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# HIGHER LEFT DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT. In this note, we extend the Brešar and Vukman's result [1, Proposition 1.6], which is well-known, to higher left derivations as follows: let $R$ be a ring. (i) Under a certain condition, the existence of a nonzero higher left derivation implies that $R$ is commutative. (ii) if $R$ is semiprime, every higher left derivation on $R$ is a higher derivation which maps $R$ into its center.

## 1. INTRODUCTION

Throughout this note, $R$ will represent an associative ring with center $Z(R)$ and we will write $[a,b]$ for the commutator $ab - ba$. It is easy to see that the identity $[a,bc] = [a,b]c + b[a,c]$ holds for all $a, b, c \in R$. Recall that $R$ is *semiprime* (resp. *prime*) if $aRa = 0$ implies $a = 0$ (resp. $aRb = 0$ implies $a = 0$ or $b = 0$).

A *derivation* is an additive mapping $\delta : R \to R$ satisfying the equation $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in R$. An additive mapping $\delta : R \to R$ is called a *left derivation* if $\delta(ab) = a\delta(b) + b\delta(a)$ holds for all $a, b \in R$.

M. Brešar and J. Vukman [1, Proposition 1.6] obtained the following results: *let $R$ be a ring, $X$ be a left $R$-module and $\delta : R \to X$ be a left derivation.*

(i) *Suppose that $aRx = 0$ with $a \in R$, $x \in X$ implies $a = 0$ or $x = 0$. If $\delta \neq 0$, then $R$ is commutative.*

(ii) *Suppose that $X = R$ is a semiprime ring. Then $\delta$ is a derivation which maps $R$ into $Z(R)$.*

Higher derivations as a generalization of derivations have been studied in rings (mainly in commutative rings), but also in noncommutative rings (see [2], [3], [4], [5]). In this note, we introduce higher left derivations to extend the above Brešar and Vukman's result [1, Proposition 1.6] to higher left derivations.

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## 2. MAIN RESULTS

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of all nonnegative integers.

**Definition 2.1.** Let $\mathcal{D} = (\delta_n)_{n\in\mathbb{N}}$ be a sequence of additive mappings on $R$. $\mathcal{D}$ is said to be:

(i) *a higher derivation* if for each $n \in \mathbb{N}$ and $i, j \in \mathbb{N}_0$,

$$\delta_n(ab) = \sum_{\substack{i+j=n \\ i\leq j}} [\delta_i(a)\delta_j(b) + c_{ij}\delta_j(a)\delta_i(b)] \quad \text{for all } a, b \in R$$

$$(\text{or } \ \delta_n(ab) = \sum_{i+j=n} \delta_i(a)\delta_j(b) \quad \text{for all } a, b \in R),$$

where $\delta_0 = id_R$ and

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

(ii) *a higher left derivation* if for each $n \in \mathbb{N}$ and $i, j \in \mathbb{N}_0$,

$$\delta_n(ab) = \sum_{\substack{i+j=n \\ i\leq j}} [\delta_i(a)\delta_j(b) + c_{ij}\delta_i(b)\delta_j(a)] \quad \text{for all } a, b \in R,$$

where $\delta_0 = id_R$ and

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

We precede the proof of our main theorem by lemmas.

**Lemma 2.2.** Let $R$ be a ring, $X$ be a left $R$-module and $\mathcal{D} = (\delta_n)_{n\in\mathbb{N}}$ be a higher left derivation of additive mappings from $R$ to $X$. If $n \in \mathbb{N}$ is such that $\delta_m = 0$ for all $m < n$, then we have

$$[a, b]\delta_n(a) = 0$$

for all $a, b \in R$.

*Proof.* Given $a, b \in R$, consider $\delta_n(aba)$. On the one hand we have

$$\delta_n(a(ba)) = \sum_{\substack{i+j=n \\ i\leq j}} [\delta_i(a)\delta_j(ba) + c_{ij}\delta_i(ba)\delta_j(a)]$$

$$
\begin{aligned}
&= \sum_{\substack{i+j=n\\ i\leq j}} \delta_i(a)\delta_j(ba) + \sum_{\substack{i+j=n\\ i\leq j}} c_{ij}\delta_i(ba)\delta_j(a)\\
&= \sum_{\substack{i+j=n\\ i\leq j}} \left[\delta_i(a) \sum_{\substack{p+q=j\\ p\leq q}} [\delta_p(b)\delta_q(a) + c_{pq}\delta_p(a)\delta_q(b)]\right]\\
&\quad + ba\delta_n(a) + \sum_{\substack{i+j=n,\, i\neq 0\\ i\leq j}} \left[c_{rs} \sum_{\substack{r+s=i\\ r\leq s}} [\delta_r(b)\delta_s(a) + \delta_r(a)\delta_s(b)]\delta_j(a)\right]\\
&= \sum_{\substack{i+p+q=n\\ i\leq p+q,\, p\leq q}} [\delta_i(a)\delta_p(b)\delta_q(a) + c_{pq}\delta_i(a)\delta_p(a)\delta_q(b)]\\
&\quad + \sum_{\substack{r+s+j=n\\ r+s\leq j,\, r\leq s}} [c_{(r+s)j}\delta_r(b)\delta_s(a)\delta_j(a) + c_{(r+s)j}c_{rs}\delta_r(a)\delta_s(b)\delta_j(a)].
\end{aligned}
$$

On the other hand, we get

$$
\begin{aligned}
\delta_n((ab)a)) &= \sum_{\substack{i+j=n\\ i\leq j}} [\delta_i(ab)\delta_j(a) + c_{ij}\delta_i(a)\delta_j(ab)]\\
&= \sum_{\substack{i+j=n\\ i\leq j}} \delta_i(ab)\delta_j(a) + \sum_{\substack{i+j=n\\ i\leq j}} c_{ij}\delta_i(a)\delta_j(ab)\\
&= ab\delta_n(a) + \sum_{\substack{i+j=n,\, i\neq 0\\ i\leq j}} \left[\sum_{\substack{r+s=i\\ r\leq s}} [\delta_r(a)\delta_s(b) + c_{rs}\delta_r(b)\delta_s(a)]\right]\delta_j(a)\\
&\quad + \sum_{\substack{i+j=n\\ i\leq j}} c_{ij}\delta_i(a)\left[\sum_{\substack{p+q=j\\ p\leq q}} [\delta_p(a)\delta_q(b) + c_{pq}\delta_p(b)\delta_q(a)]\right]\\
&= \sum_{\substack{r+s+j=n\\ r+s\leq j,\, r\leq s}} [\delta_r(a)\delta_s(b)\delta_j(a) + c_{rs}\delta_r(b)\delta_s(a)\delta_j(a)]\\
&\quad + \sum_{\substack{i+p+q=n\\ i\leq p+q,\, p\leq q}} [c_{i(p+q)}\delta_i(a)\delta_p(a)\delta_q(b) + c_{i(p+q)}c_{pq}\delta_i(a)\delta_p(b)\delta_q(a)].
\end{aligned}
$$

Comparing the two expressions for $\delta_n(aba)$, we arrive at

$$
\begin{aligned}
&\sum_{\substack{i+p+q=n\\ i\leq p+q,\, p\leq q}} [(c_{i(p+q)}c_{pq} - 1)\delta_i(a)\delta_p(b)\delta_q(a) + (c_{i(p+q)} - c_{pq})\delta_i(a)\delta_p(a)\delta_q(b)]\\
&+ \sum_{\substack{r+s+j=n\\ r+s\leq j,\, r\leq s}} [(1 - c_{(r+s)j}c_{rs})\delta_r(a)\delta_s(b)\delta_j(a) + (c_{rs} - c_{(r+s)j})\delta_r(b)\delta_s(a)\delta_j(a)] = 0
\end{aligned}
$$

which reduces to

$$\begin{aligned}(1)\quad & ab\delta_n(a) - ba\delta_n(a) \\ & + \sum_{\substack{i+p+q=n,\, i\le p+q \\ p\le q, i\ne 0, p\ne 0, q\ne n}} [(c_{i(p+q)}c_{pq} - 1)\delta_i(a)\delta_p(b)\delta_q(a) + (c_{i(p+q)} - c_{pq})\delta_i(a)\delta_p(a)\delta_q(b)] \\ & + \sum_{\substack{r+s+j=n,\, r+s\le j \\ r\le s, r\ne 0, s\ne 0, j\ne n}} [(1 - c_{(r+s)j}c_{rs})\delta_r(a)\delta_s(b)\delta_j(a) + (c_{rs} - c_{(r+s)j})\delta_r(b)\delta_s(a)\delta_j(a)] \\ & = 0.\end{aligned}$$

From the hypothesis $\delta_m = 0$ for all $m < n$, it follows that the third and fourth sum in the equation (1) are zero, respectively. Hence we have

$$[a, b]\delta_n(a) = 0$$

for all $a, b \in R$. □

**Lemma 2.3.** Let $R$ be a ring and $\mathcal{D} = (\delta_n)_{n\in\mathbb{N}}$ be a higher left derivation of additive mappings on $R$. If $n \in \mathbb{N}$ is such that $\delta_m(R) \subseteq Z(R)$ for all $m < n$, then we have

$$[a, b]\delta_n(a) = 0$$

for all $a, b \in R$.

*Proof.* According to the same process as the proof of Lemma 2.2, we arrive at the equation (1). Now the expansions of the third and fourth sum in (1) can be divided into the following cases.

Case I: $n = 4k$, $k \in \mathbb{N}$.

The indices used in the third and fourth sum of (1) can be classified among the followings:

$$\begin{aligned} p<q, &\quad i<p+q \\ p=q, &\quad i=p+q \\ p<q, &\quad i=p+q \\ p=q, &\quad i<p+q \end{aligned}$$

and

$$\begin{aligned} r<s, &\quad r+s<j \\ r=s, &\quad r+s=j \\ r<s, &\quad r+s=j \end{aligned}$$

$$r = s, \quad r + s < j.$$

Since, in cases $p < q$, $i < p + q$ and $r < s$, $r + s < j$, we have, in (1),

$$c_{i(p+q)}c_{pq} - 1 = 0 = c_{i(p+q)} - c_{pq}$$

and

$$1 - c_{(r+s)j}c_{rs} = 0 = c_{rs} - c_{(r+s)j}.$$

Thus (1) yields

$$\begin{aligned} ab\delta_n(a) - ba\delta_n(a) &- \sum_{\substack{p=q \\ i=p+q}} \delta_i(a)\delta_p(b)\delta_q(a) \\ &- \sum_{\substack{p<q \\ i=p+q}} [\delta_i(a)\delta_p(b)\delta_q(a) + \delta_i(a)\delta_p(a)\delta_q(b)] \\ &- \sum_{\substack{p=q \\ i<p+q}} [\delta_i(a)\delta_p(b)\delta_q(a) - \delta_i(a)\delta_p(a)\delta_q(b)] \\ &+ \sum_{\substack{r=s \\ r+s=j}} \delta_r(a)\delta_s(b)\delta_j(a) \\ &+ \sum_{\substack{r<s \\ r+s=j}} [\delta_r(a)\delta_s(b)\delta_j(a) + \delta_r(b)\delta_s(a)\delta_j(a)] \\ &+ \sum_{\substack{r=s, r+s<j \\ r\neq 0, s\neq 0, j\neq n}} [\delta_r(a)\delta_s(b)\delta_j(a) - \delta_r(b)\delta_s(a)\delta_j(b)] = 0. \end{aligned} \tag{2}$$

From the hypothesis $\delta_m(R) \subseteq Z(R)$ for all $m < n$, it follows that the fifth and eighth sum in (2) are zero, respectively. Since we know that $p + q = i$ and $r + s = j$ in expressions for $\delta_n(a(ba))$ and $\delta_n((ab)a)$, we deduce that $i = j$ in the third (resp. fourth) and the sixth (resp. seventh) sum of (2). Making use of the hypothesis $\delta_m(R) \subseteq Z(R)$ for all $m < n$ to compare the third (resp. fourth) sum with the sixth (resp. seventh) sum in (2), we see that the third (resp. fourth) and the sixth (resp. seventh) sum can be disappeared from (2). Hence we have

$$[a, b]\delta_n(a) = 0$$

for all $a, b \in R$.

Case II: $n = 4k - 2$, $k \in \mathbb{N}$.

The relations among indices of the the third and fourth sum in (1) are as follows:

$$p < q, \quad i < p + q$$

$$p < q, \quad i = p + q$$
$$p = q, \quad i < p + q$$

and

$$r < s, \quad r + s < j$$
$$r < s, \quad r + s = j$$
$$r = s, \quad r + s < j,$$

where $p + q, j \leq n$. Then (1) implies that

$$\begin{aligned} ab\delta_n(a) - ba\delta_n(a) &- \sum_{\substack{p<q \\ i=p+q}} [\delta_i(a)\delta_p(b)\delta_q(a) - \delta_i(a)\delta_p(a)\delta_q(b)] \\ &- \sum_{\substack{p=q \\ i<p+q}} [\delta_i(a)\delta_p(b)\delta_q(a) - \delta_i(a)\delta_p(a)\delta_q(b)] \\ &+ \sum_{\substack{r<s \\ r+s=j}} [\delta_r(a)\delta_s(b)\delta_j(a) - \delta_r(b)\delta_s(a)\delta_j(a)] \\ &+ \sum_{\substack{r=s, r+s<j \\ r\neq 0, s\neq 0, j\neq n}} [\delta_r(a)\delta_s(b)\delta_j(a) - \delta_r(b)\delta_s(a)\delta_j(b)] = 0. \end{aligned} \tag{3}$$

From the same argument as Case I, it follows that the third, the fourth, fifth and sixth sum in (3) vanish into (3). So we get

$$[a, b]\delta_n(a) = 0$$

for all $a, b \in R$.

Case III: $n$ is odd.

Indices in the the third and fourth sum of (1) are separated into the followings:

$$p < q, \quad i < p + q$$
$$p = q, \quad i < p + q$$

and

$$r < s, \quad r + s < j$$
$$r = s, \quad r + s < j,$$

where $p + q, j \leq n$. Then (1) means that

$$ab\delta_n(a) - ba\delta_n(a) - \sum_{\substack{p=q\\ i<p+q}} [\delta_i(a)\delta_p(b)\delta_q(a) - \delta_i(a)\delta_p(a)\delta_q(b)]$$
$$+ \sum_{\substack{r=s, r+s<j\\ r\neq 0, s\neq 0, j\neq n}} [\delta_r(a)\delta_s(b)\delta_j(a) - \delta_r(b)\delta_s(a)\delta_j(b)] = 0.$$

From the hypothesis $\delta_m(R) \subseteq Z(R)$ for all $m < n$, we see that that the third and the fourth sum in (4) are zero which gives

$$[a, b]\delta_n(a) = 0$$

for all $a, b \in R$. This completes the proof. □

Now we are ready to prove our main theorem.

**Theorem 2.4.** *Let $R$ be a ring, $X$ be a left $R$-module and $\mathcal{D} = (\delta_n)_{n\in\mathbb{N}}$ be a higher left derivation of additive mappings from $R$ to $X$.*

(i) *Suppose that $aRx = 0$ with $a \in R$, $x \in X$ implies $a = 0$ or $x = 0$. If $\mathcal{D} \neq 0$, then $R$ is commutative.*

(ii) *Suppose that $X = R$ is a semiprime ring. Then $\mathcal{D}$ is a higher derivation which maps $R$ into $Z(R)$.*

*Proof.* (i) We use the induction. Assume that $R$ is noncommutative. If $n = 1$, then it follows from [1, Proposition 1.6(i)] that $\delta_1 = 0$. If $n \geq 2$ and $\delta_m = 0$ for all $m < n$, then it follows from Lemma 2.2 that

$$[a, b]\delta_n(a) = 0 \tag{4}$$

for all $a, b \in R$. Replacing $b$ by $cb$ in (4), we have

$$[a, cb]\delta_n(a) = (c[a, b] + [a, c]b)\delta_n(a) = 0,$$

that is,

$$[a, c]b\delta_n(a) = 0$$

for all $a, b, c \in R$. Hence the hypothesis gives us that for each $a \in R$, either $a \in Z(R)$ or $\delta_n(a) = 0$. But then, since $Z(R)$ and $Ker\,\delta_n = \{a \in R : \delta_n(a) = 0\}$ are additive subgroups of $R$, we get either $R = Z(R)$ or $R = Ker\,\delta_n$. Thus we see that $\delta_n = 0$ since $R$ was noncommutative. This yields $\mathcal{D} = 0$.

(ii) By induction, we intend to prove that $\delta_n(R) \subseteq Z(R)$ for all $n \in \mathbb{N}$. If $n = 1$, then it follows from [1, Proposition 1.6(ii)] that $\delta_1(R) \subseteq Z(R)$. Therefore, we assume that $n \geq 2$ and $\delta_m(R) \subseteq Z(R)$ for all $m < n$. Then, from Lemma 2.3, we see that

$$[a, b]\delta_n(a) = 0$$

for all $a, b \in R$. As in (i), we obtain that

$$[a, c]b\delta_n(a) = 0 \tag{5}$$

for all $a, b, c \in R$. A linearization of (5) with $a = a + d$ gives

$$[a, c]b\delta_n(d) + [d, c]b\delta_n(a) = 0 \tag{6}$$

for all $a, b, c, d \in R$. Hence we get, from (5) and (6),

$$[a, c]b\delta_n(d)x[a, c]b\delta_n(d) = -[d, c]b\delta_n(a)x[a, c]b\delta_n(d) = 0 \tag{7}$$

for all $a, b, c, d, x \in R$. Since $R$ is semiprime, the equation (7) yields

$$[a, c]b\delta_n(d) = 0$$

for all $a, b, c, d \in R$. In particular, we have

$$[a, \delta_n(d)]b[a, \delta_n(d)] = 0$$

for all $a, b, d \in R$ which implies $[a, \delta_n(d)] = 0$ by the semiprimeness of $R$. That is, we see that $\delta_n(R) \subseteq Z(R)$. Consequently, $\mathcal{D}$ is a higher derivation. The proof of the theorem is complete. □

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