J. Appl. Math. & Informatics Vol. **28**(2010), No. 1 - 2, pp. 515 - 521
Website: http://www.kcam.biz

THE CONTINUOUS DENSITY FUNCTION OF THE LIMITING SPECTRAL DISTRIBUTION

SANG-IL CHOI

ABSTRACT. In multivariate analysis, the inversion formula of the Stieltjes transform is used to find the density of a spectral distribution of random matrices of sample covariance type. Let $B_n = \frac{1}{N} Y_n Y_n^T T_n$ where $Y_n = [Y_{ij}]_{n\times N}$ is with independent, identically distributed entries and T_n is an $n \times n$ symmetric nonnegative definite random matrix independent of the Y_{ij}'s. In the present paper, using the inversion formula of the Stieltjes transform, we will find that the limiting distribution of B_n has a continuous density function away from zero.

AMS Mathematics Subject Classification : 60E99, 26A46
Key words and phrases : Eigenvalues of random matrices, spectral distribution, Stieltjes transform

1. Introduction and preliminaries

Let M be an $m \times m$ random matrix with real eigenvalues $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_m\}$. Then the spectral distribution function of M is the distribution function $F^M(x)$ with a jump of $\dfrac{1}{m}$ at each eigenvalue defined by

$$\forall x \in \mathbb{R}, \qquad F^M(x) = \frac{1}{m}\sum_{i=1}^{m} 1_{(-\infty,x]}(\Lambda_i)$$

where 1_S is the indicator function of the set S.

We have a limit theorem found in [8].

Theorem 1 [8]. *Let $\{Y_{ij}\}_{i,j\geq 1}$ be independent, identically distributed, real-valued random variables with $E|Y_{11} - EY_{11}|^2 = 1$. For each m in $\mathbb{N}$, the set of positive integers, let $Y_m = [Y_{ij}]_{m\times n}$, where n=n(m) and $m/n \to c > 0$ as*

Received July 10, 2009. Accepted November 2, 2009.

$m \to +\infty$, and let T_m be an $m \times m$ symmetric nonnegative definite random matrix independent of the Y_{ij}'s for which there exists a sequence of positive numbers $\{\mu_k\}_{k\geq 1}$ such that for each $k \in \mathbb{N}$,

$$\int_0^{+\infty} x^k dF^{T_m}(x) = \frac{1}{m}\, trT_m^k \to \mu_k, \text{ almost surely, as } m \to +\infty$$

where trA means the trace of the matrix A. and the μ_k's satisfy Carleman's sufficiency condition,

$$\sum_{k\geq 1} \mu_{2k}^{-\frac{1}{2k}} = +\infty,$$

for the existence and the uniqueness of the distribution function H having moments $\{\mu_k\}_{k\geq 1}$, i.e., $\int x^k dH(x) = \mu_k$, for $k = 1, 2, 3, \ldots$.

Let $M_m = \frac{1}{n} Y_m Y_m^T T_m$. Then, almost surely, $\{F^{M_m}\}_{m\geq 1}$ converges weakly to a nonrandom distribution function F_0 having moments

$$\forall k \in \mathbb{N}, \qquad \nu_k = \sum_{w=1}^{k} c^{k-w} \sum \frac{k!}{m_1! m_2! \ldots m_w! w!} \mu_1^{m_1} \cdots \mu_w^{m_w}$$

where the inner sum extends over all w-tuples of nonnegative integers $(m_1, \ldots, m_w)$ such that $\sum_{j=1}^{w} m_j = k - w + 1$ and $\sum_{j=1}^{w} j m_j = k$. Moreover, these moments uniquely determine F_0.

Under the same hypothesis of Theorem 1 and for c, H and F_0 defined in Theorem 1, we have the following facts from [5],

(i) c and F_0 uniquely determine H.

(ii) Almost surely, F^{T_m} converges weakly to H as $m \to \infty$.

(iii) F_0 converges weakly to H as $c \to 0$.

Moreover, we have the continuity of F_0, from [5],

Theorem 2 [5]. *The limiting distribution function F_0 in Theorem 1 is continuous on $\mathbb{R}_+$, the set of positive real numbers. Moreover, if H places no mass at 0, then, almost surely, $\{F^{M_m}\}_{m\geq 1}$ converges to F_0 uniformly in $\mathbb{R}$.*

In [2], $B_n \equiv \frac{1}{n} Y_m^T T_m Y_m$ is studied instead of M_m, and T_m is an arbitrary diagonal matrix, i.e., H is arbitrary.

Let F denote the limiting spectral distribution function of B_n. Then

$$\forall x \in \mathbb{R}, \qquad F(x) = (1-c)1_{[0,\infty)}(x) + cF_0(x). \tag{1.1}$$

Let $\mathbb{C}_+ = \left\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\right\}$. For $z \in C_+$, let $m(z)$ be the Stieltjes transform of F, i.e.,

$$m(z) = \int \frac{dF(\lambda)}{\lambda - z}.$$

In [2], it is shown that, for each $z \in \mathbb{C}_+$, $m = m(z)$ is the unique solution for $m \in \mathbb{C}_+$ to the equation

$$m = -\left(z - c\int \frac{\lambda dH(\lambda)}{1+\lambda m}\right)^{-1}.$$

Therefore, on $\mathbb{C}_+$, $m(z)$ has an inverse, given by

$$z(m) = -\frac{1}{m} + c\int \frac{\lambda dH(\lambda)}{1+\lambda m} \qquad \text{for } m \in m(\mathbb{C}_+).$$

We will have the continuous density of F in (1.1).

Theorem 3. *For F defined as above, F has a continuous density f away from 0 and, for all $x \neq 0$,*

$$f(x) = \lim_{y \downarrow 0} \frac{1}{\pi} Im(m(x+iy)).$$

2. Preliminary results

For (1.1), we need the following theorem.

Theorem 2.1. *If A is an $n \times m$ matrix and B is an $m \times n$ matrix, then*

$$x^m P_{AB} = x^n P_{BA}$$

where P_M is the characteristic polynomial of the matrix M.

Proof. Let

$$C = \begin{pmatrix} xI_n & A \\ B & I_m \end{pmatrix} \text{ and } D = \begin{pmatrix} I_n & 0 \\ -B & xI_m \end{pmatrix}$$

be two $(m+n) \times (m+n)$ matrices, where I_k is the $k \times k$ identity matrix. Then

$$CD = \begin{pmatrix} xI_n - AB & xA \\ 0 & xI_m \end{pmatrix} \text{ and } DC = \begin{pmatrix} xI_n & A \\ 0 & xI_m - BA \end{pmatrix}.$$

Therefore, from $|CD| = |DC|$, we have

$$\begin{aligned} x^m P_{AB} &= |xI_m||xI_n - AB| \\ &= |CD| \\ &= |DC| \\ &= |xI_n||xI_m - BA| = x^n P_{BA}. \end{aligned}$$

For B_n and M_m in Chapter 1, we have $x^m P_{B_n} = x^n P_{M_m}$. Thus the difference between the eigenvalues of B_n and the eigenvalues of M_m are $|n-m|$ zeros, that is, when $m < n$, the eigenvalues of B_n has extra $n-m$ zeros, and, when $m > n$, the eigenvalues of M_m has extra $m-n$ zeros. Thus we have

$$F^{B_n}(x) = \frac{m}{n} F^{M_m}(x), \qquad \text{if } x < 0$$

and

$$F^{B_n}(x) = \frac{n-m}{n} + \frac{m}{n} F^{M_m}(x), \qquad \text{if } x \geq 0.$$

That is,

$$F^{B_n}(x) = \left(1 - \frac{m}{n}\right) 1_{[0,\infty)}(x) + \frac{m}{n} F^{M_m}(x).$$

Therefore we have (1.1).

Now let $G(\cdot)$ be an arbitrary distribution function and let $S(\cdot)$ be the Stieltjes transform of G. Let S_G be the support of G.

We can calculate G from the Inversion Formula,

$$G(x_2) - G(x_1) = \lim_{y \downarrow 0} \frac{1}{\pi} \int_{x_1}^{x_2} \mathrm{Im}(S(x+iy))dx$$

where x_1 and x_2 are continuity points of α.

Next theorem gives us the derivative of G.

Theorem 2.2. *Let $x_0 \in \mathbb{R}$. If $\lim_{z \in D \to x_0} Im(S(z))$ exists, call it $Im(S(x_0))$, then G is differentiable at x_0, and its derivative is $\frac{1}{\pi} Im(S(x_0))$.*

Proof. Given $\epsilon > 0$, let $\delta > 0$ be s.t. $|x - x_0| < \delta$, $0 < y < \delta$ implies $\frac{1}{\pi}|\mathrm{Im}(S(x+iy)) - \mathrm{Im}(S(x_0))| < \frac{\epsilon}{2}$. Since all continuity points are dense in $\mathbb{R}$, $\exists x_1, x_2$, continuity points s.t. $x_1 < x_2$ and $|x_i - x_0| < \delta$, $i = 1, 2$, and, from the Inversion Formula, we can choose y with $0 < y < \delta$ so that

$$\left| G(x_2) - G(x_1) - \frac{1}{\pi} \int_{x_1}^{x_2} \mathrm{Im}(S(x+iy))dx \right| < \frac{\epsilon}{2}(x_2 - x_1).$$

For any $x \in [x_1, x_2]$, we have $|x - x_0| < \delta$. Thus

$$\left| \frac{G(x_2) - G(x_1)}{x_2 - x_1} - \frac{1}{\pi}\mathrm{Im}(S(x_0)) \right| \leq$$
$$\frac{1}{x_2 - x_1} \left| G(x_2) - G(x_1) - \frac{1}{\pi} \int_{x_1}^{x_2} \mathrm{Im}(S(x+iy))dx \right|$$
$$+ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \left| \frac{1}{\pi} \left(\mathrm{Im}(S(x+iy)) - \mathrm{Im}(S(x_0))\right) \right| dx < \epsilon.$$

Therefore, for all $\{x_n\}$, a sequence of continuity points with $x_n \to x_0$ as $n \to \infty$,

$$\lim_{n,m\to\infty} \frac{G(x_n) - G(x_m)}{x_n - x_m} = \frac{1}{\pi}\mathrm{Im}(S(x_0)),$$

and, therefore $\{G(x_n)\}$ is Cauchy. Thus, since all continuity points are dense in $\mathbb{R}$, $\lim\limits_{x\uparrow x_0} G(x) = \lim\limits_{x\downarrow x_0} G(x)$, and, therefore G is continuous at x_0.

Therefore, by choosing the sequence $\{x_1, x_0, x_2, x_0, \dots\}$, we have

$$\lim_{n\to\infty} \frac{G(x_n) - G(x_0)}{x_n - x_0} = \frac{1}{\pi}\mathrm{Im}(S(x_0)). \tag{2.1}$$

Now, for any $x \in \mathbb{R}$ with $x \neq x_0$, $G(x_{cp})$ can be made arbitrarily close to $G(x)$, for some continuty point x_{cp}, by making x_{cp} suitably close to x. Then, $\dfrac{G(x_{cp}) - G(x_o)}{x_{cp} - x_0}$, for $x_{cp} \neq x_0$, can be made arbitrarily close to $\dfrac{G(x) - G(x_o)}{x - x_0}$, in the following way:

For any $\epsilon > 0$, if $x > x_0$, then choose x_{cp}^+, x_{cp}^-, continuity points, s.t. $x_0 < x_{cp}^- \leq x \leq x_{cp}^+$, $(1-\epsilon)(x - x_0) \leq x_{cp}^- - x_0$, and $x_{cp}^+ - x_0 \leq (1+\epsilon)(x - x_0)$, and, hence

$$\begin{aligned}(1-\epsilon)\frac{G(x_{cp}^-) - G(x_0)}{x_{cp}^- - x_0} &\leq \frac{G(x) - G(x_0)}{x - x_0}\\ &\leq (1+\epsilon)\frac{G(x_{cp}^+) - G(x_0)}{x_{cp}^+ - x_0},\end{aligned}$$

and, if $x < x_0$, then choose x_{cp}^+, x_{cp}^-, continuity points s.t. $x_0 > x_{cp}^- \geq x \geq x_{cp}^+$, $(1-\epsilon)(x_0 - x) \leq x_0 - x_{cp}^-$, and $x_0 - x_{cp}^+ \leq (x_0 - x)(1+\epsilon)$, and, hence

$$\begin{aligned}(1-\epsilon)\frac{G(x_0) - G(x_{cp}^-)}{x_0 - x_{cp}^-} &\leq \frac{G(x_0) - G(x)}{x_0 - x}\\ &\leq (1+\epsilon)\frac{G(x_0) - G(x_{cp}^+)}{x_0 - x_{cp}^+}.\end{aligned}$$

Now, for any $\{x_n\}$ with $x_n \neq x_0 \to x_0$, for each n, as above, we can choose $\left\{x_{cp}^{(n)-}\right\}$ and $\left\{x_{cp}^{(n)+}\right\}$ s.t.

$$\begin{aligned}\left(1 - \frac{1}{n}\right)\frac{G(x_{cp}^{(n)-}) - G(x_0)}{x_{cp}^{(n)-} - x_0} &\leq \frac{G(x_n) - G(x_0)}{x_n - x_0}\\ &\leq \left(1 + \frac{1}{n}\right)\frac{G(x_{cp}^{(n)+}) - G(x_0)}{x_{cp}^{(n)+} - x_0}.\end{aligned}$$

From (2.1), as $n \to \infty$, since

$$\left(1 - \frac{1}{n}\right) \frac{G(x_{cp}^{(n)-}) - G(x_0)}{x_{cp}^{(n)-} - x_0} \to \frac{1}{\pi}\mathrm{Im}(S(x_0))$$

and

$$\left(1 + \frac{1}{n}\right) \frac{G(x_{cp}^{(n)+}) - G(x_0)}{x_{cp}^{(n)+} - x_0} \to \frac{1}{\pi}\mathrm{Im}(S(x_0)),$$

we must have

$$\frac{G(x_n) - G(x_0)}{x_n - x_0} \to \frac{1}{\pi}\mathrm{Im}(S(x_0)).$$

This complete the proof.

Theorem 2.3. *Let X be an open and bounded subset of $\mathbb{R}^n$, let Y be an open and bounded subset of $\mathbb{R}^m$, and let $f : X \to Y$ be a continuous function. If, $\forall \bar{x} \in Bd(X)^1$, $\lim_{x \in X \to \bar{x}} f(x)$ exists, call it $f(\bar{x})$, then f can be extended to a continuous map on $\bar{X} = X \cup Bd(X)$.*

Proof. Let $x_0 \in Bd(X)$. Given $\epsilon > 0$, there exists $\delta > 0$ s.t.

$$\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \frac{\epsilon}{2}.$$

Therefore, $\forall \hat{x} \in Bd(X)$ with $\|\hat{x} - x_0\| < \delta$, since $\lim\limits_{x \in X \to \hat{x}} f(x)$ exists,

$$\exists \bar{x} \in X \text{ s.t. } \|\bar{x} - x_0\| < \delta \text{ and } \|f(\hat{x}) - f(\bar{x})\| < \frac{\epsilon}{2}.$$

Therefore

$$\|f(\hat{x}) - f(x_0)\| \le \|f(\hat{x}) - f(\bar{x})\| + \|f(x_0) - f(\bar{x})\| < \epsilon.$$

Therefore, for all $x \in \bar{X}$, $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$. Therefore, f is continuous for all $x \in \bar{X}$. □

Proof of the Theorem 3.

For $x_0 \neq 0$, we have the existence of $\lim\limits_{z \in \mathbb{C}_+ \to x_0} m(z)$ in [4].

From theorem 2.2, for all $x \neq 0$, the density of F is given by

$$f(x) = \lim_{y \downarrow 0} \frac{1}{\pi}\mathrm{Im}(m(x + iy)).$$

From theorem 2.3, f is continuous, for all $x \neq 0$.

This complete the proof of the Theorem 3.

[1] The set of the boundary points of X.

References

1 D. Jonsson, *Some limit theorems for the eigenvalues of a sample covariance matrix*, Journal of Multivariate Analysis **12** (March 1982), 1–38.

2 V. A. Marcenko and L. A. Pastur, *Distribution of eignvalues for some sets of random matrices*, Mathematics of the USSR-Sbornik **1, no. 4** (1967), 457–483.

3 J. W. Silverstein, *The limiting eigenvalue distribution of a multivariate F matrix*, SIAM Journal on Applied Mathematics **16, no. 3** (May 1985), 641–646.

4 J. W. Silverstein and S. I. Choi, *Analysis of limiting spectral distribution function of large dimensional random matrices*, Journal of Multivariate Analysis **54, no. 2** (August 1995), 295–309.

5 J. W. Silverstein and P. L. Combettes, *Spectral theory of large dimensional random matrices applied to signal detection*, Tech. rep , Dept. Mathematics, North Carolina State Univ., Raleigh, NC (1990).

6 J. W. Silverstein and P. L. Combettes, *Signal detection via spectral theory of large dimensional random matrices*, IEEE Trans. Signal Processing **40, no. 3** (August 1992), 2100–2105.

7 K. W. Wachter, *The strong limits of random matrix spectra for sample matrices of independent elements*, Annals of Probability **6, no. 1** (April 1985), 1–18.

8 Y. Q. Yin, *Limiting spectral distribution for a class of random matrices*, Journal of Multivariate Analysis **20, no. 1** (October 1986), 50–68.

Sang-Il Choi,
Department of Mathematics, Hanseo University, Seosan, Chungnam 356-820, Korea
e-mail: schoi@hanseo.ac.kr