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REAL RANK OF C^*-ALGEBRAS OF TYPE I

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ABSTRACT. We estimate the real rank of a composition series of closed ideals of a C^*-algebra such that its subquotients have continuous trace, which is equivalent to that the C^*-algebra is of type I.

INTRODUCTION

The real rank and the stable rank for C^*-algebras are introduced by Brown-Pedersen [2] and Rieffel [9] respectively. The ranks are viewed as noncommutative counterparts to the real and complex dimensions for topological spaces in the sense that the ranks for commutative C^*-algebras coincide with the dimensions for their spectrums. Since a stable rank estimate for an extension of C^*-algebras has been obtained by Rieffel, the stable rank for certain type I C^*-algebras has been estimated ([5], [6], [11], and [12]). On the other hand, a similar, real rank estimate for an extension of C^*-algebras has been obtained by [14] just recently. In this paper, using this remarkable estimate we consider the real rank of C^*-algebras of type I, which have composition series of closed ideals such that their subquotients have continuous trace. Refer to either [3], [7], or [8]. See also [13] for the real rank of CCR C^*-algebras, by a different method in computation.

Now recall from [2] and [9] some definitions as follows. For a C^*-algebra $\mathfrak{A}$, we denote by $\mathfrak{A}_{sa}$ the set of all self-adjoint elements of $\mathfrak{A}$. For a unital C^*-algebra $\mathfrak{A}$, denote by $L_n(\mathfrak{A})$ the set of all elements $(a_j)_{j=1}^n$ of $\mathfrak{A}^n$ such that for each $(a_j)_{j=1}^n$, we have $\sum_{j=1}^n b_j a_j = 1$ for some $(b_j)_{j=1}^n$ of $\mathfrak{A}^n$. Set $L_n(\mathfrak{A})_{sa} = L_n(\mathfrak{A}) \cap (\mathfrak{A}_{sa})^n$. The real rank of a unital C^*-algebra $\mathfrak{A}$ is defined to be the smallest non-negative integer $n = \mathrm{RR}(\mathfrak{A}) \geq 0$ such that $L_{n+1}(\mathfrak{A})_{sa}$ is dense in $(\mathfrak{A}_{sa})^{n+1}$. If no such integer exists, then set $\mathrm{RR}(\mathfrak{A}) = \infty$. The real rank of a non-unital C^*-algebra $\mathfrak{A}$ is defined by $\mathrm{RR}(\mathfrak{A}) = \mathrm{RR}(\mathfrak{A}^+)$, where $\mathfrak{A}^+$ is the unitization of $\mathfrak{A}$. The connected stable rank of

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a unital C^*-algebra $\mathfrak{A}$ is defined to be the smallest positive integer $n = \mathrm{csr}(\mathfrak{A}) \geq 1$ such that $L_m(\mathfrak{A})$ is connected for any $m \geq n$. If no such integer exists, then set $\mathrm{csr}(\mathfrak{A}) = \infty$. The connected stable rank of a non-unital C^*-algebra $\mathfrak{A}$ is defined by $\mathrm{csr}(\mathfrak{A}) = \mathrm{csr}(\mathfrak{A}^+)$.

List of Formulae

For $x \in \mathbb{R}$, set $\lceil x \rceil = \lfloor x \rfloor + 1$ if x is not an integer, where $\lfloor x \rfloor$ means the maximum integer $\leq x$, and $\lceil x \rceil = x$ if x is an integer.

Let $C(X)$ be the C^*-algebra of all continuous functions on a compact Hausdorff space X. Then

$$\mathrm{RR}(C(X)) = \dim X, \qquad ([2, \text{Proposition } 1.1]);$$
$$\mathrm{csr}(C(X)) \leq \lfloor (\dim X + 1)/2 \rfloor + 1, \qquad ([5, \text{Corollary } 2.5]).$$

Also, as for C^*-tensor products with $M_n(\mathbb{C})$ the C^*-algebra of all $n \times n$ matrix algebra over $\mathbb{C}$ and $\mathbb{K}$ the C^*-algebra of all compact operators on a Hilbert space, by [1],

$$\mathrm{RR}(C(X) \otimes M_n(\mathbb{C})) = \lceil \dim X/(2n-1) \rceil \quad \text{and} \quad \mathrm{RR}(\mathfrak{A} \otimes \mathbb{K}) \leq 1$$

for any C^*-algebra $\mathfrak{A}$. For a C^*-algebra $\mathfrak{A}$, by [10, Theorem 4.7],

$$\mathrm{csr}(\mathfrak{A} \otimes M_n(\mathbb{C})) \leq \lceil (\mathrm{csr}(\mathfrak{A}) - 1)/n \rceil + 1,$$

and $\mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq 2$ by [5, Corollary 2.5] or [11, Theorem 3.10].

For a short exact sequence of C^*-algebras: $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{J} \to 0$,

$$\max\{\mathrm{RR}(\mathfrak{J}), \mathrm{RR}(\mathfrak{A}/\mathfrak{J})\} \leq \mathrm{RR}(\mathfrak{A}), \quad \text{and}$$
$$\mathrm{RR}(\mathfrak{A}) \leq \max\{\mathrm{RR}(\mathfrak{J}), \mathrm{RR}(\mathfrak{A}/\mathfrak{J}), \mathrm{csr}(\mathfrak{A}/\mathfrak{J}) - 1\}$$

by [4] and [14] respectively. On the other hand, by [11, Theorem 3.9]

$$\mathrm{csr}(\mathfrak{A}) \leq \max\{\mathrm{csr}(\mathfrak{J}), \mathrm{csr}(\mathfrak{A}/\mathfrak{J})\}.$$

1. Real Rank Estimates

As shown in [14], we have the following:

Proposition 1.1. *For an extension of C^*-algebras: $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{J} \to 0$, we have*

$$\mathrm{RR}(\mathfrak{A}) \leq \max\{\mathrm{RR}(\mathfrak{J}), \mathrm{RR}(\mathfrak{A}/\mathfrak{J}), \mathrm{csr}(\mathfrak{A}/\mathfrak{J}) - 1\}.$$

Proof (*for completeness,* (*but slightly improved*)). We may assume that the maximum is finite since if it is infinite, then the estimate is automatic. Let n be the maximum. We also may assume that $\mathfrak{A}$ is unital, if it is non-unital, we consider its unitization $\mathfrak{A}^+$. Let $(a_j)_{j=0}^n \in \mathfrak{A}^{n+1}$ with $a_j = a_j^*$. Let U be an open neighbourhood of $(a_j)_{j=0}^n$. Let $\pi : \mathfrak{A}^{n+1} \to (\mathfrak{A}/\mathfrak{J})^{n+1}$ be the map induced from the quotient map from $\mathfrak{A}$ to $\mathfrak{A}/\mathfrak{J}$. Then there exists an element $(b_j)_{j=0}^n$ of the intersection $\pi(U) \cap L_{n+1}(\mathfrak{A}/\mathfrak{J})_{sa}$. Since $\mathrm{csr}(\mathfrak{A}/\mathfrak{J}) \leq n+1$, we have $S(b_j)_{j=0}^n = (1, 0, \cdots, 0)$ for an $(n+1) \times (n+1)$ invertible matrix S over $\mathfrak{A}/\mathfrak{J}$ connecting to the identity matrix. Let $T(c_j)_{j=0}^n$ be a lift of $S(b_j)_{j=0}^n$, so that $T(c_j)_{j=0}^n \in (\mathfrak{J}^+)^{n+1}$, where $T \in GL_{n+1}(\mathfrak{A})_0$. Let $(d_j)_{j=0}^n = \frac{1}{2}\{T(c_j)_{j=0}^n + (T(c_j)_{j=0}^n)^*\}$. Since $\mathrm{RR}(\mathfrak{J}) \leq n$, we may assume that $(d_j)_{j=0}^n \in L_{n+1}(\mathfrak{J}^+)_{sa}$. Note that any element of $\mathfrak{J}$ is mapped to zero by π, so that the set of such $(d_j)_{j=0}^n$ corresponding to an open neighbourhood of $S(b_j)_{j=0}^n$ is open relative to $(\mathfrak{J}_{sa}^+)^{n+1}$. Taking a deformation of T to the identity matrix in $GL_{n+1}(\mathfrak{A})_0$, we have $\frac{1}{2}\{(c_j)_{j=0}^n + (c_j^*)_{j=0}^n\}$ is in $U \cap L_{n+1}(\mathfrak{A})_{sa}$. □

The following follows from a rather elementary observation and might be known:

Proposition 1.2. *Let $\mathfrak{A}$ be a C^*-algebra with continuous trace. Then $\mathfrak{A}$ has a composition series of closed ideals such that its subquotients are trivial as a continuous field of elementary C^*-algebras $M_n(\mathbb{C})$ $(n \in \mathbb{N})$ or $\mathbb{K}$.*

Proof. A C^*-algebra has continuous trace if and only if it can be viewed as a continuous field of elementary C^*-algebras satisfying Fell's condition (see [3, Proposition 10.5.8]). Denote by $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ such a continuous field (vanishing at infinity), where X is a locally compact Hausdorff space and the fibers $\mathfrak{A}_t$ are elementary C^*-algebras, that are either $M_n(\mathbb{C})$ or $\mathbb{K}$. The Fell condition says that for any $s \in X$, there exists an open neighbourhood U_s of s and a continuous operator field p of $\mathfrak{A}$ such that $p(t)$ $(t \in U_s)$ are rank-one projections.

Since $\mathfrak{A}$ is non unital in general, we consider its unitization $\mathfrak{A}^+$ by adding the unit continuous operator field on X. Thus, $\mathfrak{A}^+ = \Gamma(X, \{\mathfrak{A}_t^+\}_{t\in X})$ a bounded continuous field, where $\mathfrak{A}_t^+$ are the unitizations of $\mathfrak{A}_t$.

Now take $s \in X$, $p \in \mathfrak{A}$, and U_s as above. Assume that $\mathfrak{A}$ is not trivial on U_s and at s. Then we have the isomorphism;

$$p\mathfrak{A}^+p \cong \Gamma_0(X, \{p(t)\mathfrak{A}_t^+p(t)\}_{t\in X})$$

and $p(t)\mathfrak{A}_t^+p(t) \cong \mathbb{C}$ for $t \in U_s$.

Next consider $(1-p)\mathfrak{A}^+(1-p) = \Gamma(X, \{(1-p)(t)\mathfrak{A}_t^+(1-p)(t)\}_{t\in X})$, which has continuous trace. If $\mathfrak{A}_s$ is not rank one, i.e., not isomorphic to $\mathbb{C}$, then from the Fell condition we can choose another continuous field q of $\mathfrak{A}$ and an open neighborhood V_s of s such that $q(t)$ $(t \in V_s)$ are rank-one projections. Then we have

$$q(1-p)\mathfrak{A}^+(1-p)q = \Gamma(X, \{q(t)(1-p)(t)\mathfrak{A}_t^+(1-p)(t)q(t)\}_{t\in X})$$

and $q(t)(1-p)(t)\mathfrak{A}_t^+(1-p)(t)q(t) \cong \mathbb{C}$ for $t \in U_s \cap V_s$.

By repeating this process inductively (or finitely) we conclude that the rank of $\mathfrak{A}_t$ is greater than or equal to that of $\mathfrak{A}_s$. In fact, if the rank of $\mathfrak{A}_s$ is strictly greater than that of $\mathfrak{A}_t$ $(t \neq s)$ (locally), then our argument by using the Fell condition deduces a contradiction. It follows that if $\mathfrak{A}_s = \mathbb{K}$, then the fibers $\mathfrak{A}_t$ must be $\mathbb{K}$ (locally).

It is known by [8, Proposition 5.15] that a C^*-algebra has continuous trace as $\mathfrak{A} = \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ if and only if it is locally trivial in the sense that each $t \in X$ has a compact neighborhood K such that the restriction of $\mathfrak{A}$ to K is trivial. Note also that the set of the points of X corresponding to the fiber(s) $M_k(\mathbb{C})$ for $1 \le k \le n$ and n fixed is closed in X (see [3]). Combining these facts and our observation above, we obtain a composition series of closed ideals of $\mathfrak{A}$ such that its subquotients are trivial as a continuous field of C^*-algebras. □

Theorem 1.3. *Let $\mathfrak{A}$ be a C^*-algebra of continuous trace, so that there exists a composition series of closed ideals $\{\mathfrak{I}_j\}$ of $\mathfrak{A}$ such that each subquotient $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ is trivial as a continuous field of C^*-algebras. Then we obtain*

$$\begin{aligned}
&\sup_j \mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1}) \\
&= \sup_j\{\mathrm{RR}(C_0(X_j)\otimes\mathbb{K}) \text{ or } \mathrm{RR}(C_0(X_j)\otimes M_{n_j}(\mathbb{C}))\} \\
&\le \mathrm{RR}(\mathfrak{A}) \le \sup_j\{\mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1}), \mathrm{csr}(\mathfrak{I}_j/\mathfrak{I}_{j-1}) - 1\} \\
&\le \max\{1, \sup_j\lceil \dim X_j^+/(2n_j-1)\rceil, \sup_j\lceil\lfloor(\dim X_j^+ + 1)/2\rfloor/n_j\rceil\} \\
&\le \max\{1, \sup_j\lceil \dim X_j^+/(2n_j-1)\rceil\},
\end{aligned}$$

where $\mathfrak{I}_0 = \{0\}$, and each $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ is isomorphic either to $C_0(X_j)\otimes\mathbb{K}$ or to $C_0(X_j)\otimes M_{n_j}(\mathbb{C})$ for some $n_j \ge 1$, where each X_j is a locally compact Hausdorff space, and $X_j^+ = X_j$ if X_j is compact, and if X_j is non-compact, then X_j^+ means the one-point compactification of X_j.

Furthermore, if $\mathfrak{A}$ has no infinite dimensional irreducible representations, then

1 *in the maximums above is removed, while if* $\mathfrak{A}$ *has no finite dimensional irreducible representations, then the supremums in the maximums above are removed, i.e.,* $\mathrm{RR}(\mathfrak{A}) \leq 1$.

Proof. Using the formulae in the introduction, we obtain

$$\begin{aligned}&\max\{\mathrm{RR}(\mathfrak{I}_{j-1}), \mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1})\} \leq \mathrm{RR}(\mathfrak{I}_j),\\ &\mathrm{RR}(\mathfrak{I}_j) \leq \max\{\mathrm{RR}(\mathfrak{I}_{j-1}), \mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1}), \mathrm{csr}(\mathfrak{I}_j/\mathfrak{I}_{j-1}) - 1\}\end{aligned}$$

for every j (not limit ordinals). Using these estimates inductively, we obtain

$$\begin{aligned}&\max_{1\leq j\leq n} \mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1}) \leq\\ &\mathrm{RR}(\mathfrak{I}_n) \leq \max_{1\leq j\leq n}\{\mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1}), \mathrm{csr}(\mathfrak{I}_j/\mathfrak{I}_{j-1}) - 1\}\end{aligned}$$

for every n.

If the union of $\mathfrak{I}_n$ is dense in $\mathfrak{A}$, then it follows that

$$\begin{aligned}&\sup_j \mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1}) \leq\\ &\mathrm{RR}(\mathfrak{A}) \leq \sup_j\{\mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1}), \mathrm{csr}(\mathfrak{I}_j/\mathfrak{I}_{j-1}) - 1\}.\end{aligned}$$

If not, the same estimates also hold by replacing $\mathfrak{A}$ with the norm closure of the union of $\mathfrak{I}_n$, say $\mathfrak{I}_\infty$. By passing to transfinite induction, we find a composition series of closed ideals $\mathfrak{I}_{\infty,j}$ of $\mathfrak{A}$ such that

$$\mathfrak{I}_\infty \subset \mathfrak{I}_{\infty,1} \subset \mathfrak{I}_{\infty,2} \subset \cdots \subset \mathfrak{I}_{\infty,j} \subset \mathfrak{I}_{\infty,j+1} \subset \cdots \subset \mathfrak{A},$$

and we obtain the same estimates by replacing $\mathfrak{I}_j$ and $\mathfrak{I}_\infty$ with $\mathfrak{I}_{\infty,j}$ and $\mathfrak{I}_{\infty,\infty}$ respectively, where $\mathfrak{I}_{\infty,\infty}$ is the norm closure of the union of $\mathfrak{I}_{\infty,j}$. If $\mathfrak{A} = \mathfrak{I}_{\infty,\infty}$, the induction stops and we obtain the same conclusion (by reindexing to j). If not, we again (and again) pass to transfinite induction, and we obtain the same conclusion, as desired.

Since each $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ is isomorphic either to $C_0(X_j) \otimes \mathbb{K}$ or to $C_0(X_j) \otimes M_{n_j}(\mathbb{C})$, using the formulae in the introduction we have

$$\begin{aligned}&\mathrm{RR}(C_0(X_j) \otimes \mathbb{K}) \leq 1,\\ &\mathrm{RR}(C_0(X_j) \otimes M_{n_j}(\mathbb{C})) \leq \mathrm{RR}(C(X_j^+) \otimes M_{n_j}(\mathbb{C}))\\ &= \lceil \dim X_j^+/(2n_j - 1)\rceil,\end{aligned}$$

and

$$\begin{aligned}&\operatorname{csr}(C_0(X_j)\otimes \mathbb{K}) \le 2,\\ &\operatorname{csr}(C_0(X_j)\otimes M_{n_j}(\mathbb{C})) \le \lceil(\operatorname{csr}(C_0(X_j))-1)/n_j\rceil + 1\\ &= \lceil(\operatorname{csr}(C(X_j^+))-1)/n_j\rceil + 1 \le \lceil\lfloor(\dim X_j^+ + 1)/2\rfloor/n_j\rceil + 1.\end{aligned}$$

Note that the factors $\mathbb{K}$ and $M_{n_j}(\mathbb{C})$ correspond to infinite and finite dimensional irreducible representations of $\mathfrak{A}$ respectively.

Finally, we show that

$$\lceil k/(2n-1)\rceil \ge \lceil\lfloor(k+1)/2\rfloor/n\rceil$$

for any integers $k \ge 0$ and $n \ge 1$. First suppose $k = 2k'$ for $k' \ge 0$. Then

$$\lceil\lfloor(2k'+1)/2\rfloor/n\rceil = \lceil k'/n\rceil = \lceil 2k'/2n\rceil.$$

Since $2k'/2n \le 2k'/(2n-1)$, we have $\lceil 2k'/2n\rceil \le \lceil 2k'/(2n-1)\rceil$. Next suppose $k = 2k'-1$ for $k' \ge 1$. Then

$$\lceil\lfloor(2k'-1+1)/2\rfloor/n\rceil = \lceil k'/n\rceil = \lceil 2k'/2n\rceil.$$

It follows from direct computation that $2k'/2n \le (2k'-1)/(2n-1)$ holds if and only if $k' \ge n$ holds. On the other hand, if $k' < n$, then

$$\lceil k'/n\rceil = \lceil 2k'/2n\rceil = 1 = \lceil(2k'-1)/(2n-1)\rceil.$$

□

Remark. The inequality shown in the last part of the proof is not equality in general, for instance, the case of $n = 1$ and $k \ge 2$. However, it often becomes equality, for instance, the case of $k = 1$, the cases of $k = 2, 3, 5$ and $n \ge 2$, and the case of $k = 4$ and $n \ge 3$.

Lemma 1.4. *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{A}$ has a composition series of closed ideals $\mathfrak{I}_j$, then*

$$\operatorname{csr}(\mathfrak{A}) \le \sup_j \operatorname{csr}(\mathfrak{I}_j/\mathfrak{I}_{j-1}).$$

Proof. For every j (not limit ordinals), we have

$$\operatorname{csr}(\mathfrak{I}_j) \le \max\{\operatorname{csr}(\mathfrak{I}_{j-1}), \operatorname{csr}(\mathfrak{I}_j/\mathfrak{I}_{j-1})\}.$$

Using this formula inductively, we obtain

$$\operatorname{csr}(\mathfrak{I}_n) \le \max_{1\le j\le n} \operatorname{csr}(\mathfrak{I}_j/\mathfrak{I}_{j-1})$$

for every n. If the union of $\mathfrak{I}_j$ is dense in $\mathfrak{A}$, then the conclusion follows. If not, by passing to transfinite induction (as explained above) the same holds. □

Corollary 1.5. *Let $\mathfrak{A}$ be a C^*-algebra of continuous trace as given in the theorem above. Then we obtain*

$$\mathrm{csr}(\mathfrak{A}) \leq \max\{2, \sup_j \lceil \lfloor (\dim X_j^+ + 1)/2 \rfloor / n_j \rceil + 1\}.$$

Furthermore, if $\mathfrak{A}$ has no infinite dimensional irreducible representations, then 2 *in the maximum above is removed, while if $\mathfrak{A}$ has no finite dimensional irreducible representations, then the supremum in the maximum above is removed, i.e.,* $\mathrm{csr}(\mathfrak{A}) \leq 2$.

Theorem 1.6. *Let $\mathfrak{A}$ be a C^*-algebra and $\{\mathfrak{D}_j\}$ its composition series of closed ideals. Then we obtain*

$$\sup_j \mathrm{RR}(\mathfrak{D}_j/\mathfrak{D}_{j-1}) \leq \mathrm{RR}(\mathfrak{A}) \leq \sup_j \{\mathrm{RR}(\mathfrak{D}_j/\mathfrak{D}_{j-1}), \mathrm{csr}(\mathfrak{D}_j/\mathfrak{D}_{j-1}) - 1\}.$$

Suppose that $\mathfrak{A}$ is of type I, which is equivalent to that the subquotients $\mathfrak{D}_j/\mathfrak{D}_{j-1}$ have continuous trace. Then both $\mathrm{RR}(\mathfrak{D}_j/\mathfrak{D}_{j-1})$ and $\mathrm{csr}(\mathfrak{D}_j/\mathfrak{D}_{j-1})$ are estimated as in the theorem and corollary above, respectively.

Proof. Indeed, the proof of the first part of the claim is carried out as in the proof of the theorem above. The second part follows from the theorem and corollary above. □

Remark. This remarkable result would be useful for estimating the real rank of even non-type I C^*-algebras if they have (finite or infinite) composition series of closed ideals such that the real rank of their subquotients is computable. This would be discussed somewhere else.

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