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ON THE QUOTIENT MOMENTS OF RECORD VALUES FROM THE GENERALIZED EXTREME VALUE DISTRIBUTION

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ABSTRACT. In this paper, we obtain some recurrence relations for the quotient moments of the lower record values from the generalized extereme value(GEV) distribution. We also deduce some recurrence relations for the negative moments from the GEV distribution.

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1. Introduction

Suppose that X_1, $X_2, \cdots$ is a sequence of independent and identically distributed random variables with cumulative distribution function(cdf) $F(x)$ and probability density function(pdf) $f(x)$. Let $Y_n = \min\left\{X_1, X_2, \cdots, X_n\right\}$ for $n \geq 1$. We say X_j is a lower record value of $\{X_n, n \geq 1\}$, if $Y_j < Y_{j-1}$ for $j > 1$. And we suppose that X_1 is a first lower record value. The indices at which the lower record values occur are called the record times $\{L(n), n \geq 1\}$, where $L(n) = \min\left\{j | j > L(n-1)\ ,\ X_j < X_{L(n-1)}\ ,\ n \geq 2\right\}$ with $L(1) = 1$.

There have been enormous works on obtaining recurrence relations for the moments of order statistics or record values from well known statistical distributions. Among such works are Adeyemi [1] obtained the single and product moments of order statistics from the generalized Pareto distribution. Joshi [7] discussed the recurrence relations between moments of order statistics from the exponential and truncated exponential distribution. Balarishnan et al. [4] investigated some recurrence relations for the moments of record values from the GEV

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distribution. Chang [6] established some recurrence relations for the quotient moments of the upper record values from the Weibull distribution.

In this paper, we obtain some recurrence relations for the negative and quotient moments of the lower record values from the GEV distribution.

The pdf of the GEV distribution is

$$f(x) = \begin{cases} (1-kx)^{\frac{1}{k}-1} exp\Big[-(1-kx)^{\frac{1}{k}}\Big], x < \frac{1}{k} \text{ for } k > 0, x > \frac{1}{k} \text{ for } k < 0, \\ exp(-x) exp\Big[-exp(-x)\Big], -\infty < x < \infty \text{ for } k = 0 \end{cases} \tag{1.1}$$

and the cdf of the GEV distribution is

$$F(x) = \begin{cases} exp\Big[-(1-kx)^{\frac{1}{k}}\Big], x < \frac{1}{k} \text{ for } k > 0, x > \frac{1}{k} \text{ for } k < 0, \\ exp\Big[-exp(-x)\Big], -\infty < x < \infty \text{ for } k = 0, \end{cases} \tag{1.2}$$

where k is the parameter.

From (1.1) and (1.2), we obtain the following differential equation for $k \neq 0$

$$(1-kx)f(x) = F(x)(-lnF(x)) \tag{1.3}$$

and we have the following differential equation for $k = 0$

$$f(x) = F(x)(-lnF(x)). \tag{1.4}$$

2. Recurrence relations for the negative moments

Let $X_{L(1)}, X_{L(2)}, \cdots$ be the sequence of the lower record values from the above considered GEV distribution. Then the pdf $f_{(n)}(x)$ of $X_{L(n)}$ for $n \geq 1$ is given by (see, [2, 3])

$$f_{(n)}(x) = \frac{(-lnF(x))^{n-1} f(x)}{(n-1)!}. \tag{2.1}$$

Theorem 2.1 *For $k \neq 0$, $n \geq 1$ and $s > 1$,*

$$E\Big(\frac{1}{X^s_{L(n)}}\Big) = \frac{n}{(n-ks)} E\Big(\frac{1}{X^s_{L(n+1)}}\Big) - \frac{s}{(n-ks)} E\Big(\frac{1}{X^{s+1}_{L(n)}}\Big). \tag{2.2}$$

Proof. Let us consider for $k \neq 0$, $n \geq 1$ and $s > 1$

$$E\Big(\frac{1}{X^{s+1}_{L(n)}} - \frac{k}{X^s_{L(n)}}\Big) = \int \Big(\frac{1}{x^{s+1}} - \frac{k}{x^s}\Big) f_{(n)}(x)\, dx.$$

By the pdf $f_{(n)}(x)$ (2.1), we obtain the following moment

$$E\Big(\frac{1}{X^{s+1}_{L(n)}} - \frac{k}{X^s_{L(n)}}\Big) = \frac{1}{(n-1)!} \int \frac{(1-kx)(-lnF(x))^{n-1} f(x)}{x^{s+1}}\, dx.$$

From the differential equation (1.3), we have the following moment

$$E\left(\frac{1}{X_{L(n)}^{s+1}}-\frac{k}{X_{L(n)}^{s}}\right)=\frac{1}{(n-1)!}\int\frac{F(x)(-lnF(x))^{n}}{x^{s+1}}\,dx. \tag{2.3}$$

Using integrating by parts in the right hand side of (2.3), we get the following moment

$$E\left(\frac{1}{X_{L(n)}^{s+1}}-\frac{k}{X_{L(n)}^{s}}\right)=\frac{1}{(n-1)!s}\int\frac{(-lnF(x))^{n}f(x)}{x^{s}}\,dx$$
$$-\frac{n}{(n-1)!s}\int\frac{(-lnF(x))^{n-1}f(x)}{x^{s}}\,dx.$$

By the pdf (2.1), we can write the following moment

$$E\left(\frac{1}{X_{L(n)}^{s+1}}-\frac{k}{X_{L(n)}^{s}}\right)=\frac{n}{s}\int\frac{f_{(n+1)}(x)}{x^{s}}\,dx-\frac{n}{s}\int\frac{f_{(n)}(x)}{x^{s}}\,dx.$$

Hence we have the following negative moment

$$E\left(\frac{1}{X_{L(n)}^{s}}\right)=\frac{n}{(n-ks)}E\left(\frac{1}{X_{L(n+1)}^{s}}\right)-\frac{s}{(n-ks)}E\left(\frac{1}{X_{L(n)}^{s+1}}\right).$$

This completes the proof.

Theorem 2.2 *For $k=0$, $n\geq 1$ and $s>1$,*

$$E\left(\frac{1}{X_{L(n)}^{s}}\right)=\frac{n}{(s-1)}E\left(\frac{1}{X_{L(n+1)}^{s-1}}\right)-\frac{n}{(s-1)}E\left(\frac{1}{X_{L(n)}^{s-1}}\right). \tag{2.4}$$

Proof. In the same manner as Theorem 2.1, let us consider for $k=0$, $n\geq 1$ and $s>1$

$$E\left(\frac{1}{X_{L(n)}^{s}}\right)=\frac{1}{(n-1)!}\int\frac{(-lnF(x))^{n-1}f(x)}{x^{s}}\,dx.$$

From the differential equation (1.4), we have the following moment

$$E\left(\frac{1}{X_{L(n)}^{s}}\right)=\frac{1}{(n-1)!}\int\frac{F(x)(-lnF(x))^{n}}{x^{s}}\,dx. \tag{2.5}$$

Using integrating by parts in the right hand side of (2.5), we get the following moment

$$E\left(\frac{1}{X_{L(n)}^{s}}\right)=\frac{1}{(n-1)!(s-1)}\int\frac{(-lnF(x))^{n}f(x)}{x^{s-1}}\,dx$$
$$-\frac{n}{(n-1)!(s-1)}\int\frac{(-lnF(x))^{n-1}f(x)}{x^{s-1}}\,dx.$$

By the pdf (2.1), we can write the following moment

$$E\left(\frac{1}{X^s_{L(n)}}\right)=\frac{n}{(s-1)}\int\frac{f_{(n+1)}(x)}{x^{s-1}}\,dx-\frac{n}{(s-1)}\int\frac{f_{(n)}(x)}{x^{s-1}}\,dx.$$

Hence we have the following negative moment

$$E\left(\frac{1}{x^s_{L(n)}}\right)=\frac{n}{(s-1)}E\left(\frac{1}{X^{s-1}_{L(n+1)}}\right)-\frac{n}{(s-1)}E\left(\frac{1}{X^{s-1}_{L(n)}}\right).$$

This completes the proof.

3. Recurrence relations for the quotient moments

Let $X_{L(1)},X_{L(2)},\cdots$ be the sequence of the lower record values tfrom the above considered GEV distribution. Then the joint pdf $f_{(m),(n)}(x,y)$ of $X_{L(m)}$ and $X_{L(n)}$ for $1\le m<n$ is given by (see, [2, 3])

$$f_{(m),(n)}(x,y)=\frac{(-lnF(x))^{m-1}(lnF(x)-lnF(y))^{n-m-1}f(x)f(y)}{(m-1)!(n-m-1)!F(x)}. \tag{3.1}$$

Theorem 3.1 *For $k\neq 0$, $1\le m<n$, $r\ge 1$ and $s\ge 1$*

$$E\left(\frac{X^r_{L(m)}}{X^s_{L(n)}}\right)=\frac{r}{(m+kr)}E\left(\frac{X^{r-1}_{L(m)}}{X^s_{L(n)}}\right)+\frac{m}{(m+kr)}E\left(\frac{X^r_{L(m+1)}}{X^s_{L(n)}}\right). \tag{3.2}$$

Proof. Let us consider for $k\neq 0$, $1\le m<n$, $r\ge 1$ and $s\ge 1$

$$E\left(\frac{X^{r-1}_{L(m)}}{X^s_{L(n)}}-\frac{kX^r_{L(m)}}{X^s_{L(n)}}\right)=\int\int\left(\frac{x^{r-1}}{y^s}-\frac{kx^r}{y^s}\right)f_{(m),(n)}(x,y)\,dx\,dy$$

By the joint pdf $f_{(m),(n)}(x,y)$ (3.1), we obtain the following moment

$$E\left(\frac{X^{r-1}_{L(m)}}{X^s_{L(n)}}-\frac{kX^r_{L(m)}}{X^s_{L(n)}}\right)=\frac{1}{(m-1)!(n-m-1)!}$$
$$\times\int\int\frac{x^{r-1}(1-kx)(-lnF(x))^{m-1}(lnF(x)-lnF(y))^{n-m-1}f(x)f(y)}{y^sF(x)}\,dx\,dy.$$

From the differential equation (1.3), we have the following moment

$$E\left(\frac{X^{r-1}_{L(m)}}{X^s_{L(n)}}-\frac{kX^r_{L(m)}}{X^s_{L(n)}}\right)=\frac{1}{(m-1)!(n-m-1)!}$$
$$\times\int\frac{f(y)}{y^s}\left(\int\frac{x^{r-1}(-lnF(x))^m(lnF(x)-lnF(y))^{n-m-1}}{y^s}\,dx\right)dy. \tag{3.3}$$

Using integrating by parts in the right hand side of (3.3), we get the following moment

$$E\left(\frac{X^{r-1}_{L(m)}}{X^s_{L(n)}} - \frac{kX^r_{L(m)}}{X^s_{L(n)}}\right) = \frac{m}{(m-1)!(n-m-1)!r}$$
$$\times \int\int \frac{x^r(-lnF(x))^{m-1}(lnF(x)-lnF(y))^{n-m-1}f(x)f(y)}{y^sF(x)}\,dx\,dy$$

$$-\frac{1}{(m-1)!(n-m-2)!r}$$
$$\times \int\int \frac{x^r(-lnF(x))^{m}(lnF(x)-lnF(y))^{n-m-2}f(x)f(y)}{y^sF(x)}\,dx\,dy.$$

By the joint pdf (3.1), we can write the following moment

$$E\left(\frac{X^{r-1}_{L(m)}}{X^s_{L(n)}} - \frac{kX^r_{L(m)}}{X^s_{L(n)}}\right) = \frac{m}{r}\int\int \frac{x^r f_{(m),(n)}(x,y)}{y^s}\,dx\,dy$$
$$-\frac{m}{r}\int\int \frac{x^r f_{(m+1),(n)}(x,y)}{y^s}\,dx\,dy.$$

Hence we have the following quotient moment

$$E\left(\frac{X^{r}_{L(m)}}{X^s_{L(n)}}\right) = \frac{r}{(m+kr)}E\left(\frac{X^{r-1}_{L(m)}}{X^s_{L(n)}}\right) + \frac{m}{(m+kr)}E\left(\frac{X^{r}_{L(m+1)}}{X^s_{L(n)}}\right).$$

This completes the proof.

Corollary 3.2 *For $k \neq 0$, $m \geq 1$, $r \geq 1$ and $s \geq 1$*

$$E\left(\frac{X^{r}_{L(m)}}{X^s_{L(m+1)}}\right) = \frac{r}{(m+kr)}E\left(\frac{X^{r-1}_{L(m)}}{X^s_{L(m+1)}}\right) + \frac{m}{(m+kr)}E(X^{r-s}_{L(m+1)}). \quad (3.4)$$

Proof. Upon substituting $n = m+1$ in Theorem 3.1 and simplifying, then we obtain the quotient moment (3.4) .

This completes the proof.

Theorem 3.3 *For $k = 0$, $1 \leq m < n$, $r \geq 0$ and $s \geq 1$*

$$E\left(\frac{X^{r}_{L(m)}}{X^s_{L(n)}}\right) = \frac{m}{(r+1)}E\left(\frac{X^{r+1}_{L(m)}}{X^s_{L(n)}}\right) - \frac{m}{(r+1)}E\left(\frac{X^{r+1}_{L(m+1)}}{X^s_{L(n)}}\right). \quad (3.5)$$

Proof. In the same manner as Theorem 3.1, let us consider for $k = 0$, $1 \leq m < n$, $r \geq 0$ and $s \geq 1$

$$E\left(\frac{X^r_{L(m)}}{X^s_{L(n)}}\right) = \frac{1}{(m-1)!(n-m-1)!} \times \int\int \frac{x^r(-lnF(x))^{m-1}(lnF(x) - lnF(y))^{n-m-1}f(x)f(y)}{y^sF(x)} \, dx\, dy.$$

From the differential equation (1.4), we have the following moment

$$E\left(\frac{X^r_{L(m)}}{X^s_{L(n)}}\right) = \frac{1}{(m-1)!(n-m-1)!} \times \int \frac{f(y)}{y^s}\left(\int x^r(-lnF(x))^m(lnF(x) - lnF(y))^{n-m-1} \, dx\right)dy. \tag{3.6}$$

Using integrating by parts in the right hand side of (3.6), we get the following moment

$$E\left(\frac{X^r_{L(m)}}{X^s_{L(n)}}\right) = \frac{m}{(m-1)!(n-m-1)!(r+1)} \times \int\int \frac{x^{r+1}(-lnF(x))^{m-1}(lnF(x) - lnF(y))^{n-m-1}f(x)f(y)}{y^sF(x)} \, dx\, dy$$
$$- \frac{1}{(m-1)!(n-m-2)!(r+1)} \times \int\int \frac{x^{r+1}(-lnF(x))^{m}(lnF(x) - lnF(y))^{n-m-2}f(x)f(y)}{y^sF(x)} \, dx\, dy.$$

By the joint pdf (3.1), we can write the following moment

$$E\left(\frac{X^r_{L(m)}}{X^s_{L(n)}}\right) = \frac{m}{(r+1)}\int\int \frac{x^{r+1}f_{(m),(n)}(x,y)}{y^s} \, dx\, dy - \frac{m}{(r+1)}\int\int \frac{x^{r+1}f_{(m+1),(n)}(x,y)}{y^s} \, dx\, dy.$$

Hence we have the following quotient moment

$$E\left(\frac{X^r_{L(m)}}{X^s_{L(n)}}\right) = \frac{m}{(r+1)}E\left(\frac{X^{r+1}_{L(m)}}{X^s_{L(n)}}\right) - \frac{m}{(r+1)}E\left(\frac{X^{r+1}_{L(m+1)}}{X^s_{L(n)}}\right).$$

This completes the proof.

Corollary 3.4 *For $k = 0$, $m \geq 1$, $r \geq 0$ and $s \geq 1$*

$$E\left(\frac{X^r_{L(m)}}{X^s_{L(m+1)}}\right) = \frac{m}{(r+1)}E\left(\frac{X^r_{L(m)}}{X^s_{L(m+1)}}\right) - \frac{m}{(r+1)}E(X^{r-s+1}_{L(m+1)}). \tag{3.7}$$

Proof. Upon substituting $n = m + 1$ in Theorem 3.3 and simplifying, then we obtain the quotient moment (3.7) .

This completes the proof.

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