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# DISCRETE PRESENTATIONS OF THE HOLONOMY GROUP OF A ONE-HOLED TORUS

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ABSTRACT. A one-holed torus $\Sigma(1,1)$ is a building block of oriented surfaces. In this paper we formulate the matrix presentations of the holonomy group of a one-holed torus $\Sigma(1,1)$ by the gluing method. And we present an algorithm for deciding the discreteness of the holonomy group of $\Sigma(1,1)$.

## 1. INTRODUCTION

A *hyperbolic* structure on a smooth surface $M$ is a representation of $M$ as a quotient $\Omega/\Gamma$ of a convex domain $\Omega \subset \mathbb{H}^2$ by a discrete group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ acting properly and freely. If the Euler characteristic $\chi(M)$ of $M$ is negative, then the equivalence classes of hyperbolic structures on $M$ form a deformation space $\mathfrak{T}(M)$ called the *Teichmüller space*.

Let $M$ be a compact connected smooth surface with $\chi(M) < 0$. Denote $\pi$ by the fundamental group $\pi_1(M)$ of $M$. For a given hyperbolic structure on $M$, the action of $\pi$ on the universal covering space $\tilde{M}$ of $M$ produces a homomorphism $h : \pi \to \mathbf{PSL}(2,\mathbb{R})$ called the *holonomy homomorphism* and it is well-defined up to conjugation. Hence the Teichmüller space $\mathfrak{T}(M)$ has a natural topology which identified with the open dense subset of the orbit space $\mathrm{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))/\mathbf{PSL}(2,\mathbb{R})$ corresponding to irreducible representations. Since the holonomy homomorphism $h : \pi \to \mathbf{PSL}(2,\mathbb{R})$ is an *isomorphism onto its image* $\Gamma = h(\pi)$ called the *holonomy group*, the generators of $\pi$ can be presented by the matrices in $\mathbf{PSL}(2,\mathbb{R})$ up to conjugation.(Goldman [2], Johnson and Millson [4]) Therefore giving a hyperbolic structure on $M$ is equivalent to finding a discrete subgroup $\Gamma$ of $\mathbf{PSL}(2,\mathbb{R})$ up to conjugation.(Matsuzaki and Taniguchi [8])

Let $M = \Sigma(g,n)$ be a compact connected oriented surface with $g$-genus and $n$-boundary components. If $\chi(M) = 2-2g-n < 0$, then the Teichmüller space $\mathfrak{T}(M)$ is diffeomorphic to $\mathbb{R}^{6g-6+3n}$. And $M$ can be decomposed as a disjoint union

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of $g$ one-holed tori $\Sigma(1,1)$ and $g-2+n$ pairs of pants $\Sigma(0,3)$. Thus a one-holed torus $\Sigma(1,1)$ and a pair of pants $\Sigma(0,3)$ are building blocks of an oriented surface $M$.(Wolpert [11])

The purposes of this paper are the followings: First we formulate the matrix presentations of the holonomy group of a one-holed torus $\Sigma(1,1)$ by the gluing method. The matrix presentations of the holonomy group of a pair of pants $\Sigma(0,3)$ in [6] will be used for the gluing method. Second we give an algorithm for deciding the discreteness of the holonomy group of $\Sigma(1,1)$.

## 2. PRELIMINARIES

Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half plane. The Lie group $\mathbf{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by

$$A \cdot z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}. \tag{2.1}$$

An element $A$ of $\mathbf{SL}(2,\mathbb{R})$ is said to be *hyperbolic* if $A$ has two distinct real eigenvalues. Since $f(x) = x^2 - tx + 1$ is the characteristic polynomial of $A \in \mathbf{SL}(2,\mathbb{R})$ where $t = \mathrm{tr}(A)$, $A$ is hyperbolic if and only if $\mathrm{tr}(A)^2 > 4$. An element $A$ of $\mathbf{PSL}(2,\mathbb{R})$ is said to be *hyperbolic* if $A$ has two distinct fixed points on $\partial\mathbb{H}^2$. Since the absolute value of trace is still defined, $A$ is hyperbolic if and only if $|\mathrm{tr}(A)| > 2$. A hyperbolic element $A$ of $\mathbf{PSL}(2,\mathbb{R})$ can be expressed by the diagonal matrix

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \overset{\text{let}}{=} \pm \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \tag{2.2}$$

via an $\mathbf{SL}(2,\mathbb{R})$-conjugation where $\alpha > 1$.

The following theorem is due to Kuiper [7].

**Theorem 2.1.** *Suppose $M$ is a compact oriented hyperbolic surface. Then every nontrivial element of the holonomy group $\Gamma$ is hyperbolic.*

A hyperbolic manifold $M$ can be *developed* into $\mathbb{H}^2$ as follows.(Thurston [10]) Since the universal covering space $\tilde{M}$ is simply connected, the coordinate charts on $\tilde{M}$ can globalize to define a hyperbolic map $\mathbf{dev} : \tilde{M} \to \mathbb{H}^2$, called the *developing map*. Let $\Omega = \mathbf{dev}(\tilde{M})$ be the *developing image* in $\mathbb{H}^2$. For a non-trivial element $A$ of the holonomy group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$, the *translation length* $\ell(A)$ is defined by $\ell(A) := \inf_{z\in\Omega} d_P(z, A(z))$ where $d_P$ is the Poincaré metric on $\Omega$. From Beardon's book [1], we get the relation

$$\left|\frac{\mathrm{tr}(A)}{2}\right| = \cosh\left(\frac{\ell(A)}{2}\right). \tag{2.3}$$

Suppose that $|\mathrm{tr}(A)| = \alpha + \alpha^{-1}$ with $\alpha > 1$. Since $\cosh^{-1}(t) = \log(t + \sqrt{t^2-1})$, Equation (2.3) becomes

$$\ell(A) = \log(\alpha^2) \tag{2.4}$$

for a hyperbolic element $A \in \mathbf{PSL}(2,\mathbb{R})$.

The *principal line* of a hyperbolic element $A \in \mathbf{PSL}(2,\mathbb{R})$ is the $A$-invariant unique geodesic in $\mathbb{H}^2$. It is the line joining the *repelling* and *attracting* fixed points of $A$. For easy understanding, see Figure 1, 2, and 3 or Beardon's book [1]. We now consider the location of the principal line of $A$ and the relations of entries of $A$. The following Theorem 2.2 is some results in [6].

**Theorem 2.2.** *Let $z_a, z_r$ be the attracting and repelling fixed points of a hyperbolic element* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{\text{let}}{=} \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *in* $\mathbf{PSL}(2,\mathbb{R})$. *Suppose both fixed points are finite (not infinite). Then we have the following relations.*

(1) $0 < z_a < z_r \iff a^2 < d^2,\, b\,c < 0,\, b\,d > 0$
(2) $0 < z_r < z_a \iff a^2 > d^2,\, b\,c < 0,\, a\,c > 0$
(3) $z_a < z_r < 0 \iff a^2 > d^2,\, b\,c < 0,\, a\,c < 0$
(4) $z_r < z_a < 0 \iff a^2 < d^2,\, b\,c < 0,\, b\,d < 0$
(5) $z_a < 0 < z_r \iff b\,c > 0\,,\, a\,c < 0,\, b\,d < 0$
(6) $z_r < 0 < z_a \iff b\,c > 0\,,\, a\,c > 0,\, b\,d > 0$

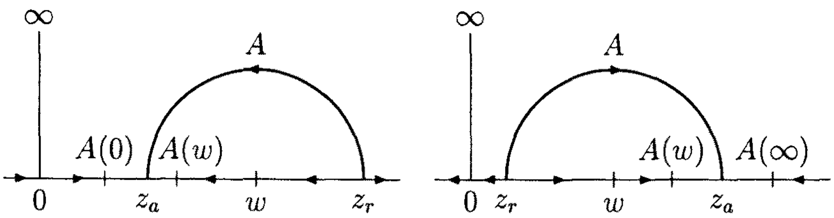


*Figure* 1. Fixed points with $0 < z_a < z_r$ and $0 < z_r < z_a$

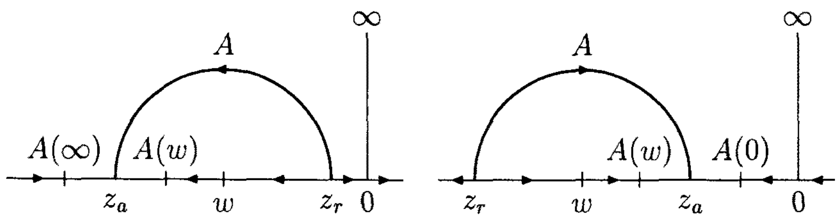


*Figure* 2. Fixed points with $z_a < z_r < 0$ and $z_r < z_a < 0$

**Proposition 2.3.** *Let $\{x, y\}$ and $\{-b, b\}$ be the fixed points of hyperbolic elements $A, B \in \mathbf{PSL}(2,\mathbb{R})$ respectively. Then the principal lines of $A$ and $B$ are perpendicular if and only if $b^2 = xy$.*

*Proof.* (Case I) In the case $x < y$. Consider the linear fractional transformation $f(z) = \frac{z-x}{-z+y}$. This transformation $f : \mathbb{H}^2 \to \mathbb{H}^2$ is well-defined since $\det(f) =$

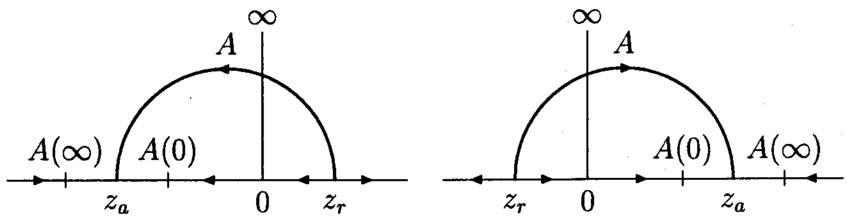


*Figure* 3. Fixed points with $z_a < 0 < z_r$ and $z_r < 0 < z_a$

$y - x > 0$. Since $f$ is a conformal map which send the fixed points $\{x, y\}$ of $A$ to $\{0, \infty\}$, the principal lines of $A$ and $B$ are perpendicular if and only if those of $f(A)$ and $f(B)$ are perpendicular if and only if $f(-b) = -f(b)$; i.e. $\left(\frac{-b-x}{b+y}\right) = -\left(\frac{b-x}{-b+y}\right)$. After some calculation we get the result $b^2 = xy$.

(Case II) In the case $y < x$. Consider $f(z) = \frac{z-x}{z-y}$. Then $f : \mathbb{H}^2 \to \mathbb{H}^2$ is well-defined since $\det(f) = -y + x > 0$. Similarly we can get the result $b^2 = xy$. □

From the condition $xy = b^2 > 0$, we know both fixed points $\{x, y\}$ of $A$ should be positive or negative.

## 3. HOLONOMY GROUP OF A ONE-HOLED TORUS $\Sigma(1,1)$

Suppose a one-holed torus $\Sigma(1,1)$ is equipped with a hyperbolic structure. Since the holonomy homomorphism $h : \pi \to \mathbf{PSL}(2, \mathbb{R})$ is an isomorphism onto its image $\Gamma = h(\pi)$, we will identified the fundamental group $\pi$ of $\Sigma(1,1)$ with the holonomy group $\Gamma$; i.e. $\pi = \Gamma = \langle\, A, B, C \in \mathbf{PSL}(2, \mathbb{R}) \mid R = CB^{-1}A^{-1}BA = I \,\rangle$.

Above argument is true for the hyperbolic structures. (See Goldman [2], Johnson and Millson [4].) But for a general geometric structure theory, the holonomy homomorphism $h$ may not be an isomorphism onto its image. We can find examples in Sullivan and Thurston's paper [9].

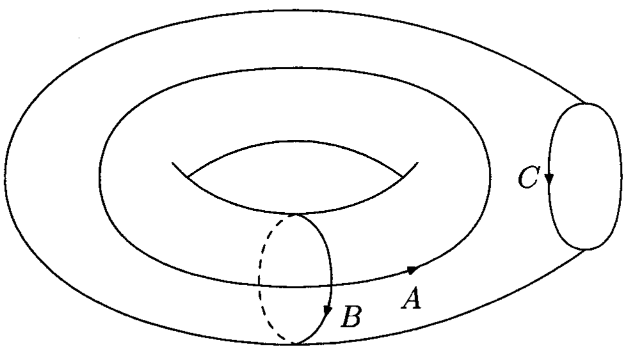


*Figure* 4. A one-holed torus $M = \Sigma(1,1)$

Let $A, B, C \in \mathbf{PSL}(2,\mathbb{R})$ represent elements of the fundamental group of $M$ as in Figure 4. We will find the expression of the generators $A, B$ and $C$ of $\pi$ in terms of $\mathbf{SL}(2,\mathbb{R})$ instead of $\mathbf{PSL}(2,\mathbb{R})$ because $\mathbf{SL}(2,\mathbb{R})$ is more convenient to compute and understand than $\mathbf{PSL}(2,\mathbb{R})$.

We now explain about the gluing method. Let $C_1, C_2, C_3 \in \mathbf{SL}(2,\mathbb{R})$ represent the boundary components of a pair of pants $\Sigma(0,3)$ as in Figure 5. Then the fundamental group $\pi$ of $\Sigma(0,3)$ is identified with $\pi = \langle\, C_1, C_2, C_3 \in \mathbf{SL}(2,\mathbb{R}) \mid R = C_3C_2C_1 = I\,\rangle$.

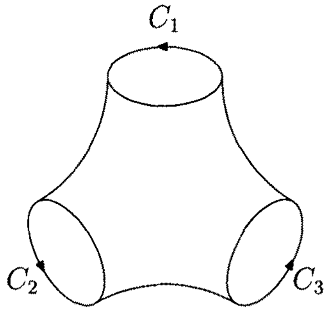


*Figure* 5. A pair of pants $M = \Sigma(0,3)$

Suppose two boundary components $C_1, C_2$ of a pair of pants $\Sigma(0,3)$ have the same translation lengths ; i.e. $\ell(C_1) = \ell(C_2)$. Then a one-holed torus $\Sigma(1,1)$ can be obtained by gluing two boundaries $C_1, C_2$ of a pair of pants $\Sigma(0,3)$. By the orientations of boundary components $C_1$ and $C_2$, the boundary $C_1$ is identified with $C_2^{-1}$ up to conjugation. For an easy understanding, see the Figure 6. Thus there exists a matrix $Q \in \mathbf{SL}(2,\mathbb{R})$ such that $C_1 = Q^{-1}C_2^{-1}Q$.

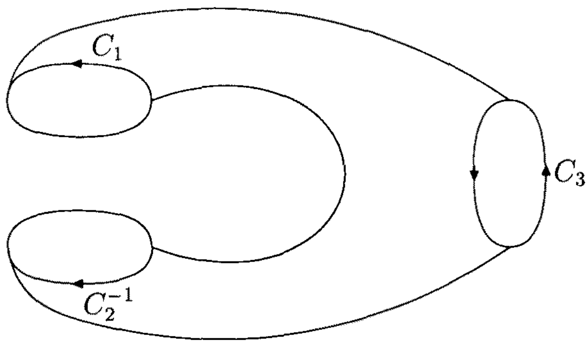


*Figure* 6. Gluing boundary components $C_1$ with $C_2^{-1}$

Without loss of generality, we may assume that $\mathrm{tr}(C_1) > 2$ and $\mathrm{tr}(C_2) > 2$. In the cases $\mathrm{tr}(C_1) < -2$ or $\mathrm{tr}(C_2) < -2$, we replace $C_1$ to $-C_1$ or $C_2$ to $-C_2$. Suppose $\lambda, \mu$ are the eigenvalues of $C_1, C_2$ respectively with $\lambda > 1$ and $\mu > 1$. Since

$\ell(C_1) = \log(\lambda^2)$ and $\ell(C_2) = \log(\mu^2)$ in Equation (2.4), the condition $\ell(C_1) = \ell(C_2)$ induces $\lambda = \mu$.

**Theorem 3.1.** *Suppose $C_1, C_2, C_3 \in \mathbf{SL}(2,\mathbb{R})$ are the generators of the fundamental group of a pair of pants $\Sigma(0,3)$ with $\ell(C_1) = \ell(C_2)$. If $Q \in \mathbf{SL}(2,\mathbb{R})$ is a hyperbolic matrix such that $C_1 = Q^{-1}C_2^{-1}Q$, then $A := Q$, $B := C_2^{-1}$, $C := C_3$ are the generators of the fundamental group of a one-holed torus $\Sigma(1,1)$.*

*Proof.* By assumption, we have $C_1 = Q^{-1}C_2^{-1}Q$. If we define $A = Q$, $B = C_2^{-1}$, $C = C_3$, then they are hyperbolic matrices and satisfy

$$CB^{-1}A^{-1}BA = C_3C_2Q^{-1}C_2^{-1}Q = C_3C_2C_1 = I.$$

Therefore $A$, $B$, $C$ form the generators of the fundamental group of $\Sigma(1,1)$. □

Now we find the matrix presentations of the holonomy group of a one-holed torus $\Sigma(1,1)$ by Theorem 3.1. The following Theorem 3.2 is one of the main results in [6]. Since the matrices $C_1, C_2, C_3 \in \mathbf{SL}(2,\mathbb{R})$ are represented up to conjugation, without loss of generality, we may assume $C_2$ is diagonal.

**Theorem 3.2.** *The following matrices $C_1$, $C_2$, $C_3 \in \mathbf{SL}(2,\mathbb{R})$ with*

$$\lambda > 1,\ \mu > 1,\ a < \lambda^{-1},\ c \neq 0 \tag{3.1}$$

*form the generators of the holonomy group of a pair of pants $\Sigma(0,3)$.*

$$C_1 = \begin{pmatrix} a & -(\lambda - a)(\lambda^{-1} - a)c^{-1} \\ c & \lambda + \lambda^{-1} - a \end{pmatrix},\quad C_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \tag{3.2}$$

*and*

$$C_3 = \begin{pmatrix} \mu^{-1}(\lambda + \lambda^{-1} - a) & \mu(\lambda - a)(\lambda^{-1} - a)c^{-1} \\ -\mu^{-1}c & \mu\, a \end{pmatrix}. \tag{3.3}$$

*Proof.* See Kim's paper [6]. □

The conditions in (3.1) are from the locations of principal lines of $C_1, C_2$, and $C_3$. See the Figure 7. In the case $c < 0$ ($c > 0$), the fixed points of $C_1$ and $C_3$ are positive (negative) respectively. (Compare with the results (1) and (4) in Theorem 2.2.)

We now remind some relations between two hyperbolic elements $A$ and $\widetilde{A}$ in $\mathbf{SL}(2,\mathbb{R})$. See [6] for detail. For a hyperbolic element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2,\mathbb{R})$, we denote $\widetilde{A} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Then $z$ is a fixed point of $A$ if and only if $-z$ is a fixed point of $\widetilde{A}$ since $\widetilde{A}(-z) = -A(z)$. And the principal lines of $A$ and $\widetilde{A}$ are symmetric with respect to the imaginary axis. See the Figure 8.

Without loss of generality, we assume the fixed points of hyperbolic matrices $C_1, C_3$ in (3.2) and (3.3) are positive ; i.e. we shall assume $c < 0$ from now on. In the case the fixed points are negative, we just consider $\widetilde{C}_1, \widetilde{C}_3$ instead of $C_1, C_3$. Then we can reformulate the matrix presentations of $C_1, C_2, C_3$ as follows.

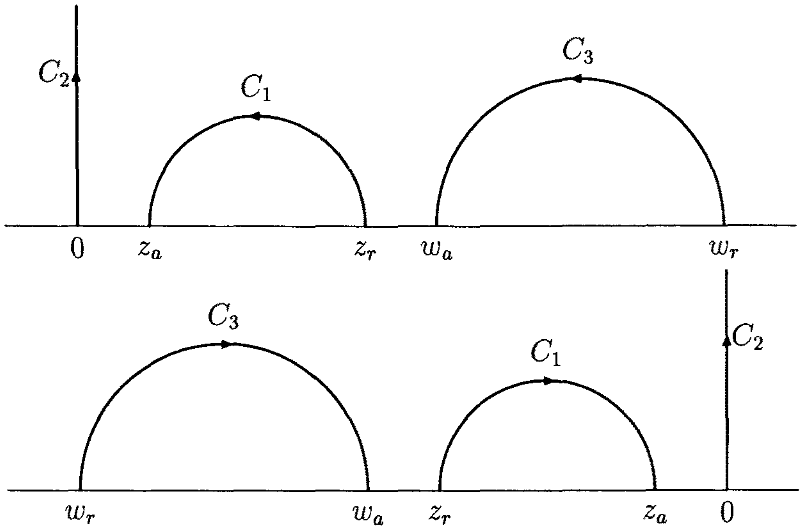


*Figure* 7. The locations of the principal lines of a pair of pants $\Sigma(0,3)$

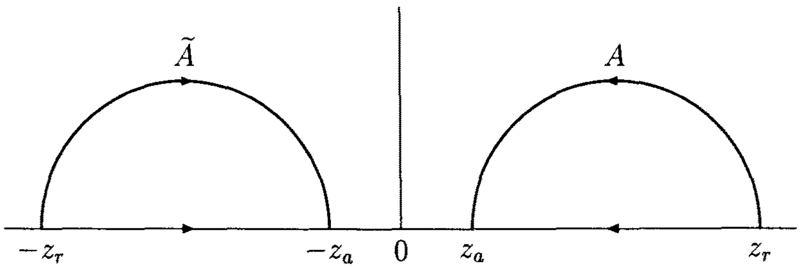


*Figure* 8. The fixed points of the matrices $A$ and $\widetilde{A}$

**Theorem 3.3.** *The following matrices* $C_1$, $C_2$, $C_3 \in \mathbf{SL}(2,\mathbb{R})$ *with*

$$\lambda > 1,\ \mu > 1,\ a < \lambda^{-1} \tag{3.4}$$

*form the generators of the holonomy group of a pair of pants* $\Sigma(0,3)$.

$$C_1 = \begin{pmatrix} a & (\lambda - a)(\lambda^{-1} - a) \\ -1 & \lambda + \lambda^{-1} - a \end{pmatrix},\quad C_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \tag{3.5}$$

*and*

$$C_3 = \begin{pmatrix} \mu^{-1}(\lambda + \lambda^{-1} - a) & -\mu(\lambda - a)(\lambda^{-1} - a) \\ \mu^{-1} & \mu\, a \end{pmatrix}. \tag{3.6}$$

*Proof.* We rename the matrices in (3.2) and (3.3) as $B_1, B_2$, and $B_3$. Let $P = \begin{pmatrix} \sqrt{-c} & 0 \\ 0 & \sqrt{-c^{-1}} \end{pmatrix}$. It is well-defined since we assume $c < 0$. Then we can calculate

$$PAP^{-1} = \begin{pmatrix} \sqrt{-c} & 0 \\ 0 & \sqrt{-c^{-1}} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \sqrt{-c^{-1}} & 0 \\ 0 & \sqrt{-c} \end{pmatrix} = \begin{pmatrix} \alpha & -c\beta \\ -c^{-1}\gamma & \delta \end{pmatrix}.$$

Thus we get the results $PB_1P^{-1} = C_1, PB_2P^{-1} = C_2$, and $PB_3P^{-1} = C_3$ which are the matrices in (3.5) and (3.6). Since the generators of the holonomy group of surfaces are up to conjugation, we can think the matrices $C_1, C_2$ and $C_3$ form the generators of the holonomy group of a pair of pants $\Sigma(0,3)$. □

From now on, we denote $C_1, C_2, C_3$ as the matrices (3.5) and (3.6) in Theorem 3.3 instead of the matrices (3.2) and (3.3) in Theorem 3.2.

Consider the hyperbolic matrices $C_1$, $C_2 \in \mathbf{SL}(2,\mathbb{R})$ in (3.5). Suppose that $C_1$ and $C_2$ have the same translation lengths; i.e. $\ell(C_1) = \ell(C_2)$. Since $\lambda > 1$ and $\mu > 1$, it is equivalent to $\lambda = \mu$ by Equation (2.4). Now we shall find a hyperbolic matrix $Q \in \mathbf{SL}(2,\mathbb{R})$ such that $C_1 = Q^{-1}C_2^{-1}Q$.

Let $Q = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. After some calculations, the relation $QC_1 = C_2^{-1}Q$ induces $y = -(\lambda^{-1} - a)x$ and $w = -(\lambda - a)z$ ; i.e. $Q$ becomes

$$Q = \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & -(\lambda^{-1} - a)x \\ z & -(\lambda - a)z \end{pmatrix}.$$

Then the condition $\det(Q) = 1$ implies that $1 = xw - yz = -xz(\lambda - \lambda^{-1})$. Plug in $x = 1/\sqrt{\lambda - \lambda^{-1}}$ and $z = -x = -1/\sqrt{\lambda - \lambda^{-1}}$. Then the following matrix $Q \in \mathbf{SL}(2,\mathbb{R})$ satisfies the condition $C_1 = Q^{-1}C_2^{-1}Q$ ;

$$Q = \frac{1}{\sqrt{\lambda - \lambda^{-1}}} \begin{pmatrix} 1 & -(\lambda^{-1} - a) \\ -1 & (\lambda - a) \end{pmatrix}. \tag{3.7}$$

**Proposition 3.4.** *Suppose we have another $\bar{Q} \in \mathbf{SL}(2,\mathbb{R})$ such that $C_1 = \bar{Q}^{-1}C_2^{-1}\bar{Q}$. Then there exists a diagonal matrix $D \in \mathbf{SL}(2,\mathbb{R})$ such that $\bar{Q} = DQ$.*

*Proof.* From the condition $Q^{-1}C_2^{-1}Q = C_1 = \bar{Q}^{-1}C_2^{-1}\bar{Q}$, we have $(\bar{Q}Q^{-1})C_2^{-1} = C_2^{-1}(\bar{Q}Q^{-1})$. Since $C_2^{-1}$ is a diagonal matrix, the commutativity of $(\bar{Q}Q^{-1})$ with $C_2^{-1}$ implies $(\bar{Q}Q^{-1})$ should be diagonal. Therefore there exists a diagonal matrix $D \in \mathbf{SL}(2,\mathbb{R})$ such that $\bar{Q} = DQ$. □

Let $D \in \mathbf{SL}(2,\mathbb{R})$ be a diagonal matrix with entries $D_{11} = t$ and $D_{22} = t^{-1}$. Then we have

$$\bar{Q} = DQ = \frac{1}{\sqrt{\lambda - \lambda^{-1}}} \begin{pmatrix} t & -t(\lambda^{-1} - a) \\ -t^{-1} & t^{-1}(\lambda - a) \end{pmatrix}. \tag{3.8}$$

Now we shall show that the matrix $\bar{Q}$ in (3.8) is hyperbolic.

**Proposition 3.5.** *Let $\bar{Q}$ be the matrix in (3.8). Then*

(1) $\mathrm{tr}(\bar{Q}) > 2$ *if and only if* $t > 0$.
(2) $\mathrm{tr}(\bar{Q}) < -2$ *if and only if* $t < 0$.

*Proof.* From Theorem 3.3, we have the conditions $a < \lambda^{-1} < 1 < \lambda$. Suppose $t > 0$. Then we also have $t^{-1}(\lambda - a) > 0$. Thus

$$\mathrm{tr}(\bar{Q}) = \frac{t + t^{-1}(\lambda - a)}{\sqrt{\lambda - \lambda^{-1}}} \geq \frac{2\sqrt{\lambda - a}}{\sqrt{\lambda - \lambda^{-1}}} > 2$$

since $(\lambda - a) > (\lambda - \lambda^{-1})$. Conversely, suppose $\mathrm{tr}(\bar{Q}) > 2$. Since $(\lambda - a) > 0$, the sign of $t$ should be positive. Similarly we can show $t < 0$ if and only if $\mathrm{tr}(\bar{Q}) < 2$. □

Thus the matrix $\bar{Q}$ in (3.8) is hyperbolic and satisfies $C_1 = Q^{-1}C_2^{-1}Q$. Consider the fixed points and the principal line of $\bar{Q}$. Let $w_r$ and $w_a$ be the repelling and attracting fixed points of $\bar{Q}$. We denote $\sqrt{\lambda - \lambda^{-1}}\,\bar{Q}$ by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then we have $\beta\gamma = (\lambda^{-1} - a)$, $\alpha\gamma = -1$, and $\beta\delta = -(\lambda^{-1} - a)(\lambda - a)$. From the results in Theorem 2.2, the conditions $\beta\gamma > 0$, $\alpha\gamma < 0$, $\beta\delta < 0$ induces $w_a < 0 < w_r$.

**Proposition 3.6.** *Suppose $\bar{Q}$ is the hyperbolic matrix in (3.8). Then the attracting and repelling fixed points $w_a$ and $w_r$ of $\bar{Q}$ are*

$$w_a = \frac{-D - \sqrt{E}}{2}, \quad w_r = \frac{-D + \sqrt{E}}{2}$$

*where $D = [t^2 - (\lambda - a)]$ and $E = D^2 + 4t^2(\lambda^{-1} - a)$.*

*Proof.* Since $w_a, w_r$ are the fixed points of the transformation $\bar{Q}(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$, they are the roots of the equation $\gamma z^2 + (\delta - \alpha)z - \beta = 0$. Thus the fixed points $w_a, w_r$ of $\bar{Q}$ are $\frac{(\alpha - \delta) \pm \sqrt{(\alpha-\delta)^2 + 4\beta\gamma}}{2\gamma} = \frac{[t - t^{-1}(\lambda - a)] \pm \sqrt{[t - t^{-1}(\lambda - a)]^2 + 4(\lambda^{-1} - a)}}{-2t^{-1}}$ $= \frac{-[t^2 - (\lambda - a)] \mp \sqrt{[t^2 - (\lambda - a)]^2 + 4t^2(\lambda^{-1} - a)}}{2}$. From the above argument, we know the condition $w_a < 0 < w_r$. Therefore we get the results $w_a = \frac{-D - \sqrt{E}}{2}$ and $w_r = \frac{-D + \sqrt{E}}{2}$. □

**Proposition 3.7.** *The principal line of $\bar{Q}$ intersects those of $C_1$ and $C_2$.*

*Proof.* Since the principal line of $C_2$ is the upper half imaginary axis, the condition $w_a < 0 < w_r$ implies that the principal line of $\bar{Q}$ intersects that of $C_2$.

The fixed points of $C_1$ in (3.5) are $z_a = (\lambda^{-1} - a)$ and $z_r = (\lambda - a)$ with $0 < z_a < z_r$. From the conditions $0 < z_a < z_r$ and $w_a < 0 < w_r$, the principal line of $\bar{Q}$ intersects that of $C_1$ if and only if $z_a < w_r < z_r$

$$\begin{aligned}
&\iff 2(\lambda^{-1} - a) < -D + \sqrt{E} < 2(\lambda - a) \\
&\iff 2(\lambda^{-1} - a) + D < \sqrt{E} < 2(\lambda - a) + D = [t^2 + (\lambda - a)] \\
&\iff [2(\lambda^{-1} - a) + D]^2 < E < [t^2 + (\lambda - a)]^2 = D^2 + 4t^2(\lambda - a) \\
&\iff 4(\lambda^{-1} - a)^2 + 4(\lambda^{-1} - a)D < 4t^2(\lambda^{-1} - a) < 4t^2(\lambda - a) \\
&\iff (\lambda^{-1} - a) + D < t^2 < t^2(\lambda - a)(\lambda^{-1} - a)^{-1}.
\end{aligned}$$

Both inequalities hold since $a < \lambda^{-1} < \lambda$. □

Suppose that the trace of $\bar{Q}$ is *positive* ; i.e. $t > 0$. Now we find when $\bar{Q}$ has the smallest trace. From the proof of Proposition 3.5, we know the smallest value of the trace of $\bar{Q}$ is $2\frac{\sqrt{\lambda-a}}{\sqrt{\lambda-\lambda^{-1}}}$. And it takes when $t = t^{-1}(\lambda - a)$. i.e. $t = \sqrt{\lambda - a}$ since $t > 0$. Let $Q_0$ be the $\bar{Q}$ plug in $t = \sqrt{\lambda - a}$. Then

$$Q_0 = \bar{Q}\big|_{t=\sqrt{\lambda-a}} = \frac{\sqrt{\lambda-a}}{\sqrt{\lambda-\lambda^{-1}}}\begin{pmatrix} 1 & -(\lambda^{-1}-a) \\ -(\lambda-a)^{-1} & 1 \end{pmatrix}. \tag{3.9}$$

**Proposition 3.8.** *Suppose that $Q_0$ is the hyperbolic matrix in (3.9). Then the attracting and repelling fixed points $w_a$ and $w_r$ of $Q_0$ are*

$$w_a = -\sqrt{(\lambda^{-1}-a)(\lambda-a)},\ w_r = \sqrt{(\lambda^{-1}-a)(\lambda-a)}.$$

*Proof.* We denote $Q_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then the fixed points of $Q_0$ are $\frac{(\alpha-\delta)\pm\sqrt{(\alpha-\delta)^2+4\beta\gamma}}{2\gamma}$ $= \frac{\pm 2\sqrt{\beta\gamma}}{2\gamma} = \mp\sqrt{(\lambda^{-1}-a)(\lambda-a)}$. Since $w_a < 0 < w_r$, we get the results. □

**Theorem 3.9.** *The the principal line of $Q_0$ orthogonally intersects those of $C_1$ and $C_2$.*

*Proof.* Denote $b = \sqrt{(\lambda^{-1}-a)(\lambda-a)} > 0$. Then the fixed points of $Q_0$ are $\pm b$. Thus the principal line of $Q_0$ orthogonally intersect $C_2$. Since the fixed points $x, y$ of $C_1$ are $(\lambda^{-1} - a)$ and $(\lambda - a)$, we have $b^2 = xy$. From Proposition 2.3, the principal line of $Q_0$ orthogonally intersect $C_1$. □

Since $\bar{Q}$ is hyperbolic such that $C_1 = \bar{Q}^{-1}C_2^{-1}\bar{Q}$, we have the following theorem.

**Theorem 3.10.** *The generators of the holonomy group of a one-holed torus $\Sigma(1,1)$ are expressed by*

$$A = \frac{1}{\sqrt{\lambda-\lambda^{-1}}}\begin{pmatrix} t & -t(\lambda^{-1}-a) \\ -t^{-1} & t^{-1}(\lambda-a) \end{pmatrix},\quad B = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \tag{3.10}$$

*and*

$$C = \begin{pmatrix} \lambda^{-1}(\lambda+\lambda^{-1}-a) & -\lambda(\lambda-a)(\lambda^{-1}-a) \\ \lambda^{-1} & \lambda a \end{pmatrix} \tag{3.11}$$

*with $\lambda > 1$, $t > 0$, $a < \lambda^{-1}$ up to conjugation.*

*Proof.* Consider the matrices $C_1, C_2, C_3$ in (3.5) and (3.6). First let $\mu = \lambda$ since $\ell(C_1) = \ell(C_2)$. From Theorem 3.1, $A = \bar{Q}$, $B = C_2^{-1}$, $C = C_3$ are the generators of the holonomy group of a one-holed torus $\Sigma(1,1)$ up to conjugation. □

## 4. Application : Algorithm for Deciding the Discreteness

Finally we can present an algorithm for deciding the discreteness of the holonomy group of a one-holed torus $\Sigma(1,1)$. Let $A_r, B_r, A_a, B_a$ be the repelling and attracting fixed points of hyperbolic matrices $A$ and $B$ respectively. We define $\mathsf{CR}(A,B)$ by the cross ratio of $B_a, A_r, A_a, B_r$ ; that is

$$\mathsf{CR}(A,B) = [B_a, A_r, A_a, B_r] = \frac{(B_a - A_a)(A_r - B_r)}{(B_a - A_r)(A_a - B_r)}.$$

Then $\mathsf{CR}(A,B) = \mathsf{CR}(B,A)$ and represents the relations between the principal lines of $A$ and $B$.

**Definition 4.1.** The principal lines of two hyperbolic elements $A$ and $B$ are said to be *intersect* if they intersect in $\mathbb{H}^2$, *intersect at infinity* if they intersect in $\partial\mathbb{H}^2$, *separated with the same orientation (separated with the opposite orientation)* if they do not intersect in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ and $A_r < A_a < B_r < B_a$ or $A_a < A_r < B_a < B_r$ ($A_r < A_a < B_a < B_r$ or $A_a < A_r < B_r < B_a$) up to conjugation.

Consider Figure 9. The principal lines of $A, B$ are intersect, those of $B, C$ are separated with the opposite orientation, and those of $A, C$ are separated with the same orientation.

**Theorem 4.2.** *Suppose $A, B$ are hyperbolic matrices in $\mathbf{SL}(2,\mathbb{R})$. Then the principal lines of $A, B$ are*

(1) *intersect* $\iff \mathsf{CR}(A,B) < 0$
(2) *intersect at infinity* $\iff \mathsf{CR}(A,B) = 0$ *or* $\infty$
(3) *separated with the opposite orientation* $\iff 0 < \mathsf{CR}(A,B) < 1$
(4) *separated with the same orientation* $\iff \mathsf{CR}(A,B) > 1$

*Proof.* Suppose $f$ is a linear fractional transformation such that $f(B_r) = \infty$ and $f(B_a) = 0$. Since the cross ratio is invariant under the linear fractional transformations,

$$\mathsf{CR}(A,B) = \frac{(0 - z_a)(z_r - \infty)}{(0 - z_r)(z_a - \infty)} = \frac{z_a}{z_r}$$

where $z_a = f(A_a)$ and $z_r = f(A_r)$. If $\mathsf{CR}(A,B) = z_a/z_r < 0$, then the fixed points of $f(A)$ have the opposite signs. Thus the principal lines of $f(A), f(B)$ are intersect. Therefore those of $A, B$ are also intersect since they are invariant under linear fractional transformations. If $\mathsf{CR}(A,B) = z_a/z_r = 0$, then $z_a = 0$ or $z_r = \infty$. Then the principal lines of $f(A), f(B)$ are intersect at infinity. Thus those of $A, B$ have the same result. We can prove similarly the case $\mathsf{CR}(A,B) = z_a/z_r = \infty$. If $0 < \mathsf{CR}(A,B) = z_a/z_r < 1$, then both fixed point have the same signs. Thus if $z_r > 0$ then $0 < z_a < z_r$, and if $z_r < 0$ then $z_r < z_a < 0$. Therefore they are separated with the opposite orientation. The cases $\mathsf{CR}(A,B) > 1$ can be similarly proved. □

**Remark 4.3.** Since $A$ is hyperbolic, $A$ has two distinct fixed points. Thus the case $\mathsf{CR}(A,B) = z_a/z_r = 1$ can not be happen.

Suppose $A, B, C$ are hyperbolic elements. Then the holonomy group

$$\pi = \langle A, B, C \mid R = CB^{-1}A^{-1}BA = I \rangle$$

is discrete if and only if the principal lines of $A, B, C$ are located as in Figure 9 up to conjugation. (Keen [5], Goldman [3])

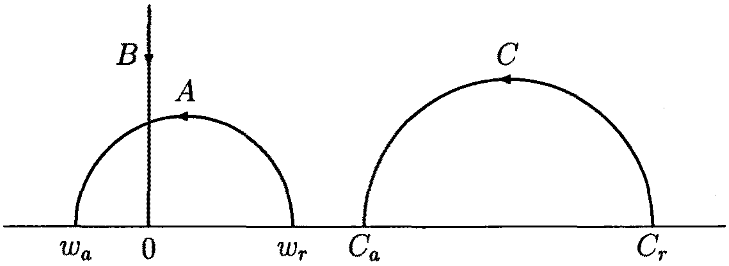


*Figure* 9. The locations of the principal lines of a one-holed torus $\Sigma(1,1)$

**Proposition 4.4.** *Suppose that the matrix $C$ in (3.11) is hyperbolic. Let $C_a, C_r$ be the fixed point of $C$. Then we have*

$$C_a,\ C_r = \frac{F \pm \sqrt{G}}{2}$$

*where* $F = (\lambda + \lambda^{-1} - a - \lambda^2 a)$ *and* $G = \left(\lambda + \lambda^{-1} - a + \lambda^2 a\right)^2 - 4\lambda^2$.

*Proof.* We give the same proof in Proposition 3.6. We denote $C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then the fixed points of $C$ are $\frac{(\alpha-\delta)\pm\sqrt{(\alpha-\delta)^2+4\beta\gamma}}{2\gamma} = \frac{(\alpha-\delta)\pm\sqrt{(\alpha+\delta)^2-4}}{2\gamma}$. Thus $C_a$, $C_r$ are $\frac{[\lambda^{-1}(\lambda+\lambda^{-1}-a)-\lambda a]\pm\sqrt{[\lambda^{-1}(\lambda+\lambda^{-1}-a)+\lambda a]^2-4}}{2\lambda^{-1}} = \frac{[(\lambda+\lambda^{-1}-a)-\lambda^2 a]\pm\sqrt{[(\lambda+\lambda^{-1}-a)+\lambda^2 a]^2-4\lambda^2}}{2}$.
Therefore we get the results. □

**Theorem 4.5** (Main Theorem). *Suppose $A, B$ are hyperbolic matrices in* $\mathbf{SL}(2,\mathbb{R})$ *with* $\mathrm{tr}(A) > 2$ *and* $\mathrm{tr}(B) > 2$. *Let* $\pi = \langle A, B, C \mid CB^{-1}A^{-1}BA = I \rangle$. *Then $\pi$ is discrete if and only if* $\mathsf{CR}(A,B) < 0$ *and* $\mathrm{tr}(C) < -2$.

*Proof.* ($\Rightarrow$) Without loss of generality we may assume $A, B, C$ are the matrices in (3.10) and (3.11). Suppose $\pi$ is discrete. Since the principal lines of $A, B$ are intersect, we have $\mathsf{CR}(A,B) < 0$ by Theorem 4.2. Let $C_a, C_r$ and $C_{ij}$ stand for the fixed points and the $(i,j)$-th entry of the matrix $C$. Since $0 < C_a < C_r$, we have $|C_{11}| < |C_{22}|$ and $C_{12}C_{22} > 0$ by Theorem 2.2. Then the condition

$$C_{12}C_{22} = -\lambda^2 a(\lambda - a)(\lambda^{-1} - a) > 0$$

implies $a < 0$ since $\lambda > 1$ and $a < \lambda^{-1}$. And the condition

$$|C_{11}| = |\lambda^{-1}|\,|\lambda + \lambda^{-1} - a| < |C_{22}| = |\lambda|\,|a|$$

induces $\lambda^{-1}(\lambda+\lambda^{-1}-a) < \lambda(-a)$. Thus

$$\mathrm{tr}(C) = \lambda^{-1}(\lambda+\lambda^{-1}-a)+\lambda a < 0.$$

Since $C$ is hyperbolic, the trace of $C$ should be less than $-2$.

($\Leftarrow$) Since $\mathsf{CR}(A,B) < 0$, the principal lines of $A, B$ are intersect. Without loss of generality we may assume the principal line of $B$ is the upper half of the imaginary axis. Thus the fixed points of $A$ are $w_a < 0 < w_r$ up to conjugation. To show the discreteness, we have to show that if $\mathrm{tr}(C) < -2$, then $w_r < C_a < C_r$.

The condition $\mathrm{tr}(C) = \lambda^{-1}(\lambda+\lambda^{-1}-a)+\lambda a < -2$ implies $a < 0$ since $\lambda > 1$ and $a < \lambda^{-1}$. Therefore we have the followings ;

$$\begin{aligned} C_{12}C_{21} &= -(\lambda-a)(\lambda^{-1}-a) < 0 \\ C_{12}C_{22} &= -\lambda^2 a(\lambda-a)(\lambda^{-1}-a) > 0 \\ |C_{11}|-|C_{22}| &= \lambda^{-1}(\lambda+\lambda^{-1}-a)-\lambda(-a) = \mathrm{tr}(C) < -2. \end{aligned}$$

By Theorem 2.2, we get $0 < C_a < C_r$. Hence, from Proposition 4.4, the fixed point $C_a$ and $C_r$ of $C$ should be

$$C_a = \frac{F-\sqrt{G}}{2} \quad \text{and} \quad C_r = \frac{F+\sqrt{G}}{2}.$$

In the proof of Proposition 3.7, we showed $w_r < z_r = (\lambda - a)$. Thus to show $w_r < C_a < C_r$, it is enough to show that $z_r < C_a$. This is equivalent to $2(\lambda-a) < F-\sqrt{G}$. Since

$$\sqrt{G} < F-2(\lambda-a) = -\lambda+\lambda^{-1}+a-\lambda^2 a,$$

and $-\lambda+\lambda^{-1}+a-\lambda^2 a > -\lambda-\lambda^{-1}+a-\lambda^2 a = -\mathrm{tr}(C)\lambda > 0$, it is equivalent to show that

$$G = \left(\lambda+\lambda^{-1}-a+\lambda^2 a\right)^2 - 4\lambda^2 < \left(-\lambda+\lambda^{-1}+a-\lambda^2 a\right)^2.$$

After some calculations, we can get the equivalent condition

$$(\lambda^2-1)(\lambda-a) > 0.$$

This is true since $\lambda > 1$ and $\lambda > a$. It proves the main theorem. □

We give an algorithm for deciding the discreteness of a holonomy group of a one-holed torus $\Sigma(1,1)$. For given two hyperbolic matrices $A, B$ in $\mathbf{SL}(2,\mathbb{R})$,

**Step 1:** Compute $\mathrm{tr}(A)$ and $\mathrm{tr}(B)$. If $\mathrm{tr}(A) < -2$, then replace $A$ by $-A$. Similarly if $\mathrm{tr}(B) < -2$, then replace $B$ by $-B$.

**Step 2:** By step 1, without loss of generality, we may assume that $\mathrm{tr}(A) > 2$ and $\mathrm{tr}(B) > 2$. Compute the attracting and repelling fixed points $A_a, A_r$ of $A$ and $B_a, B_r$ of $B$.

**Step 3:** Compute $\mathsf{CR}(A,B) = [B_a, A_r, A_a, B_r]$. If $\mathsf{CR}(A,B) < 0$, then go to step 4. Otherwise the hyperbolic matrices $A, B$ can not generate a discrete holonomy group of $\Sigma(1,1)$.

**Step 4:** Compute $C = A^{-1}B^{-1}AB$. If $\mathrm{tr}(C) < -2$, then

$$\pi = \langle\ A, B, C \in \mathbf{SL}(2,\mathbb{R}) \mid R = CB^{-1}A^{-1}BA = I\ \rangle$$

is a discrete group. If $\mathrm{tr}(C) \geq -2$, then $\pi$ is not discrete.

Using above algorithm we can make a computer program determine the discreteness of a holonomy group.

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