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STABILITY OF FUNCTIONAL EQUATIONS RELATED TO THE EXPONENTIAL AND BETA FUNCTIONS

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ABSTRACT. In this paper we obtain the Hyers-Ulam stability of functional equations

$$f(x+y) = f(x) + f(y) + \ln a^{2xy-1}$$

and

$$f(x+y) = f(x) + f(y) + \ln \beta(x,y)^{-1}$$

which is related to the exponential and beta functions.

1. INTRODUCTION

In 1940, S. M. Ulam gave a wide ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems [19]. One of those was the question concerning the stability of homomorphisms;

> *Let G_1 be a group and G_2 a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all x, $y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?*

In the next year, D. H. Hyers [7] answered the Ulam's question for the case of the additive mapping on the Banach spaces G_1, G_2 as follows;

> *Let G_1 and G_2 are Banach spaces. Assume that a mapping $f : G_1 \to G_2$ satisfies the inequality*

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon$$

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for all $x, y \in G_1$. *Then the limit* $g(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for all* $x \in G_1$ *and* g *is the unique additive mapping satisfying*

$$||f(x) - g(x)|| \le \varepsilon$$

for all $x \in G_1$.

In 1978, Th. M. Rassias [16] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded;

Let G_1 *be a vector space and* G_2 *a Banach space. Assume that a mapping* $f : G_1 \to G_2$ *satisfies*

$$||f(x+y) - f(x) - f(y)|| \le \varepsilon(||x||^p + ||y||^p)$$

for all $x, y \in G_1$, $\varepsilon > 0$ *and* $p < 1$. *Then the limit* $g(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for all* $x \in G_1$ *and* g *is the unique additive mapping satisfying*

$$||f(x) - g(x)|| \le \frac{2\varepsilon}{2 - 2^p}||x||^p$$

for all $x \in G_1$.

In 1991, Z. Gajda [3] gave an affirmative solution to this question by the same approach as in Th. M. Rassias [16]. It was also shown by Z. Gajda [3], as well as by Th. M. Rassias and P. Šemrl [17] that one cannot prove the Th. M. Rassias' type theorem when $p = 1$. These results provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept.

P. Găvruta [4] provided a further generalization of Th. M. Rassias' Theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [1]-[19]).

Gilányi [5] and Rätz [18] showed that if satisfies the functional inequality

$$||2f(x) + 2f(y) - f(xy^{-1})|| \le ||f(xy)||,$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

Gilânyi [6] and Fechner [2] proved the generalized Hyers-Ulam stability of this functional inequality. H. Kim and J. Oh [11] proved the Hyers-Ulam stability of the following functional inequality with $abc \neq 0$

$$||af(x) + bf(y) + cf(z)|| \le ||f(ax + by + cz)|| + \phi(x, y, z).$$

Now, we consider the following functional equations (1) and (2)

$$f(x+y) = f(x) + f(y) + \ln a^{2xy-1} \tag{1}$$

and

$$f(x+y) = f(x) + f(y) + \ln \beta(x,y)^{-1}. \tag{2}$$

Then functional inequalities with a perturbing term δ are represented by

$$|f(x+y) - (f(x) + f(y) + \ln a^{2xy-1})| \leq \delta$$

and

$$|f(x+y) - (f(x) + f(y) + \ln \beta(x,y)^{-1})| \leq \delta.$$

The purpose of this paper is to prove that if f satisfies the above inequalities, then we can find the stable mappings satisfying the equation (1) and (2) near an approximate mapping f, and thus we prove the Hyers-Ulam stability of the above functional inequalities.

2. Solutions of the Equation (1) and (2)

Let $f(x) = \ln a^{x^2+x+1}$. Then $f(x+y) = f(x) + f(y) + \ln a^{2xy-1}$. Thus this function is a solution of the functional equation (1). Now consider the gamma and beta functions

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

and

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

It is well known that these gamma and beta functions satisfy the equation

$$\beta(x,y)\Gamma(x+y) = \Gamma(x)\Gamma(y).$$

Thus if we let $f(x) = \ln \Gamma(x)$, then

$$f(x+y) = f(x) + f(y) + \ln \beta(x,y)^{-1},$$

and so this logarithm of the gamma function is a solution of the functional equation (2).

3. Stability of the Functional Equation (1) and (2)

The following theorem is the Hyers-Ulam stability of the functional equation

$$f(x+y) = f(x) + f(y) + \ln a^{2xy-1} \qquad (a > 0)$$

which is related to the exponential function.

Theorem 1. *Let $\delta > 0$ and $a > 0$ be given. Assume that a mapping $f : R \to R$ satisfies the functional inequality*

$$|f(x+y) - f(x) - f(y) - \ln a^{2xy-1}| < \delta \tag{3}$$

for all $x, y \in R$. Then there exists a unique mapping $g : R \to R$ such that

$$g(x+y) = g(x) + g(y) + \ln a^{2xy-1}$$

for all $x, y \in R$ and

$$|f(x) - g(x)| \leq \delta$$

for all $x \in R$. In particular, g is defined by

$$g(x) := \lim_{n\to\infty} P_n(x)$$

where

$$P_n(x) = \frac{f(2^n x)}{2^n} - \ln \prod_{i=1}^{n} a^{\frac{2^{2i-1}x^2-1}{2^i}}$$

for all $x \in R$.

Proof. If we replace y by x and dividing 2 in (3), we get

$$\left|\frac{f(2x)}{2} - \ln a^{\frac{2x^2-1}{2}} - f(x)\right| \leq \frac{\delta}{2} \tag{4}$$

for all $x \in R$. We use induction on n to prove

$$\left|\frac{f(2^n x)}{2^n} - \ln a^{\sum_{i=1}^{n} \frac{2^{2i-1}x^2-1}{2^i}} - f(x)\right| \leq \delta \sum_{i=1}^{n} \frac{1}{2^i} \tag{5}$$

for all $x \in R$. On account of (4), the inequality holds for $n = 1$. Suppose that the inequality (5) holds true for some integer $n > 1$. Then (4) and (5) imply

$$\begin{aligned}
&\left|\frac{f(2^{n+1}x)}{2^{n+1}} - \ln a^{\sum_{i=0}^{n} \frac{2^{2i+1}x^2-1}{2^{i+1}}} - f(x)\right| \\
&\leq \left|\frac{f(2^n \cdot 2x)}{2\cdot 2^n} - \frac{1}{2}\ln a^{\sum_{i=1}^{n} \frac{2^{2i-1}(2x)^2-1}{2^i}} - \frac{f(2x)}{2}\right| + \left|\frac{f(2x)}{2} - \ln a^{\frac{2x^2-1}{2}} - f(x)\right| \\
&\leq \delta \sum_{i=1}^{n+1} \frac{1}{2^i}
\end{aligned}$$

which ends the proof of (5). For any $x \in R$ and for every positive integer n we define that

$$P_n(x) = \frac{f(2^n x)}{2^n} - \ln \prod_{i=1}^{n} a^{\frac{2^{2i-1}x^2-1}{2^i}}.$$

Let $m, n > 0$ be integers with $n > m$. Then it follows from (3)

$$\begin{aligned} &|P_n(x) - P_m(x)| \\ &= \frac{1}{2^m}\left|\frac{f(2^{n-m}(2^m x))}{2^{n-m}} - \ln\left(\prod_{i=1}^{n-m} a^{\frac{2^{2i-1}(2^m x)^2-1}{2^i}}\right) - f(2^m x)\right| \\ &\leq \delta \sum_{i=1}^{n-m} \frac{1}{2^i} \to 0 \end{aligned}$$

as $m \to \infty$. Therefore, the sequence $\{P_n(x)\}$ is a Cauchy sequence, and we may define a function $g : R \to R$ by

$$g(x) := \lim_{n\to\infty} P_n(x)$$

for all $x \in R$. Now we prove that

$$g(x+y) = g(x) + g(y) + \ln a^{2xy-1}$$

for all $x, y \in R$. For this, we consider the following property.

$$\begin{aligned} &\ln \frac{\prod_{i=1}^n a^{\frac{2^{(2i-1)}x^2-1}{2^i}} \prod_{i=1}^n a^{\frac{2^{(2i-1)}y^2-1}{2^i}}}{a^{2xy-1}\prod_{i=1}^n a^{\frac{2^{(2i-1)}(x+y)^2-1}{2^i}}} = \ln \frac{a^{-(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n})}}{a^{2xy-1}a^{2xy(1+2+\cdots+2^{n-1})}} \\ &\leq \frac{1}{2^n}\ln\left[\frac{1}{a^{2(2^n x)(2^n y)-1}}\right] \end{aligned}$$

for all $x, y \in R$. Thus we have

$$\begin{aligned} &\left|g(x+y) - g(x) - g(y) - \ln a^{2xy-1}\right| \\ &= \lim_{n\to\infty}\left|\frac{f(2^n x + 2^n y)}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} - \frac{1}{2^n}\ln a^{2(2^n x)(2^n y)-1}\right| \\ &= \lim_{n\to\infty}\frac{\delta}{2^n} = 0, \end{aligned}$$

and so

$$g(x+y) = g(x) + g(y) + \ln a^{2xy-1}$$

for all $x, y \in R$. From the inequality (5), we have

$$|f(x) - g(x)| \leq \delta$$

for all $x \in R$. Now suppose that h satisfies the equation

$$h(x+y) = h(x) + h(y) + \ln a^{2xy-1}$$

for all $x, y \in R$ and

$$|f(x) - h(x)| \leq \delta$$

for all $x \in R$. Then

$$\begin{aligned}&|g(x) - h(x)|\\ &\leq \lim_{n\to\infty} \frac{1}{2^n}\left|f(2^n x) - h(2^n x)\right| + \lim_{n\to\infty}\left|\frac{h(2^n x)}{2^n} - \ln\prod_{i=1}^{n} a^{\frac{2^{2i-1}x^2-1}{2^i}} - h(x)\right|\\ &= \lim_{n\to\infty}\frac{\delta}{2^n} + 0 = 0\end{aligned}$$

as $n \to \infty$ and for all $x \in R$, and thus g is unique. □

Corollary 2. *Let A be a Banach algebra and $\delta > 0$ be given. Assume that a mapping $f : A \to A$ satisfies the functional inequality*

$$||f(x+y) - f(x) - f(y) - (2xy-1)|| < \delta$$

for all $x, y \in A$. Then there exists a unique mapping $g : A \to A$ such that

$$g(x+y) = g(x) + g(y) + 2xy - 1$$

for all $x, y \in A$ and

$$|f(x) - g(x)| \leq \delta$$

for all $x \in A$. In particular, g is defined by

$$g(x) := \lim_{n\to\infty} P_n(x)$$

where

$$P_n(x) = \frac{f(2^n x)}{2^n} - \sum_{i=1}^{n}\left(\frac{2^{2i-1}x^2 - 1}{2^i}\right)$$

for all $x, y \in A$.

Proof. From Theorem 1 with $a = e$, we complete the proof. □

The following theorem is the Hyers-Ulam stability of the functional equation

$$f(x+y) = f(x) + f(y) + \ln\beta(x,y)^{-1}$$

which is related to the beta function.

Theorem 3. *Let $\delta > 0$ and $a > 0$ be given. Assume that a mapping $f : (0,\infty) \to (0,\infty)$ satisfies the functional inequality*

$$|f(x+y) - f(x) - f(y) - \ln\beta(x,y)^{-1}| < \delta \tag{6}$$

for all $x, y \in (0,\infty)$. Then there exists a unique mapping $g : (0,\infty) \to (0,\infty)$ such that

$$g(x+y) = g(x) + g(y) + \ln\beta(x,y)^{-1}$$

for all $x, y \in (0, \infty)$ and

$$|f(x) - g(x)| \leq \delta$$

for all $x \in (0, \infty)$. In particular, g is defined by

$$g(x) := \lim_{n \to \infty} P_n(x)$$

where

$$P_n(x) = \frac{f(2^n x)}{2^n} - \ln \prod_{i=0}^{n-1} \beta(2^i x, 2^i x)^{\frac{1}{2^{i+1}}}$$

for all $x, y \in (0, \infty)$.

Proof. If we replace y by x and dividing 2 in (6), we get

$$\left| \frac{f(2x)}{2} + \ln \beta(x, x)^{\frac{1}{2}} - f(x) \right| \leq \frac{\delta}{2} \tag{7}$$

for all $x \in R$. We use induction on n to prove

$$\left| \frac{f(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} \beta(2^i x, 2^i x)^{\frac{1}{2^{i+1}}} - f(x) \right| \leq \delta \sum_{i=1}^{n-1} \frac{1}{2^{i+1}} \tag{8}$$

for all $x > 0$. On account of (7), the inequality holds for $n = 1$. Suppose that inequality (8) holds true for some integer $n > 1$. Then (7) and (8) imply

$$\begin{aligned}
&\left| \frac{f(2^{n+1} x)}{2^{n+1}} + \ln \prod_{i=0}^{n} \beta(2^i x, 2^i x)^{\frac{1}{2^{i+1}}} - f(x) \right| \\
&\leq \left| \frac{f(2^n \cdot 2x)}{2 \cdot 2^n} + \frac{1}{2} \ln \prod_{i=0}^{n-1} \beta(2^i \cdot 2x, 2^i \cdot 2x)^{\frac{1}{2^{i+1}}} - \frac{f(2x)}{2} \right| \\
&\quad + \left| \frac{f(2x)}{2} + \ln \beta(x, x)^{\frac{1}{2}} - f(x) \right| \\
&\leq \delta \sum_{i=0}^{n} \frac{1}{2^{i+1}}
\end{aligned}$$

for any $x > 0$, which ends the proof of (8). For any $x > 0$ and for every positive integer n we define that

$$P_n(x) = \frac{f(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} \beta(2^i x, 2^i x)^{\frac{1}{2^{i+1}}}$$

for all $x, y > 0$. Let $m, n > 0$ be integers with $n > m$. Then it follows from (6) that for all $x > 0$

$$\begin{aligned}
&|P_n(x) - P_m(x)| \\
&= \frac{1}{2^m}\left|\frac{f(2^{n-m}(2^m x))}{2^{n-m}} + \ln\prod_{i=0}^{n-1}\beta(2^i x, 2^i x)^{\frac{1}{2^{i-m+1}}} - f(2^m x)\right| \\
&= \frac{1}{2^m}\left|\frac{f(2^{n-m}(2^m x))}{2^{n-m}} + \ln\prod_{i=0}^{n-m-1}\beta(2^i 2^m x, 2^i 2^m x)^{\frac{1}{2^{i+1}}} - f(2^m x)\right| \\
&\leq \delta\sum_{i=m}^{n-1}\frac{1}{2^{i+1}} \to 0
\end{aligned}$$

as $m \to \infty$. Therefore, the sequence $\{P_n(x)\}$ is a Cauchy sequence, and we may define a function $g : (0, \infty) \to (0, \infty)$ by

$$g(x) := \lim_{n\to\infty} P_n(x)$$

for all $x > 0$. Now we prove that

$$g(x + y) = g(x) + g(y) + \ln\beta(x, y)^{-1}$$

for all $x, y > 0$. For this, we consider the following property of the beta function.

$$\beta(x + y, x + y) = \frac{\beta(x, x + y)\beta(y, y + 2x)}{\beta(x, y)} = \frac{\beta(x, x)\beta(y, y)\beta(2x, 2y)}{\beta(x, y)^2}$$

for all $x, y > 0$. By this property, we have the equation

$$\begin{aligned}
&\prod_{i=0}^{n-1}\left[\frac{\beta(2^i(x + y), 2^i(x + y))}{\beta(2^i x, 2^i x)\beta(2^i y, 2^i y)}\right]^{\frac{1}{2^{i+1}}} = \prod_{i=0}^{n-1}\left[\frac{\beta(2^{i+1}x, 2^{i+1}y)}{\beta(2^i x, 2^i y)^2}\right]^{\frac{1}{2^{i+1}}} \\
&= \left[\frac{\beta(2x, 2y)}{\beta(x, y)^2}\right]^{\frac{1}{2}} \cdot \left[\frac{\beta(2^2x, 2^2y)}{\beta(2x, 2y)^2}\right]^{\frac{1}{2^2}} \cdots \left[\frac{\beta(2^n x, 2^n y)}{\beta(2^{n-1}x, 2^{n-1}y)^2}\right]^{\frac{1}{2^n}} \qquad (9) \\
&= \frac{\beta(2^n x, 2^n y)^{\frac{1}{2^n}}}{\beta(x, y)}
\end{aligned}$$

for all x, $y > 0$. From this equation (9) we get

$$\begin{aligned}
&\left|g(x + y) - g(x) - g(y) - \ln\beta(x, y)^{-1}\right| \\
&= \lim_{n\to\infty}\left|\frac{f(2^n x + 2^n y)}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} - \frac{1}{2^n}\ln\beta(2^n x, 2^n y)^{-1}\right| \\
&\leq \lim_{n\to\infty}\frac{\delta}{2^n} = 0
\end{aligned}$$

for all $x, y > 0$ and thus

$$g(x+y) = g(x) + g(y) + \ln \beta(x,y)^{-1}$$

for all $x, y > 0$. From the inequality (8), we have

$$|f(x) - g(x)| \leq \delta$$

for all $x > 0$. Now suppose that h satisfies the equation

$$h(x+y) = h(x) + h(y) + \ln \beta(x,y)^{-1}$$

for all $x, y > 0$ and

$$|f(x) - h(x)| \leq \delta$$

for all $x > 0$. Then for all $x > 0$

$$\begin{aligned} &|g(x) - h(x)| \\ &\leq \lim_{n\to\infty} \frac{1}{2^n}\left|f(2^n x) - h(2^n x)\right| + \lim_{n\to\infty}\left|\frac{h(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} \beta(2^i x, 2^i y)^{\frac{1}{2^{i+1}}} - h(x)\right| \\ &\leq \lim_{n\to\infty} \frac{\delta}{2^n} + 0 = 0 \end{aligned}$$

as $n \to \infty$ and for all $x > 0$, and thus g is unique. □

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