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Optimal maintenance procedure for multi-state deteriorated system with incomplete monitoring

L. Jin*
Department of Informatics, The University of Electro-Communications, Tokyo

K. Suzuki
Department of Informatics, The University of Electro-Communications, Tokyo

Abstract. The optimal replacement problem was investigated for a multi-state deteriorated system for which the true internal state cannot be observed directly except when the system breaks down completely. The internal state was assumed to be monitored incompletely by a monitor that gives information related to the true state of the system. The problem was formulated as a partially observable Markov decision process. The optimal procedure was found to be a monotone procedure with respect to stochastic increasing ordering of the state probability vectors under some assumptions. Limiting the optimal procedure to a monotone procedure would greatly reduce the tremendous amount of calculation time required to find the optimal procedure.

Key Words: *Condition monitoring maintenance, monotone procedure, partially observable Markov decision process, stochastic increasing, totally positive of order 2*

1. INTRODUCTION

1.1 Background and previous research

Since the breakdowns that occur in huge, complex systems can have a great impact on society, it is necessary to improve the reliability of such systems. However, it is sometimes difficult to make improvements at the system design stage for technological reasons. Condition monitoring maintenance, which can prevent breakdowns in advance, plays an important role in the field of reliability. Derman (1963) investigated an optimal replacement problem for a multi-state deteriorated system with complete observation. For the incomplete monitoring case, the optimal maintenance problem is usually formulated as

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* Corresponding Author.
E-mail address: *jinlu@se.uec.ac.jp*

a partially observable Markov decision process (POMDP). This approach has many applications and has attracted the attention of many researchers. Monahan (1982) provided an excellent survey on the theory, models, and algorithms of POMDP. Smallwood and Sondik (1973) showed that the optimal expected cost for the finite horizon becomes a piecewise-linear and convex function of the state probability vector and formulated an algorithm to calculate the optimal procedure and expected cost. Ivy and Pollock (2005) developed the concept of "marginally monotonic", which requires component-wise partial ordering, and proved that the optimal procedure has a marginally monotonic structure when the transition probability of the system follows a geometric distribution and the conditional probability of the monitor follows a binomial distribution. Ohnishi, Kawai, and Mine (1986) investigated a system monitored incompletely and derived a sufficient condition for the optimality of a monotone procedure. Maehara and Suzuki (2005) considered the optimal maintenance policy for an incomplete monitoring system with uncertain repair after inspection.

In the previous work of Ohnishi et al. (1986) and Maehara et al. (2005), the optimal maintenance problem was investigated on the basis of totally positive of order 2 (TP_2) ordering of the prior state probability vectors. This research deals with the optimal maintenance problem on the basis of stochastic increasing (SI) ordering of the state probability vectors.

1.2 Model description

In this research, the true internal state cannot be observed directly except when the system breaks down completely. This means the true internal state can be known exactly only when the system is down completely. Let X_t denote the system's internal state at time *t*; its value comes from a finite set {0, 1, ⋯, *i*, ⋯, *N*} in which the numbers are ordered to reflect the degree of system deterioration. State 0 denotes the best state, i.e., the system is like new, and state *N* denotes the most deteriorated state. Let $\Pi = (\pi_0, \pi_1, \cdots, \pi_N)$ be the prior state probability vector of *X*, where $\pi_i = \Pr(X = i)$, $\sum_{i=1}^{N} \pi_i = 1$, and $0 \le \pi_i \le 1$ for any *i*.

The state deteriorates in accordance with a stationary discrete-time Markov chain having a known transition law. Let $\boldsymbol{P}$ be the transition probability matrix; element P_{ij} denotes the one-step probability of transition from state *I* to state *j*. At each time period, the state is monitored incompletely by a monitor that gives information related to the true state of the system. M_t is the outcome of the monitor at time *t*; it comes from a finite set{0, 1, ⋯, θ, ⋯, *M*}. Let$\Gamma = \{\gamma_{j\theta} = \Pr(M_t = \theta | X_t = j)\}$ be a conditional probability matrix that describes the relationships between the system's true states and the monitor's output:

$$\gamma_{N\theta} = \begin{cases} 0 & (\theta = 0, \cdots, M-1) \\ 1 & (\theta = M) \end{cases}$$

Two actions, "Keep" and "Replace," are considered. "Keep" means an action that continues the system's operation with incomplete monitoring, and the operating cost per period in state *I* is given by C_i. "Replace" means an action to replace the system with a new one, and replacement cost *R* (>0) is assumed to be constant. At any given time period, only one action can be selected as the optimal one.

This model can be used to describe the preventive maintenance of many systems. In the case of an aircraft, the true state (X_t) of an engine is unobservable except when it breaks down completely. The engine's degradation can be described by transition probability matrix ***P***. The spectrometric oil analysis program (SOAP) is an effective way to predict engine degradation and to detect engine problems early. Oil samples are taken from the aircraft engine and analyzed using spectrometric tools to determine the concentration of sub micrometer particles of worn metal suspended in the oil. The results of SOAP (*Mt*) provide information related to the engine's true state (X_t). The relationship between M_t and X_t can be described by conditional probability matrix Γ.

2. DEFINITIONS AND ASSUMPTIONS

2.1 Partial ordering

Stochastic Increasing (SI) (Marshall and Olkin, 1979)

- For vectors X = $(x_1, x_2, \cdots, x_n)$ and ***y*** = $(y_1, y_2, \cdots, y_n)$, if
$$\sum_{i=k}^{n} x_i \leq \sum_{i=k}^{n} y_i \text{ for any } 1 \leq k \leq n,$$
x is stochastically smaller (or smaller in distribution) than ***y***, denoted by $\boldsymbol{x} \prec_s \boldsymbol{y}$.

- For $(n \times m)$ matrix Γ, if
$$\sum_{j=k}^{m} \gamma_{ij} \leq \sum_{j=k}^{m} \gamma_{i'j} \text{ for any } 1 \leq i < i' \leq n \text{ and } 1 \leq k \leq m,$$
Γ has an SI property, denoted by $\Gamma \in \mathrm{SI}$.

Totally Positive of order 2 ($\mathbf{TP_2}$) (Karlin, 1968)

- For vectors ***x*** = $(x_1, x_2, \cdots, x_n)$ and ***y*** = $(y_1, y_2, \cdots, y_3)$, if
$$\frac{x_i}{y_i} \geq \frac{x_j}{y_j} \quad \text{for any } 1 \leq i < j \leq n,$$
x is said to be smaller than ***y*** in the monotone likelihood ratio order, denoted by $\boldsymbol{x} \prec_{LR} \boldsymbol{y}$.

- If $(n \times m)$ matrix Γ satisfies
$$\begin{vmatrix} \gamma_{ij} & \gamma_{ij'} \\ \gamma_{i'j} & \gamma_{i'j'} \end{vmatrix} \geq 0 \text{ for any } 1 \leq i<i'<n, \text{ and } 1 \leq j < j' \leq m,$$
we say that Γ has a property of $\mathrm{TP_2}$, denoted by $\Gamma \in \mathrm{TP_2}$.

Clearly, the monotone likelihood ratio is a special case of $\mathrm{TP_2}$. In the following, $\boldsymbol{x} \prec_T \boldsymbol{y}$ is used instead of $\boldsymbol{x} \prec_{LR} \boldsymbol{y}$ for simplicity.

SI is a milder condition that can be used to compare more cases than $\mathrm{TP_2}$. For example, for a system with three states, Π^1 = (0.1, 0.5, 0.4) and Π^2 = (0.1, 0.2, 0.7) describe the system's state at two different time periods, t_1 and t_2; Π^2 describes a more deteriorated state than Π^1. In this case, $\Pi^1 \prec S\Pi^2$ holds; it is impossible to order them from TP_2 ordering. For this system, if the state transition probability matrix ***P*** is given as

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0 & 0.4 & 0.6 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \tag{2.1}$$

P has an SI property but does not satisfy the condition of TP_2.

2.2 Assumptions

In this research, the following assumptions are made.

(A-1) Transition probability matrix ***P*** ($=\{p_{ij}\}$) has a property of SI (***P***$\in$SI).

Assumption A-1 implies that the more a system has deteriorated, the more likely it is to deteriorate further or fail.

(A-2) Observed conditional probability matrix Γ ($=\{\gamma j_\theta\}$) has a property of TP_2, and the form is given as

$$\Gamma = \begin{array}{c} \\ 0 \\ \vdots \\ i \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & \cdots & \theta & \cdots & M \end{array} \\ \begin{pmatrix} \gamma_{00} & \cdots & \gamma_{0\theta} & \cdots & \gamma_{0M} \\ \vdots & \ddots & & & \vdots \\ \gamma_{i1} & \cdots & \gamma_{i\theta} & \cdots & \gamma_{iM} \\ \vdots & & & \ddots & \vdots \\ \gamma_{N-1,0} & \cdots & \gamma_{N-1,\theta} & \cdots & \gamma_{N-1,M} \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \end{array} \in TP_2.$$

This assumption implies that higher states of the system give rise to higher output levels of the monitoring probabilistically and that the true internal state of the system can be known exactly when the system is down completely. This is a natural condition in actual situations.

(A-3) Discount factor β satisfies

$$\beta \leq \frac{R-C_{N-1}}{R-C_0}$$

(A-4) "Keep" cost C_i is a nondecreasing function of state *i*(denoted by $C_i\uparrow$ (*i*)), and the relationship between "Keep" cost C_i and "Replace" cost *R* is

$$C_0 \leq C_1 \leq \cdots \leq C_{N-1} \leq R \leq C_N$$

A typical example of the cost functions under assumptions A-3 and A-4 is shown in Figure2.1. "Keep" cost C_i increases gradually with the deterioration of the system and rises rapidly if the system breaks down. That is,C_N is quite large compared with C_0, C_1, ···, C_{N-1} and *R*. "Replace" cost *R* is between C_{N-1} and C_N.

In the case of an aircraft, for example, a fatigue crack in the fuselage can lead to a crash (failure). Horizontal a x is II in Figure 2.1 represents the state (the crack length) of the fuselage. When the length exceeds the limit criterion, the fuselage breaks, possibly leading to a crash. In states 0 − (*N* −1) (the crack is shorter than the limit criterion), there is almost no increase in the "Keep" cost since the aircraft can still fly safely. However, "Keep" cost $C_N$for state *N* (crash due to break in fuselage) becomes immeasurably huge. C_N is also hugecompared with *R*.

A multiple load path structure and damage tolerance design are commonly used to minimize fatigue crack extension and improve fuselage reliability. Therefore, we can divide fuselage deterioration into several stages (e.g., $N + 1$ states) in accordance with the crack length and take a preventive countermeasure at an early stage before failure occurs (state N).

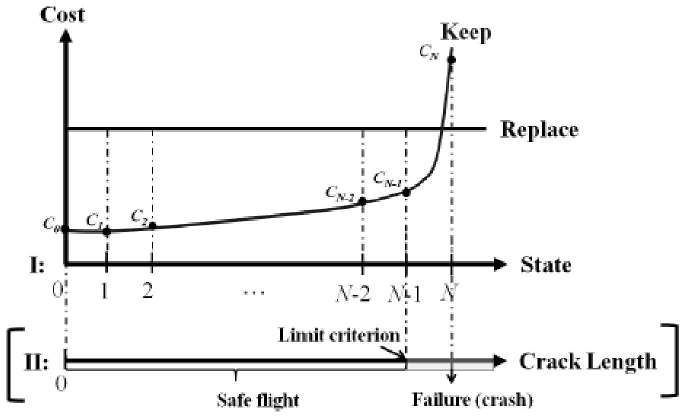

Figure 2.1. Cost functions under assumptions A-3 and A-4

2.3 Lemmas

Several lemmas were used in this research. For notation convenience, let $\phi^{(k)} = (\phi_0^{(k)}, \phi_1^{(k)}, \cdots, \phi_N^{(k)})$ in which $\phi_j^{(k)} := \sum_{i=0}^{N} \pi_i^{(k)} p_{ij}$, and $h_j \uparrow (j)$ denote the nondecrease of h_j in j.

- **Lemma 1**
  For $\Pi^{(1)} \prec_s \Pi^{(2)}$
  $$\phi^{(1)} \prec_s \phi^{(2)}$$
  holds if $\boldsymbol{P} \in \mathrm{SI}$. Here, $\Pi^{(1)} = (\pi_0^{(1)}, \pi_1^{(1)}, \cdots \pi_N^{(1)})$, $\Pi^{(2)} = (\pi_0^{(2)}, \pi_1^{(2)}, \cdots \pi_N^{(2)})$, and $P = \{p_{ij}\}$.

- **Lemma 2 (Lehmann, 1959)**
  For $h_j \uparrow (j)$,
  $$\sum_{j=0}^{N} h_j \phi_j^{(1)} \le \sum_{j=0}^{N} h_j \phi_j^{(2)}$$
  if $\phi^{(1)} \prec_s \phi^{(2)}$.

- **Lemma 3 (Lehmann, 1959)**
  For matrix $\Gamma \in TP_2$, if the sign of h_j changes once in j, the sign of $\sum_{j=0}^{N} h_j \gamma_{ij}$ changes once at most in i.

2.4 Monotone procedure

Since the monitor information is incomplete, we select one action on the basis of prior state probability vector Π. Let A= {1(Keep), 2(Replace)} be the action space, and let the set of all maintenance procedures Δ be the set of all functions $\delta : \Omega = \{\Pi \in \Re^{n+1}\} \rightarrow A$. The previous history of any fixed sequence of monitor output sand actions is summarized into current state probability vector Π (Sawaragi and Yoshikawa (1970).

- **Monotone Procedure:**
  The procedure $\delta \in \Delta$ is said to be "monotone" if $\Pi \prec \Pi'$ implies $\delta(\Pi) \leq \delta(\Pi')$, where $\delta(\Pi), \delta(\Pi') \in \{1$(Keep), 2(Replace)$\}$.

Here, "$\prec$" represents partial ordering. We use partial ordering in the sense of SI.

If the optimal procedure is a monotone procedure, the tremendous amount of calculation time required to find the optimal procedure can be greatly reduced. This enables the optimal decision to be identified in a much shorter period of time.

For example, if the system has two states, prior state probability vector $\mathbf{\Pi}$ is given as $(1 - \pi, \pi)$. Therefore, $\mathbf{\Pi}$ can be described by a function of π. For the two-actions case, we have to select the optimal procedure out of $2^{100} \approx 10^{30}$ procedures if state probability π is divided into 100 intervals. If the optimal procedure is given as

$$\begin{cases} \pi \leq i^* & \text{Keep} \\ \pi \leq i^* & \text{Replace'} \end{cases}$$

we say the optimal procedure is a "monotone procedure," and we need to consider at most only 100 + 1 procedures instead of all 10^{30} (see Figure 2.2). This enables the optimal decision to be identified in a much shorterperiod of time.

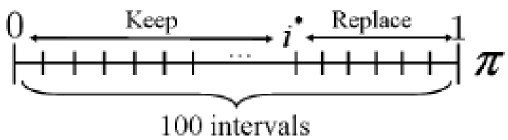

Figure 2.2. Monotone procedure (two states)

3. OPTIMAL MAINTENANCE PROBLEM

3.1 Partially observable Markov decision process (POMDP)

At the beginning of any time period, "Keep" or "Replace" is selected as the action. The optimal procedure is the collection of actions that minimize the discounted total expected cost incurred in both current and future time periods for every possible given state. The problem is how to minimize the discounted total expected cost over an infinite horizon. This problem is formulated as a partially observable Markov decision process(POMDP).

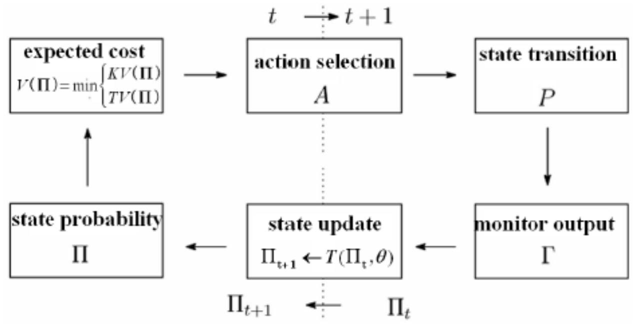

Figure 3.1. Partially observable Markov decision process (POMDP)

A schematic representation of POMDP is given by Figure 3.1. It starts from the left lower frame with "**Π**." At any time period, since the state information obtained from the monitor is incomplete, we describe the system's exact state using prior state probability **Π**, in which the monitoring information and the system history are aggregated. At the beginning of time period *t*, an action that minimizes the total cost over both the current and future time periods should be selected. The discounted total expected cost *V*(**Π**) can be calculated on the basis of **Π**, and the decision maker determines the most suitable action, the one that minimizes the discounted total expected cost *V*(**Π**) (refer to δ3.2 for details).

If "Keep" is selected as the optimal action, the true internal state X_t transitions to $Xt+1$ in accordance with transition probability matrix ***P*** (common for *t*). At the next time period *t* + 1, the monitor gives an observation($Mt = \theta$) based on conditional probability matrix Γ. We can obtain posterior state probability *T*(**Π**, θ) based on observation θ and prior state probability Π, and the posterior state probability *T*(**Π**, θ) of time period "*t*" is updated to be prior state probability Π of time period "*t* + 1" with probability *P*(θ| **Π**).

$$P(\theta|\boldsymbol{\Pi}) = \Pr(Mt+1 = \theta|\boldsymbol{\Pi}) = \sum_{j=0}^{N}\sum_{i=0}^{N}\pi_i\, p_{ij}\gamma_{j\theta}, \quad (3.1)$$

and the *j*-th element of posterior state probability vector ***T***(**Π**, θ) is

$$T(j|\boldsymbol{\Pi}, \theta) = \Pr(Xt+1 = j|\boldsymbol{\Pi}, Mt+1 = \theta)$$

$$= \frac{\sum_{i=0}^{N}\pi_i p_{ij}\gamma_{j\theta}}{\sum_{j=0}^{N}\sum_{i=0}^{N}\pi_i p_{ij}\gamma_{j\theta},} = \frac{\phi_j\gamma_{j\theta}}{P(\theta|\boldsymbol{\Pi})}. \quad (3.2)$$

If "Replace" is selected as the optimal action, the system is replaced by a new one, and the true internal state of the next time period ($Xt+1$) becomes "0." The state probability is updated from **Π** to e^0 with probability1 in the next time period, where

$$e^0 = (1, 0, \cdots, 0, \cdots, 0).$$

A POMDP can be applied to the preventive maintenance of all kinds of systems. For example, a nuclear power plant has a refueling period (about 40 days in Japan) after 13 months of operation. One refueling period together with one operation period is treated as one time period. The overall state of the operation unit is checked during the refueling period. The true state of the operation unit, which cannot be observed directly, is estimated on the basis of monitor information and the operation history. The estimated true state is described by prior state probability Π. The optimal maintenance action ("Keep" or "Replace") for the operation unit isidentified on the basis of Π before every operation period.

If the optimal procedure, which is the collection of actions that minimize the discounted total expected cost $V(\mathbf{\Pi})$ for every $\mathbf{\Pi}$, is a monotone procedure, the optimal maintenance action for the operation unit can beidentified quickly.

3.2 Total expected cost

Let $V(\mathbf{\Pi})$ denote the discounted optimal total expected cost function over an infinite time period with initialstate probability $\mathbf{\Pi}$.

$$V(\mathbf{\Pi}) = \min\begin{cases} \sum_{i=0}^{N} C_i\pi_i + \beta\cdot\sum_{\theta=0}^{M} P(\theta|\mathbf{\Pi})V\big(T(\mathbf{\Pi},\theta)\big) := KV(\mathbf{\Pi}) \\ R+\beta\cdot V(e^0) := RV(\mathbf{\Pi}) \end{cases} \quad (3.3)$$

This is a recursive function that can be calculated on the basis of state probability Π; β (0 < β <1) is the discount factor. The first (second) term, $KV(\mathbf{\Pi})$ ($RV(\mathbf{\Pi})$), on the right corresponds to the total expected costover the current and future time periods when "Keep" ("Replace") is selected at the beginning and the optimal procedure is then followed.

The action that minimizes the right side of (3.2) is the optimal one and should be selected for prior state probability vector $\mathbf{\Pi}$. Hence, the optimal procedure is obtained by selecting the action that minimizes the discounted total expected cost for each $\mathbf{\Pi}$.

Next, we examine the properties of the optimal total expected cost function. On the basis of (3.3), we consider functions $V^{(n)}(\mathbf{\Pi})$(n = 0, 1, 2, ⋯), which are defined inductively as

$$V^{(n)}(\mathbf{\Pi}) = 0,$$

$$V^{(n)}(\mathbf{\Pi}) = \min\begin{cases} \sum_{i=0}^{N} C_i\pi_i + \beta\cdot\sum_{\theta=0}^{M} P(\theta|\mathbf{\Pi})V^{(n-1)}\big(T(\mathbf{\Pi},\theta)\big) := KV^{(n-1)}(\mathbf{\Pi}) \\ R+\beta\cdot V^{(n-1)}(e^0) := RV^{(n-1)}(\mathbf{\Pi}) \end{cases} \quad (n \geq 1). \quad (3.4)$$

$V^{(n)}(\Pi)$is interpreted as the optimal expected cost over n(≥1) time periods. From the standard argument of contraction mapping theory, $V^{(n)}(\mathbf{\Pi})$must converge to $V(\mathbf{\Pi})$ as n tends to infinity.

3.3 Preliminary properties

Given the assumptions in Section 2.2 and the lemmas in Section 2.3, we obtain the following preliminary properties for function $C_j^{(n-1)}$, which are used in the following subsections.

$$C_j^{(n-1)} = \begin{cases} 0 & (n=1) \\ C_j + \sum_{\theta=\theta_1^*+1}^{M}(R-C_j)\gamma_{j\theta} & (n=2), \\ (C_j + \beta\sum_{\kappa=0}^{N} C_k^{(n-2)} p_{jk}) + C_j^{(n-1)*}\sum_{\theta=\theta_{n-1}^*+1}^{M}\gamma_{j\theta} & (n\geq 3) \end{cases} \quad (3.5)$$

and

$$C_j^{(n-1)*} = \left\{\left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k}\right) - \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk}\right)\right\} \ (n \geq 3). \quad (3.6)$$

- **Preliminary Property 1**:

  $C_j^{(n-1)*} \geq 0$ *for* $n \geq 3$ *and* $j \leq N-1$.

- **Preliminary Property 2**:

  $C_j^{(n-1)} \uparrow (j)$ *for any* $n \geq 1$ *and* $j \leq N$.

- **Preliminary Property 3**:

  The total expected cost function for n (≥ 3) *time periods is given by*

$$V^{(n)}(\boldsymbol{\Pi}) = \min\begin{cases} \sum_{i=0}^{N} C_i \pi_i + \beta \sum_{j=0}^{N} C_j^{(n-1)} \phi_j \\ R + \beta C_0 + \beta^2 \sum_{j=0}^{N} C_j^{(n-2)} p_{0j} \end{cases}$$

The proofs of these preliminary properties are given in the appendix.

3.4 Properties of optimal total expected cost function

In this subsection, we examine the properties of the optimal total expected cost function given the preliminary properties in Section 3.3. Let $\mathcal{F}$ denote the class of all functions of $V(\boldsymbol{\Pi})$ that satisfy $V(\boldsymbol{\Pi}^1) \leq V(\boldsymbol{\Pi})$ for $\boldsymbol{\Pi}^1 <_S \boldsymbol{\Pi}^2$.

- **Property 1**:

  The total expected cost function over one time period is given by

$$V^{(1)}(\boldsymbol{\Pi}) = \min\begin{cases} \sum_{i=0}^{N} \pi_i C_i, \\ R \end{cases} \quad (3.7)$$

  and $V^{(1)}(\boldsymbol{\Pi}) \in \mathcal{F}$.

  Property 1 can clearly be derived from assumption A-4 and Lemma 2.

- **Property 2**:

  The total expected cost function over two time periods is given by

$$V^{(2)}(\boldsymbol{\Pi}) = \min\begin{cases} \sum_{i=0}^{N} \pi_i C_i + \beta \sum_{j=0}^{N} C_j^{(1)} \phi_j, \\ R + \beta C_0 \end{cases} \quad (3.8)$$

  and $V^{(2)}(\boldsymbol{\Pi}) \in \mathcal{F}$.

Proof:

From (3.4), we get

$$V^{(2)}(\mathbf{\Pi}) = \min\begin{cases}\sum_{i=0}^{N} C_i\pi_i + \beta\sum_{\theta=0}^{M} P(\theta|\mathbf{\Pi})V^{(1)}\big(T(\mathbf{\Pi},\theta)\big) := KV^{(1)}(\mathbf{\Pi}) \\ R + \beta V^{(1)}(e^0) := RV^{(1)}(\mathbf{\Pi})\end{cases}. \quad (3.9)$$

$KV^{(1)}(\mathbf{\Pi})$ in (3.9) can be written as

$$KV^{(1)}(\mathbf{\Pi}) = \sum_{i=0}^{N} C_i\pi_i + \beta\cdot\sum_{\theta=0}^{M}\min\begin{cases}\sum_{j=0}^{N} C_j\phi_j\gamma_{j\theta} \\ \sum_{j=0}^{N} R\phi_j\gamma_{j\theta}\end{cases} \quad (3.10)$$

from (3.1), (3.2), and (3.7). Taking the difference between the two terms after "min" in (3.10), we get $\sum_{j=0}^{N}(R - C_j)\phi_j\gamma_{j\theta}$, the sign of which changes at most once in θ given the assumptions in Section 2.2 and the lemmas in Section 2.3. This means that there exists at most one $\theta_1^*(0 \le \theta_1^* \le M)$ for which the sign of $\sum_{j=0}^{N}(R - C_j)\phi_j\gamma_{j\theta}$ changes. Therefore, (3.10) can be written as

$$KV^{(1)}(\mathbf{\Pi}) = \sum_{i=0}^{N}\pi_i C_i + \beta\sum_{j=0}^{N} C_j^{(1)}\phi_j\,, \quad (3.11)$$

Where

$$C_j^{(1)} = C_j + \sum_{\theta=\theta_i^*+1}^{M}(R - C_j)\gamma_{j\theta}. \quad (3.12)$$

Therefore, the total expected cost function can be written as

$$V^{(2)}(\mathbf{\Pi}) = \min\begin{cases}\sum_{i=0}^{N}\pi_i C_i + \beta\sum_{j=0}^{M} C_j^{(1)}\phi_j = KV^{(1)}(\mathbf{\Pi}) \\ R + \beta C_0 = RV^{(1)}(\mathbf{\Pi})\end{cases} \quad (3.13)$$

since

$$RV^{(1)}(\mathbf{\Pi}) = R + \beta V^{(1)}(e^0) = R + \beta\min\begin{cases}C_0 \\ R\end{cases} = R + \beta C_0.$$

From assumption A-4 and Lemma 2, we get

$$\sum_{i=0}^{N}\pi_i^{(1)}C_i \le \sum_{i=0}^{N}\pi_i^{(2)}C_i, \quad \mathbf{\Pi}^{(1)} < s\mathbf{\Pi}^{(2)}. \quad (3.14)$$

Since $C_j^{(1)}\uparrow$ (*j*) from Preliminary Property 2, and $\phi^{(1)} <_S \phi^{(2)}$ from assumption A-1 based on Lemma1, we get

$$\sum_{j=0}^{N} C_j^{(1)}\phi_j^{(1)} \le \sum_{j=0}^{N} C_j^{(1)}\phi_j^{(2)}, \quad \mathbf{\Pi}^{(1)} < s\mathbf{\Pi}^{(2)}. \quad (3.15)$$

Therefore, $KV^{(1)}(\mathbf{\Pi}^{(1)}) \le KV^{(1)}(\mathbf{\Pi}^{(2)})$ for $\mathbf{\Pi}^{(1)} < s\mathbf{\Pi}^{(2)}$ from (3.11), (3.14), and (3.15). $RV^{(1)}(\mathbf{\Pi})$ in (3.13) is a constant function of $\mathbf{\Pi}$. Therefore, $V^{(2)}(\mathbf{\Pi}) \in \mathcal{F}$ from the assumptions in Section 2.2.

- **Property 3**:

$V^{(n)}(\Pi)(n \ge 3)$*, which is the total expected cost over n* (≥ 3) *time periods, is an element of* $\mathcal{F}$.

Proof:

From Preliminary Property 3, we get

$$V^{(n)}(\mathbf{\Pi}) = \min\begin{cases} \sum_{i=0}^{N} C_i\pi_i + \beta\sum_{j=0}^{N} C_j^{(n-1)}\phi_j := KV^{(n-1)}(\mathbf{\Pi}) \\ R + \beta C_0 + \beta^2\sum_{j=0}^{N} C_j^{(n-2)}p_{0j} := RV^{(n-1)}(\mathbf{\Pi}) \end{cases}. \quad (3.16)$$

Similar to the discussion of Property 2, we obtain $KV^{(n-1)}(\mathbf{\Pi}^{(1)}) \leq KV^{(n-1)}(\mathbf{\Pi}^{(2)})$ for$\Pi^{(1)} < s\mathbf{\Pi}^{(2)}$from the lemmas in Section 2.3 and Preliminary Property 2. $RV^{(n-1)}(\mathbf{\Pi})$ in (3.16) is a constant function of Π.
Therefore, $V^{(n)}(\mathbf{\Pi}) \in \mathcal{F}$from the assumptions in Section 2.2.

Given properties 1, 2, and 3, we derive the following theorems.

- **Theorem 1:**

The total expected cost over infinite time periods,$V(\mathbf{\Pi})$*, is an element of* $\mathcal{F}$.

Proof:

From properties 1, 2, and 3, we have $V^{(n)}(\mathbf{\Pi}) \in \mathcal{F}$for any $n \geq 1$. This guarantees $V(\mathbf{\Pi}) \in \mathcal{F}$since $V^{(n)}(\mathbf{\Pi})$ converges to $V(\mathbf{\Pi})$as *n* tends to infinity.

Theorem 1 means that, given the assumptions in Section 2.2, the optimal monotone procedure holds with respect to SI ordering of the state probability vectors $\mathbf{\Pi}$. Arranging the content of Theorem 1, we obtain the following structural property as Theorem 2.

- **Theorem 2: Structural Property**

There exists an optimal procedure that is a monotone procedure. That is, the optimal procedure is determined by Π^**, such that the system is kept operating for* $\mathbf{\Pi} <_s \mathbf{\Pi}^*$*nd is replaced for* $\mathbf{\Pi}^* < s\ \mathbf{\Pi}$.The structural property of the monotone procedure in **Theorem 2** is illustrated in Figure 3.2. It is assumed thatΠ locates on the horizontal axis, ordered by SI ordering. As shown in Figure 3.2, if the optimal procedure is given by a monotone procedure as Theorem 2, we need to determine at most one threshold $\mathbf{\Pi}^*$atwhich the optimal action changes. Since we do not need to consider the optimal actions for each $\mathbf{\Pi}$, the calculation time to find an optimal procedure is reduced.

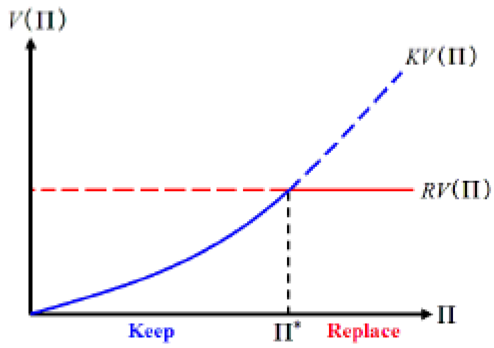

Figure 3.2. Structural property of monotone procedure

3.5 Discussion

The optimal procedure described above is similar to one introduced elsewhere (Derman, 1963, Karlin and Rubin, 1956). Derman (1963) investigated the optimal replacement problem for a multi-state deteriorated system for which the true internal state could be known exactly. This means that conditional probability matrix Γ was an identify matrix. The optimalprocedure was found to be a monotone procedure when state transition probability matrix $\boldsymbol{P}$ had an SI property. Karlin and Rubin (1956) investigated the case of incomplete observations without state transitions. This means that state transition probability matrix $\boldsymbol{P}$ was an identify matrix. They proved that the TP_2 property of conditional probability matrix Γ is a sufficient condition for the optimal procedure to be a monotone procedure with respect to TP2 ordering of the state probability vectors $\mathbf{\Pi}$. This research investigated the optimal maintenance problemfor a general model of a monitoring system with state transitions and incomplete observations and found that the optimal procedure is a monotone procedure with respect to SI ordering of $\mathbf{\Pi}$ when $\boldsymbol{P}$ has an SI property andΓ has a TP_2 property.

4. CONCLUSION

This research investigated the optimal replacement problem for a multi-state deteriorated system for which the true internal state cannot be observed directly except when the system breaks down completely. It found that the optimal procedure is a monotone procedure with respect to Stochastic Increasing (SI) ordering of the state probability vectors Π when the transition probability matrix has an SI property and the conditional probability matrix has a Totally Positive of order 2 (TP2) property under some assumptions. This finding can be used to greatly reduce the tremendous amount of calculation time required to find the optimal procedure, enabling the optimal decision to be identified in a much shorter period of time.

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APPENDIX

A.1 Proof of Preliminary Property 1

$C_j^{(n-1)*} \geq 0$ *for* $n \geq 3$ *and* $j \leq N-1$, *where*

$$C_j^{(n-1)*} = \left\{\left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k}\right) - \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk}\right)\right\}.$$

Proof:

We prove Preliminary Property 1 using mathematical induction.

- Step 1) For $n = 3$, $C_j^{(2)*} \geq 0$ ($j \leq N-1$).

$$\begin{aligned} C_j^{(2)*} &= (R + \beta C_0) - \left(C_j + \beta \sum_{k=0}^{N} C_k^{(1)} p_{jk}\right) \\ &\geq (R + \beta C_0) - \left(C_{N-1} + \beta \sum_{k=0}^{N} C_k^{(1)} p_{jk}\right) \\ &\geq \beta\left(R - \sum_{k=0}^{N} C_k^{(1)} p_{jk}\right) = \beta \sum_{k=0}^{N} (R - C_k^{(1)}) p_{jk} \end{aligned} \quad (A.1)$$

  since $C_j \leq C_{N-1}$ ($j \leq N-1$) and $\beta \leq \frac{R - C_{N-1}}{R - C_0}$. From (A.1), we derive $C_j^{(2)*} \geq 0$ ($j \leq N-1$) since $R \geq C_k^{(1)}$ from (3.12).

- Step 2) For $n = 4$, $C_j^{(3)*} \geq 0$ ($j \leq N-1$).

$$\begin{aligned} C_j^{(3)*} &= \left\{\left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(1)} p_{0k}\right) - \left(C_j + \beta \sum_{k=0}^{N} C_k^{(2)} p_{jk}\right)\right\} \\ &\geq \left\{\left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(1)} p_{0k}\right) - \left(C_{N-1} + \beta \sum_{k=0}^{N} C_k^{(2)} p_{jk}\right)\right\} \\ &\geq \beta R + \beta^2 \sum_{k=0}^{N} C_k^{(1)} p_{0k} - \beta \sum_{k=0}^{N} C_k^{(2)} p_{jk} \end{aligned} \quad (A.2)$$

  Since $C_j \leq C_{N-1}$ ($j \leq N-1$) and $\beta \leq \frac{R - C_{N-1}}{R - C_0}$. We get $C_k \leq C_k^{(1)}$ from (3.12). From (3.5) and (3.6), we get

$$C_j^{(2)} = \sum_{\theta=0}^{\theta_2^*} \left(C_j + \beta \sum_{k=0}^{N} C_k^{(1)} p_{jk}\right) \gamma_{j\theta} + \sum_{\theta=\theta_2^*+1}^{M} (R + \beta C_0) \gamma_{j\theta} . \quad (A.3)$$

  Then we obtain $C_k^{(2)} \leq R + \beta C_0$ from (A.3) since $C_j^{(2)*} \geq 0$ in Step 1. Therefore, (A.2) can be written as

$$C_j^{(3)*} \geq \beta R + \beta^2 \sum_{k=0}^{N} C_k p_{0k} - \beta \sum_{k=0}^{N} (R + \beta C_0) p_{jk} = \beta^2 \sum_{k=0}^{N} (C_k - C_0) p_{0k} \geq 0.$$

- Step 3) For $n \geq 5$, $C_j^{(n-1)*} \geq 0$ if $C_j^{(n-2)*} \geq 0$ and $C_j^{(n-3)*} \geq 0$ ($j \leq N-1$).

  From (3.5), we get

$$C_j^{(n-2)} = \sum_{\theta=0}^{\theta^*_{n-2}} \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-3)} p_{jk}\right) \gamma_{j\theta} + \sum_{\theta=\theta^*_{n-3}+1}^{M} \left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-4)} p_{0k}\right) \gamma_{j\theta}. \quad (A.4)$$

$$C_j^{(n-3)} = \sum_{\theta=0}^{\theta^*_{n-3}} \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-4)} p_{jk}\right) \gamma_{j\theta} + \sum_{\theta=\theta^*_{n-3}+1}^{M} \left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-5)} p_{0k}\right) \gamma_{j\theta}. \quad (A.5)$$

  For $j \leq N-1$, if $C_j^{(n-2)*} \geq 0$, we have

$$C_j^{(n-2)} \leq R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-4)} p_{0k} \quad (A.6)$$

  from (A.4), and, if $C_j^{(n-3)*} \geq 0$, we get

$$C_j + \beta \sum_{k=0}^{N} C_k^{(n-4)} p_{jk} \leq C_j^{(n-3)} \quad (A.7)$$

  from (A.5). From (A.6), we derive

$$C_j^{(n-1)*} = \left\{\left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k}\right) - \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk}\right)\right\},$$

$$\geq \left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k}\right) - \left\{C_j + \beta \left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-4)} p_{0k}\right)\right\}$$

$$\geq \left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k}\right) - \left\{C_{N-1} + \beta \left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-4)} p_{0k}\right)\right\}$$

$$\geq \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k} - \beta^2 C_0 - \beta^3 \sum_{k=0}^{N} C_k^{(n-4)} p_{0k}, \quad (A.8)$$

  since $\beta \leq \frac{R-C_{N-1}}{R-C_0}$. From (A.7) and (A.8), we derive

$$C_j^{(n-1)*} \geq \beta^2 \sum_{k=0}^{N} \left(C_k + \beta \sum_{l=0}^{N} C_l^{(n-4)} p_{kl}\right) p_{0k} - \beta^2 C_0 - \beta^3 \sum_{k=0}^{N} C_k^{(n-4)} p_{0k},$$

$$= \beta^2 \sum_{k=0}^{N} (C_k - C_0) p_{0k} \geq 0$$

Therefore, $C_j^{(n-1)*} \geq 0$ for $n \geq 3$ and $j \leq N-1$ from steps 1, 2, and 3.

A.2 Proof of Preliminary Property 2

$C_j^{(n-1)} \uparrow (j)$ *for any* $n \geq 1$ *and* $j \leq N$.

Proof:

We prove **Preliminary Property 2** using mathematical induction.

- Step 1) For $n = 2$, $C_j^{(1)} \uparrow (j)$ ($j \le N$).

Taking the difference $C_{j'}^{(1)} - C_j^{(1)}$ ($j<j' <N$), we get

$$\begin{aligned} C_{j'}^{(1)} - C_j^{(1)} &= \left(C_{j'} - C_j\right) + \left(\sum_{\theta=\theta_i^*+1}^{M} R\gamma_{j'\theta} - \sum_{\theta=\theta_i^*+1}^{M} R\gamma_{j\theta}\right) \\ &\quad - \sum_{\theta=\theta_i^*+1}^{M} C_{j'}\gamma_{j'\theta} + \sum_{\theta=\theta_i^*+1}^{M} C_j\gamma_{j\theta} \\ &\ge \left(C_{j'} - C_j\right) + \left(\sum_{\theta=\theta_i^*+1}^{M} C_{j'}\gamma_{j'\theta} - \sum_{\theta=\theta_i^*+1}^{M} C_{j'}\gamma_{j\theta}\right) \\ &\quad - \sum_{\theta=\theta_i^*+1}^{M} C_{j'}\gamma_{j'\theta} + \sum_{\theta=\theta_i^*+1}^{M} C_j\gamma_{j\theta} \\ &= \sum_{\theta=0}^{\theta_1^*} C_{j'}\gamma_{j'\theta} - \sum_{\theta=0}^{\theta_1^*} C_j\gamma_{j\theta} \ge 0 \end{aligned}$$

since $\Gamma \in TP_2$ from assumption A-2 and $C_j \le C_{j'} \le R$ from assumption A-4. Therefore, $C_j^{(1)} \uparrow$ (j) for $j <N$. For $j = N$, $C_N^{(1)} = R$ since $\sum_{\theta=\theta_i^*+1}^{M} \gamma_{j\theta} = 1$ from assumption A-2. Therefore, $C_j^{(1)} \uparrow$ (j) for any $j \le N$ since

$$\max\{C_j^{(1)}\} = R(j <N)$$

from (3.12).

- Step 2) For $n \ge 3$, $C_j^{(n-1)} \uparrow (j)$ if $C_j^{(n-2)} \uparrow (j)$ ($j \ge N$).

Taking the difference $C_{j'}^{(n-1)} - C_j^{(n-1)}$ ($j<j' <N$), we get

$$\begin{aligned} C_{j'}^{(n-1)} - C_j^{(n-1)} &= \sum_{\theta=0}^{\theta_{n-1}^*} \left(C_{j'} + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{j'k}\right) \gamma_{j'\theta} \\ &\quad - \sum_{\theta=0}^{\theta_{n-1}^*} \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk}\right) \gamma_{j\theta} \\ &\quad + \left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k}\right) \left(\sum_{\theta=\theta_{n-1}^*+1}^{M} \gamma_{j'\theta} - \sum_{\theta=\theta_{n-1}^*+1}^{M} \gamma_{j\theta}\right). \end{aligned}$$

Since $R + \beta C0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k} \ge C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk}$ from $C_j^{(n-1)*} \ge 0$ (**Preliminary Property 1**), we get

$$\begin{aligned} C_{j'}^{(n-1)} - C_j^{(n-1)} &\ge \sum_{\theta=0}^{\theta_{n-1}^*} \left(C_{j'} + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{j'k}\right) \gamma_{j'\theta} \\ &\quad - \sum_{\theta=0}^{\theta_{n-1}^*} \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk}\right) \gamma_{j\theta} \\ &\quad + \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk}\right) \left(\sum_{\theta=\theta_{n-1}^*+1}^{M} \gamma_{j'\theta} - \sum_{\theta=\theta_{n-1}^*+1}^{M} \gamma_{j\theta}\right) \\ &= \left\{\left(C_{j'} + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{j'k}\right) - \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk}\right)\right\} \sum_{\theta=0}^{\theta_{n-1}^*} \gamma_{j'\theta}. \end{aligned}$$

On the basis of Lemma 2, we obtain $C_{j'}^{(n-1)} - C_j^{(n-1)} \ge 0$ ($j<j' <N$) from the following relationships:

– $C_j \uparrow (j)$ (from assumption A-4),

$-C_j^{(n-2)} \uparrow (j)$ (from assumption in Step 2),
$-\boldsymbol{P} \in$ SI (from assumption A-1).

Therefore, $C_j^{(n-1)}$ is a nondecreasing function of j ($<N$), and

$$\max\left\{C_j^{(n-1)}\right\} = R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k} \; (j < N)$$

since $R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k} \geq C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{0k}$ from $C_j^{(n-1)*} \geq 0$ (Preliminary Property 1).
For $j = N$, we have

$$C_N^{(n-1)} = R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k}$$

since $\sum_{\theta=\theta_{n-1}^*+1}^{M} \gamma_{j\theta} = 1$. Therefore, $C_j^{(n-1)}$ is a nondecreasing function for any $j \leq N$.

From (3.5), $C_j^{(0)} = 0$. Therefore, we derive that $C_j^{(n-1)} \uparrow (j)$ for any $n \geq 1$ and $j \leq N$ from steps 1 and 2.

A.3 Proof of Preliminary Property 3

Total expected cost function for n (≥ 3) *time periods is given by*

$$V^{(n)}(\boldsymbol{\Pi}) = \min\begin{cases} \sum_{i=0}^{N} C_i \pi_i + \beta \sum_{j=0}^{N} C_j^{(n-1)} \phi_j \\ R + \beta C_0 + \beta^2 \sum_{j=0}^{N} C_j^{(n-2)} p_{0j} \end{cases}.$$

Proof:

We prove Preliminary Property 3 using mathematical induction.

- Step 1) For $n = 3$:

From (3.3) and (3.4), $V^{(3)}(\boldsymbol{\Pi})$ can be written as

$$V^{(3)}(\boldsymbol{\Pi}) = \min\begin{cases} \sum_{i=0}^{N} C_i \pi_i + \beta \cdot \sum_{\theta=0}^{M} P(\theta|\boldsymbol{\Pi}) V^{(2)}\left(T(\boldsymbol{\Pi}, \theta)\right) := KV^{(2)}(\boldsymbol{\Pi}\Pi) \\ R + \beta V^{(2)}(e^0) := RV^{(2)}(\boldsymbol{\Pi}) \end{cases}. \quad \text{(A.9)}$$

The first term of (A.9), $KV^{(2)}(\boldsymbol{\Pi})$, can be written as

$$KV^{(2)}(\boldsymbol{\Pi}) = \sum_{i=0}^{N} C_i \pi_i + \beta \sum_{\theta=0}^{M} \min\begin{cases} \sum_{j=0}^{N} \left(C_j + \beta \sum_{k=0}^{N} C_k^{(1)} p_{jk}\right) \phi_j \gamma_{j\theta} \\ \sum_{j=0}^{N} (R + \beta C_0) \phi_j \gamma_{j\theta} \end{cases} \quad \text{(A.10)}$$

from (3.8). Taking the difference between the two items after "min" in (A.10), we get

$$\sum_{j=0}^{N}\left(R+\beta C_0-C_j-\beta\sum_{k=0}^{N}C_k^{(1)}p_{jk}\right)\phi_j\gamma_{j\theta}. \quad (A.11)$$

Since $\Gamma(=\{\gamma_{j\theta}\})$ has a property of TP2 and $\phi_j \geq 0$, we get $\{\phi_j\gamma_{j\theta}\} \in$TP2. On the basis of Lemma 2, we get $\sum_{k=0}^{N}C_k^{(1)}p_{jk}\uparrow$ (*j*) since $C_j^{(1)}\uparrow$ (*j*) and ***P*** $\in$SI. Therefore, (A.10) can be written as (A.12) since the sign of (A.11) changes once at most in accordance with Lemma 3.

$$KV^{(2)}(\mathbf{\Pi})=\sum_{i=0}^{N}C_i\pi_i+\beta\sum_{j=0}^{N}C_j^{(2)}\phi_j \ , \quad (A.12)$$

where

$$C_j^{(2)}=\left(C_j+\beta\sum_{k=0}^{N}C_k^{(1)}p_{jk}\right)+\left\{(R+\beta C_0)-\left(C_j+\beta\sum_{k=0}^{N}C_k^{(1)}p_{jk}\right)\right\}\sum_{\theta=\theta_2^*+1}^{M}\gamma_{j\theta} \ .$$

The second term of (A.9), $RV^{(2)}(\mathbf{\Pi})$, can be written as

$$RV^{(2)}(\mathbf{\Pi})=R+\beta\min\begin{cases}C_0+\beta\sum_{j=0}^{N}C_j^{(1)}p_{0j}\\ R+\beta C_0\end{cases} .$$

From (3.12), we have

$$C_j \leq C_j^{(1)} \leq R \, . \quad (A.13)$$

From assumptions A-3 and A-4 and (A.13), we get

$$(R+\beta C_0)-\left(C_0+\beta\sum_{j=0}^{N}C_j^{(1)}p_{0j}\right)\geq 0.$$

Then,

$$RV^{(2)}(\mathbf{\Pi})=R+\beta C_0+\beta\sum_{j=0}^{N}C_j^{(1)}p_{0j} \, .$$

Therefore, the total expected cost when *n* = 3 can be written as

$$V^{(3)}(\mathbf{\Pi}) = \min\begin{cases}\sum_{i=0}^{N}C_i\pi_i+\beta\sum_{j=0}^{N}C_j^{(2)}\phi_j\\ R+\beta C_0+\beta^2\sum_{j=0}^{N}C_j^{(1)}p_{0j}\end{cases}$$

- Step 2) For *n* >3:

We suppose

$$V^{(n-1)}(\mathbf{\Pi}) = \min\begin{cases}\sum_{i=0}^{N}C_i\pi_i+\beta\sum_{j=0}^{N}C_j^{(n-2)}\phi_j\\ R+\beta C_0+\beta^2\sum_{j=0}^{N}C_j^{(n-3)}p_{0j}\end{cases} \quad (A.14)$$

and prove

$$V^{(n)}(\mathbf{\Pi}) = \min\begin{cases}\sum_{i=0}^{N}C_i\pi_i+\beta\sum_{j=0}^{N}C_j^{(n-1)}\phi_j\\ R+\beta C_0+\beta^2\sum_{j=0}^{N}C_j^{(n-2)}p_{0j}\end{cases}$$

From (3.3) and (3.4), the total expected cost for *n* time periods can be written as

$$V^{(n)}(\mathbf{\Pi})=\min\begin{cases}\sum_{i=0}^{N}C_i\pi_i+\beta\sum_{\theta=0}^{M}P(\theta|\mathbf{\Pi})V^{(n-1)}\left(T(\mathbf{\Pi},\theta)\right)\coloneqq KV^{(n-1)}(\mathbf{\Pi})\\ R+\beta V^{(n-1)}(e^0)\coloneqq RV^{(1)}(\mathbf{\Pi})\end{cases} . \quad (A.15)$$

From (A.14), $KV^{(n-1)}(\mathbf{\Pi})$, the first term of (A.15), can be written as

$$KV^{(n-1)}(\boldsymbol{\Pi}) = \sum_{i=0}^{N} C_i \pi_i + \beta \sum_{\theta=0}^{M} \min \begin{cases} \sum_{j=0}^{N} \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk} \right) \phi_j \gamma_{j\theta} \\ \sum_{j=0}^{N} \left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k} \right) \phi_j \gamma_{j\theta} \end{cases}. \quad \text{(A.16)}$$

Taking the difference after "min" in (A.16), we get

$$\sum_{j=0}^{N} \left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k} \right) \phi_j \gamma_{j\theta} - \sum_{j=0}^{N} \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk} \right) \phi_j \gamma_{j\theta}. \quad \text{(A.17)}$$

From the following relationships,

- $C_j \uparrow (j)$ (from assumption A-4),
- $C_k^{(n-2)} \uparrow (k)$ (from **Preliminary Property 2**),
- $\boldsymbol{P} \in SI$ (from assumption A-1),
- $\{\phi_j \gamma_{j\theta}\} \in TP2$ (from assumption A-2),
- $R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-2)} p_{0k}$ is a constant function of j,

We derive that the sign of (A.17) changes at most once in accordance with Lemma 3. Therefore, (A.16) can be written as

$$KV^{(n-1)}(\boldsymbol{\Pi}) = \sum_{i=0}^{N} C_i \pi_i + \beta \sum_{j=0}^{N} C_j^{(n-1)} \phi_j \ , \quad \text{(A.18)}$$

where

$$C_j^{(n-1)} = \sum_{\theta=0}^{\theta_{n-1}^*} \left(C_j + \beta \sum_{k=0}^{N} C_k^{(n-2)} p_{jk} \right) \gamma_{j\theta} + \sum_{\theta=\theta_{n-1}^*+1}^{M} \left(R + \beta C_0 + \beta^2 \sum_{k=0}^{N} C_k^{(n-3)} p_{0k} \right) \gamma_{j\theta}$$

The second term of (A.15) can be written as

$$RV^{(n-1)}(\boldsymbol{\Pi}) = R + \beta \min \begin{cases} C_0 + \beta \sum_{j=0}^{N} C_j^{(n-2)} p_{0j} \\ R + \beta C_0 + \beta^2 \sum_{j=0}^{N} C_j^{(n-3)} p_{0j} \end{cases} = R + \beta C_0 + \beta^2 \sum_{j=0}^{N} C_j^{(n-2)} p_{0j} \quad (36)$$

since

$$\left(R + \beta C_0 + \beta^2 \sum_{j=0}^{N} C_j^{(n-3)} p_{0j} \right) - \left(C_0 + \beta \sum_{j=0}^{N} C_j^{(n-2)} p_{0j} \right) \geq 0$$

from Preliminary Property 1. Therefore, the total expected cost for n (>3) time periods can be written as

$$V^{(n)}(\Pi) = \min \begin{cases} \sum_{i=0}^{N} C_i \pi_i + \beta \sum_{j=0}^{N} C_j^{(n-1)} \phi_j \\ R + \beta C_0 + \beta^2 \sum_{j=0}^{N} C_j^{(n-2)} p_{0j} \end{cases}$$

from (A.18) and (A.19).

Therefore, Preliminary Property 3 is proved from the results of steps 1 and 2.

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