 1$, $g(x) = q|x|^{q-1}sgn(x)$ and $q = \frac{p}{p-1}$. Then $\lambda = m^{\frac{1}{p}}pq^{\frac{1}{q}}$, $G(x) = |x|^q$, $(G(x))^{\frac{1}{q}} = |x|$ and

$$\frac{d\tau}{dt} = q^{\frac{1}{q}} = \text{constant} > 0.$$

If there exists a constant $\epsilon \in [0, 1)$ such that for all $x \in \mathbb{R}$, $0 \leq F(x) \leq \epsilon\lambda|x|$, then by Theorem 2 and Remark 2, all solutions of system (1) are oscillatory, and hence system (1) fails to have property $Z_1^+$.

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