J. Appl. Math. & Informatics Vol. 28(2010), No. 1 - 2, pp. 499 - 507
Website: http://www.kcam.biz

LOCAL SPECTRAL PROPERTIES OF SEMI-SHIFTS

JONG-KWANG YOO* AND YONG IL KIM

ABSTRACT. In this note, we study the local spectral properties of semi-shifts. If $T \in L(X)$ is a semi-shift on a complex Banach space X, then T is admissible. We also prove that if $T \in L(X)$ is subadmissible, then $X_T(F) = E_T(F)$ for all closed $F \subseteq \mathbb{C}$. In particular, every subscalar operator on a Banach space is admissible.

AMS Mathematics Subject Classification : 47A11, 47A53
Key words and phrases : Local spectrum, Algebraic spectral subspace, Analytic spectral subspace, Semi-shifts, Bishop's property (β)

1. Introduction

We first recall some basic notions and results from local spectra theory. Let X be a complex Banach space and $L(X)$ denotes the Banach algebra of all bounded linear operators of X itself, equipped with the usual operator norm. For $T \in L(X)$, TX and $KerT$ will denote the range and kernel, respectively. Given an operator $T \in L(X)$, $\sigma_p(T)$, $\sigma(T)$ and $\rho(T)$ denotes the point spectrum, the spectrum and resolvent set of T and let $Lat(T)$ stand for the collection of all $T-$invariant closed linear subspaces of X, and for $Y \in Lat(T)$, $T|Y$ denotes the restriction of T on Y. For $T \in L(X)$, we denote by $R_T : \lambda \in \rho(T) \to R_T(\lambda) := (T - \lambda I)^{-1} \in L(X)$ its resolvent map. It is well known that if $\lambda \in \rho(T)$ then

$$\|R_T(\lambda)\| \geq \frac{1}{dist(\lambda, \sigma(T))},$$

where $dist(\lambda, \sigma(T))$ denotes the distance of the complex number λ from $\sigma(T)$. This implies that the resolvent map is never bounded. For an operator $T \in L(X)$ and arbitrary $x \in X$, we define $f : \rho(T) \to X$ by $f(\lambda) := R_T(\lambda)x$. Then f may have analytic extensions, solutions of the equation $(T - \lambda)f(\lambda) = x$. If for every

Received August 3, 2009. Revised August 30, 2009. October 21, 2009. * Corresponding author.

$x \in X$ any two extensions of $R_T(\lambda)x$ agree on their common domain, $T \in L(X)$ is said to have the *single-valued extension property(SVEP)*. In this case, let $\rho_T(x)$ be the maximal domain of such extensions. The set $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ is called the *local spectrum* of T at x. Evidently, $\sigma_T(x)$ is closed with $\sigma_T(x) \subseteq \sigma(T)$.

The resolvent set $\rho(T)$ is always a subset of $\rho_T(x)$, so the analytic solutions occurring in the definition of the local resolvent set may be thought of as local extensions of the function $(T-\lambda)^{-1}x$. It is obvious that T has the SVEP if and only if the zero function is the only analytic function that satisfies $(T-\lambda)f(\lambda) = 0$. By the Liouville theorem, it is clear that T has the SVEP if and only if for any non-zero $x \in X$, we have $\sigma_T(x) \neq \phi$, see [1], [8] and [10] for more details.

In this note, we proved that if $T \in L(X)$ is a semi-shift on a complex Banach space X, then T is admissible and $X_T(F) = E_T(F)$ for all closed $F \subseteq \mathbb{C}$ and $E_T(\mathbb{C} \setminus \{0\}) = \bigcap_{n=1}^{\infty} T^n X$. We also prove that if $T \in L(X)$ is a semi-shift on a reflexive Banach space X and $\tilde{x}(\lambda)$ is bounded, then $x \in \bigcap_{\lambda \in \partial\sigma_T(x)} (T-\lambda)E_T(\sigma_T(x))$. Finally, we proved that if $T \in L(X)$ is subadmissible, then $X_T(F) = E_T(F)$ for all closed $F \subseteq \mathbb{C}$. In particular, every subscalar operator on a Banach space is admissible.

We shall also need some closely related notions. An operator $T \in L(X)$ is said to have *Bishop's property* (β) if for every open subset U of $\mathbb{C}$ and for every sequence of analytic functions $f_n : U \to X$ for which $(T-\lambda)f_n(\lambda)$ converges uniformly to zero on each compact subset of U, it follows that $f_n(\lambda) \to 0$ as $n \to \infty$, locally uniformly on U.

For every closed subset F of $\mathbb{C}$, let $X_T(F) = \Big\{x \in X : \sigma_T(x) \subseteq F\Big\}$ denote the corresponding *analytic spectral subspace* of T, that is, $x \in X_T(F)$ if and only if every $\lambda \in \mathbb{C} \setminus F$ has an open neighborhood V and an analytic function $f : V \to X$ such that $(T-\mu)f(\mu) = x$ for all $\mu \in V$.

It is easy to see that $X_T(F)$ is a $T-$invariant linear subspace of X and also hyperinvariant for T, but need not be closed.

An operator $T \in L(X)$ is said to have *Dunford's property (C)* if $X_T(F)$ is closed for every closed $F \subseteq \mathbb{C}$. It is well known that the following implication hold:

$$T \text{ has property } (\beta) \;\Rightarrow\; T \text{ has property (C)} \;\Rightarrow\; T \text{ has SVEP}. \tag{1}$$

Note that neither of the implications (1) may be reversed in general, see [1], [13].

Associated with the operator T and each closed subset F of $\mathbb{C}$ is also an *algebraic spectral subspace* $E_T(F)$, defined to be the linear span of the collection of all (not necessarily closed) linear subspaces Y of X for which

$$(T-\lambda)Y = Y \quad \text{for each} \quad \lambda \in \mathbb{C} \setminus F,$$

Evidently, $E_T(F)$ is the largest linear subspace Y for which $(T-\lambda)Y = Y$ for all $\lambda \in \mathbb{C} \setminus F$. These spaces, with an equivalent definition, were introduced in

[4] in connection with certain problems in automatic continuity. It follows from Proposition 1.2.16 in [10] that $X_T(F) \subseteq E_T(F)$ for every $T \in L(X)$ and closed set $F \subseteq \mathbb{C}$.

Thus if T has no non-trivial divisible subspace in the sense that $E_T(\phi) = \{0\}$, then clearly T has SVEP. By the open mapping theorem, we observe, for a closed set $F \subseteq \mathbb{C}$ that if $E_T(F)$ is closed, we have $E_T(F) = X_T(F)$, see [11]. It is clear that $x \in E_T(F)$ if for every $\lambda \in \mathbb{C} \setminus F$, there exists (x_n) in X such that $(T-\lambda)x_{n+1} = x_n$ and $x = x_0$ for all $n = 0, 1, 2, \cdots$.

An operator $T \in L(X)$ on a Banach space X is said to be *admissible* if, for each closed $F \subseteq \mathbb{C}$, the algebraic spectral subspace $E_T(F)$ is closed.

Examples 1. 1) Recall from [7] that an operator $T \in L(X)$ is said to be a generalized scalar operator if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to L(X)$ satisfying $\Phi(1) = I$ and $\Phi(z) = T$ where I is the identity operator on X and z denotes the identity function on $\mathbb{C}$. In [15], it is shown that if $T \in L(X)$ is a generalized scalar operator then $E_T(F)$ is closed, for any closed $F \subseteq \mathbb{C}$. Hence all generalized operators and, in particular, all normal operators on a Hilbert spaces are admissible.

2) Recall from [7] that an operator $T \in L(X)$ is said to be super-decomposable operator if for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$ there is an operator $R \in L(X)$ commuting with T such that $\sigma\Big(T|\overline{R(X)}\Big) \subseteq U$ and $\sigma\Big(T|\overline{(I-R)(X)}\Big) \subseteq V$. It follows from [10] that if T is super-decomposable and $E_T(\phi) = \{0\}$, then the algebraic spectral subspace $E_T(F)$ is closed for any closed $F \subseteq \mathbb{C}$. Thus super-decomposable operators with no non-trivial divisible subspaces are admissible.

3) Recall from [6] that an operator $T \in L(X)$ on a Banach space X is said to be a totally paranormal operator(TPN) if $\|(T-\lambda)x\|^2 \leq \|(T-\lambda)^2 x\|$ for every $\lambda \in \mathbb{C}$ and every $x \in X$. In particular, every hyponormal operator is totally paranormal operator. It follows from [6] that if $T \in L(H)$ on a Hilbert space H is TPN and $\sigma_p(T) = \phi$, then the algebraic spectral subspace $E_T(F)$ is closed, for any closed $F \subseteq \mathbb{C}$. Thus TPN operators without eigenvalues are admissible.

2. Local spectral properties of semi-shifts

We say that an operator $T \in L(X)$ is *semi-shift* if T is an isometry for which

$$\bigcap_{n=1}^{\infty} T^n X = \{0\}.$$

Evidently, a semi-shift on a non-trivial Banach space is a non-invertible isometry.

Natural examples include, for arbitrary $1 \leq p \leq \infty$, the unilateral right shifts of arbitrary multiplicity $\ell^p(\mathbb{N})$, and the right translation operators on $L^p([0,\infty))$. Moreover, it follows easily from the von Neumann-Wold decomposition that, on Hilbert spaces, the semi-shifts are precisely the pure isometries.

In [15], Vrbová proved that if $T \in L(X)$ is a generalized scalar operator on a complex Banach space X, then

$$X_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} (T-\lambda)^p X$$

for all sufficiently large integers p and closed sets $F \subseteq \mathbb{C}$. From this equality, we have

$$E_T(F) \subseteq \bigcap_{\lambda \in \mathbb{C} \setminus F, n \in \mathbb{N}} (T-\lambda)^n X \subseteq \bigcap_{\lambda \in \mathbb{C} \setminus F} (T-\lambda)^p X = X_T(F)$$

for all closed $F \subseteq \mathbb{C}$, since $E_T(F) = (T-\lambda)^n E_T(F) \subseteq (T-\lambda)^n X$ for all $\lambda \in \mathbb{C} \setminus F$ and for all $n \in \mathbb{N}$. Hence

$$X_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} (T-\lambda)^p X, \quad \text{for all closed } F \subseteq \mathbb{C}.$$

Since every generalized scalar operator has SVEP, Vrbová's result shows that

$$\bigcap_{\lambda \in \mathbb{C}} (T-\lambda)^p X = E_T(\phi) = X_T(\phi) = \{0\},$$

i.e., every generalized scalar operator has no divisible subspace different from zero and there exists an integer $p \in \mathbb{N}$ such that the intersection of the ranges $(T-\lambda)^p X$ over all $\lambda \in \mathbb{C}$ is trivial.

An operator $T \in L(X)$ on a complex Banach space X is said to be *subscalar* provided that T is similar to the restriction of a generalized scalar operator to a closed invariant subspace.

Theorem 1. *If $T \in L(X)$ is a semi-shift on a complex Banach space X, then T is admissible. Furthermore, $X_T(F) = E_T(F)$ for all closed $F \subseteq \mathbb{C}$ and $E_T(\mathbb{C} \setminus \{0\}) = \bigcap_{n=1}^{\infty} T^n X$.*

Proof. Let $F \subseteq \mathbb{C}$ be a given closed set. If $E_T(F) = \{0\}$, then $E_T(F)$ is closed. Suppose that $E_T(F) \neq \{0\}$ and suppose that there is $\lambda \in \mathbb{C} \setminus F$ with $|\lambda| < 1$. Since $(T-\lambda)E_T(F) = E_T(F)$ and $T-\lambda$ is bounded below, it follows that

$$(T-\lambda)\overline{E_T(F)} = \overline{E_T(F)}.$$

This implies that the restriction of $T-\lambda$ to $\overline{E_T(F)}$ is invertible.

However, if an isometry on any Banach space is non-invertible, then $\sigma(T)$ is the entire unit disc. Thus $T|\overline{E_T(F)}$ is invertible, and hence $\sigma(T|\overline{E_T(F)}) \subseteq \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle. From this it follows that $E_T(F) \subseteq \overline{E_T(F)} \subseteq E_T(\mathbb{T})$. However, if $\overline{E_T(F)} \subseteq E_T(\mathbb{T})$ then we may assume that isometry T is invertible. Let $S := T|\overline{E_T(F)}$. Then S is invertible and so $E_S(F)$ is closed in $\overline{E_T(F)}$. Hence $E_T(F) = E_S(F)$ is closed. But if an isometry T is invertible, then by Corollary 4.6 [7] T is generalized scalar, and hence $E_T(F)$ is closed. This part of the argument was done under the assumption that there is $\lambda \in \mathbb{C} \setminus F$ with $|\lambda| < 1$. If this is not the case, then $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq F$ and hence $E_T(F) = X$ is closed. Finally, we will show that

$$E_T(\mathbb{C} \setminus \{0\}) = \bigcap_{n=1}^{\infty} T^n X.$$

Let $Z := \bigcap_{n=1}^{\infty} T^n X$. It is clear that $E_T(\mathbb{C} \setminus \{0\}) \subseteq Z$. It remains to show that $Z \subseteq E_T(\mathbb{C} \setminus \{0\})$. It suffices to show that $TZ = Z$. Clearly, $TZ \subseteq Z$. If $x \in Z$ and $x = T^n x_n$, $n = 1, 2, 3, \cdots$, then $T(x_1 - Tx_2) = 0$ and $T(Tx_2 - T^2x_3) = 0$. Since T is injective, $x_1 = Tx_2 = T^2x_2$. By iterating this procedure, we have

$$x_1 = Tx_2 = T^2x_3 = T^3x_4 = \cdots,$$

and so $x_1 \in Z$ and hence $x = Tx_1 \in Z$. This means that $TZ = Z$. By definition of $E_T(\mathbb{C} \setminus \{0\})$, we have $Z \subseteq E_T(\mathbb{C} \setminus \{0\})$.

It is well known that if T is an isometry on a Banach space X then, by Theorem 1, $E_T(\mathbb{C}\setminus\{0\}) = \bigcap_{n=1}^{\infty} T^n X$. Let $Y := E_T(\mathbb{C}\setminus\{0\})$. Since T is an isometry, $T^n X$ is closed for all $n \in \mathbb{N}$. Thus Y is closed and $T|Y$ is invertible. This means that $T|Y$ is an invertible isometry and consequently $T|Y$ is generalized scalar. It follows that $E_T(\phi) = E_{T|Y}(\phi) = \{0\}$ so that an isometry has no non-trivial divisible subspace.

Denote in the sequel for $A \subseteq \mathbb{C}$ the closure by $\overline{A}$ and by A° the interior.

Proposition 2. *Suppose that $T \in L(X)$ is a semi-shift on a reflexive Banach space X. If $\tilde{x}(\lambda)$ is bounded, then $x \in \bigcap_{\lambda \in \partial\sigma_T(x)} (T - \lambda)E_T(\sigma_T(x))$.*

Proof. Clearly, $X_T(\sigma_T(x)) = E_T(\sigma_T(x))$ for all $x \in X$, by Theorem 1. Suppose that $\tilde{x}(\lambda)$ is bounded. Let $\lambda \in \partial\sigma_T(x)$. Then there exists $\lambda_n \in \rho_T(x)$ such that $\lambda_n \to \lambda$ as $n \to \infty$. Since X is a reflexive Banach space, we can choose λ_n so that $\tilde{x}(\lambda_n)$ is convergent sequence, let y be its limit. Thus we have $(T - \lambda)y = x$. It follows from Proposition 1.2.16 [10] that $\sigma_T(x) = \sigma_T(y)$. This means that $y \in X_T(\sigma_T(x)) = E_T(\sigma_T(x))$ and hence $x = (T - \lambda)y \in (T - \lambda)E_T(\sigma_T(x))$.

It is clear from Proposition 2 that if $x \notin \bigcap_{\lambda \in \partial\sigma_T(x)} (T - \lambda)E_T(\sigma_T(x))$, then the local resolvent function $\tilde{x}(\lambda)$ is unbounded. It is clear that the local resolvent function $\tilde{x}(\lambda)$ is analytic on $\rho_T(x)$.

Lemma 3. *Suppose that $T \in L(X)$ is a semi-shift on a reflexive Banach space X and all derivatives of $\tilde{x}(\lambda)$ are bounded. If $\lambda_0 \in \partial\sigma_T(x)$, then there exists a sequence (x_n) in X such that $x = x_0$, $(T-\lambda_0)x_{n+1} = x_n$ and $\sigma_T(x_{n+1}) = \sigma_T(x_n)$ for all $n = 0, 1, 2, \cdots$.*

Proof. Clearly, $X_T(\phi) = E_T(\phi)$ is closed, by Proposition 1.1 [9] and hence T has SVEP. Suppose that all derivatives of $\tilde{x}(\lambda)$ are bounded. Since $\{\tilde{x}'(\lambda)\}$ is bounded, there exists a positive constant $m > 0$ such that

$$\|\tilde{x}(\lambda) - \tilde{x}(\mu)\| \leq m|\lambda - \mu|$$

for all $\lambda, \mu \in \rho_T(x)$. If $\lambda_0 \in \partial\sigma_T(x)$, then there exists $\{\lambda_n\} \subseteq \rho_T(x)$ that is converging to λ_0. Since $\{\lambda_n\}$ is a Cauchy sequence, $\tilde{x}(\lambda_n)$ is also a Cauchy sequence in X. Let $x_1 := \lim_{n\to\infty} \tilde{x}(\lambda_n)$. Then we have $(T-\lambda_0)x_1 = x$ and $\sigma_T(x) = \sigma_T(x_1)$. Since $(T-\lambda_0)(T-\lambda)\tilde{x}_1(\lambda) = (T-\lambda)\tilde{x}(\lambda)$, and by the SVEP of T, we have

$$(T - \lambda_0)\tilde{x}_1(\lambda) = \tilde{x}(\lambda).$$

Thus we obtain

$$\tilde{x}_1(\lambda) = \frac{\tilde{x}(\lambda) - x_1}{\lambda - \lambda_0} = \lim_{n\to\infty} \frac{\tilde{x}(\lambda) - \tilde{x}(\lambda_n)}{\lambda - \lambda_n}$$

Using the preceding inequality, we have $\|\tilde{x}_1(\lambda)\| \leq m$. Since X is a reflexive Banach space, we can choose λ_n so that $\tilde{x}(\lambda_n)$ is convergent sequence, let $\lim_{n\to\infty} \tilde{x}(\lambda_n) := x_2$. As before, we have $(T-\lambda_0)x_2 = x_1$ and $\sigma_T(x_2) = \sigma_T(x_1)$. By our assumption, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\|\tilde{x}(\lambda) - \tilde{x}(\mu) - (\lambda - \mu)\tilde{x}(\mu)\| \leq C_1|\lambda - \mu|^2,$$
$$\|\tilde{x}'(\lambda) - \tilde{x}'(\mu)\| \leq C_2|\lambda - \mu|$$

for all $\lambda, \mu \in \rho_T(x)$. We derive that $\tilde{x}'(\lambda_n)$ is also a Cauchy sequence, denote z its limit. By the preceding inequality, we have

$$\|\tilde{x}_1(\lambda) - z\| \leq C_1|\lambda - \lambda_0|.$$

Thus $z = x_2$ and $\|\tilde{x}_2(\lambda)\| \leq C_2$. Hence we construct by induction a sequence (x_n) in X such that $x = x_0$, $(T-\lambda_0)x_{n+1} = x_n$ and $\sigma_T(x_{n+1}) = \sigma_T(x_n)$ for all $n = 0, 1, 2, \cdots$. □

Theorem 4. *Suppose that $T \in L(X)$ is a semi-shift on a reflexive Banach space X. If all the derivatives of $\tilde{x}(\lambda)$ are bounded, then $\sigma_T(x) = \overline{\sigma_T(x)^\circ}$.*

Proof. It is clear that $\overline{\sigma_T(x)^\circ} \subseteq \sigma_T(x)$, since $\sigma_T(x)$ is closed. It follows from Theorem 1 that T is admissible. Thus we have $E_T(\overline{\sigma_T(x)^\circ}) = X_T(\overline{\sigma_T(x)^\circ})$. Let $\lambda_0 \in \partial\sigma_T(x)$. Then, by Lemma 3 there exists a sequence (x_n) in X such that

$$x = x_0, (T - \lambda_0)x_{n+1} = x_n \quad \text{and} \quad \sigma_T(x_{n+1}) = \sigma_T(x_n)$$

for all $n = 0, 1, 2, \cdots$. Thus $x \in E_T(\sigma_T(x)^\circ) \subseteq E_T(\overline{\sigma_T(x)^\circ}) = X_T(\overline{\sigma_T(x)^\circ})$. This means that $X_T(\sigma_T(x)) \subseteq X_T(\overline{\sigma_T(x)^\circ})$ and hence $\sigma_T(x) \subseteq \overline{\sigma_T(x)^\circ}$.

Lemma 5 ([11]). *Let $T \in L(X)$ be a bounded linear operator on a Banach space X. Suppose that $F \subseteq \mathbb{C}$ is closed and $E_T(F)$ is closed. Then $X_T(F) = E_T(F)$.*

Recall that a linear subspace Y of X is said to be $T-$*divisible subspace* if

$$(T - \lambda)Y = Y \quad \text{for all } \lambda \in \mathbb{C}.$$

Evidently, $E_T(\phi)$ is the largest $T-$divisible linear subspace.

Lemma 6. *Every admissible operator cannot have non-trivial divisible subspaces. In particular, if $T \in L(X)$ is admissible then T has SVEP.*

Proof. If $T \in L(X)$ is admissible, then by Lemma 5 $E_T(F) = X_T(F)$ is closed for every closed $F \subseteq \mathbb{C}$. In particular, $E_T(\phi) = X_T(\phi)$ is closed. It follows from Proposition 1.1 [8] that T has SVEP and $E_T(\phi) = X_T(\phi) = \{0\}$ is closed. If Z is a $T-$divisible subspace of T, then $(T - \lambda)Z = Z$ for all $\lambda \in \mathbb{C}$. By the maximality of $E_T(\phi)$, $Z \subseteq E_T(\phi) = \{0\}$, and hence $Z = \{0\}$.

An operator $T \in L(X)$ is called *semi-admissible* if there is an admissible operator $S \in L(X)$ on some Banach space Y and an injective continuous linear map $A \in L(X, Y)$ for which $SA = AT$. If the injection A has closed range then we shall call T *subadmissible.* This means that an operator T is subadmissible if, up to similarity, it is the restriction to an invariant subspace of an admissible operator.

Theorem 7. *If $T \in L(X)$ on a complex Banach space X is subadmissible then $X_T(F) = E_T(F)$ for all closed $F \subseteq \mathbb{C}$.*

Proof. Assume that $S \in L(Y)$ is an admissible extension of T. Then, by Lemma 6 S has SVEP. Let $A : X \to Y$ be a continuous linear injection with closed range for which $AT = SA$. Let $F \subseteq \mathbb{C}$ be closed. It follows from Proposition 1.2.17 [10] that $AX_T(F) \subseteq Y_S(F)$. Since $(S - \lambda)AE_T(F) = A(T - \lambda)E_T(F) = AE_T(F)$ for all $\lambda \in \mathbb{C} \setminus F$, we have $AE_T(F) \subseteq E_S(F)$, by maximality of $E_S(F)$. If $x \in E_T(F)$ then $Ax \in Y_S(F)$, so there is an analytic function $f : \mathbb{C} \setminus F \to Y_S(F)$ for which $(S - \lambda)f(\lambda) = Ax$ for all $\lambda \in \mathbb{C} \setminus F$. On the other hand, by definition of $E_T(F)$, for every $\lambda \in \mathbb{C} \setminus F$ there is $x_\lambda \in E_T(F)$ such that $(T - \lambda)x_\lambda = x$. Thus we have $Ax_\lambda \in Y_S(F)$. Since $(S - \lambda)(Ax_\lambda - f(\lambda)) = 0$, it follows from Proposition 1.2.16 [10] that $Ax_\lambda - f(\lambda) \in Ker(S - \lambda) \subseteq Y_S(\{\lambda\})$. Hence

$$Ax_\lambda - f(\lambda) \in Y_S(F) \cap Y_S(\{\lambda\}) = Y_S(F \cap \{\lambda\}) = Y_S(\phi) = \{0\},$$

by the SVEP of S. Hence $f(\lambda) = Ax_\lambda$ for all $\lambda \in \mathbb{C} \setminus F$. By the open mapping theorem, the inverse $A^{-1} : AX \to X$ is continuous, and hence the mapping given by $x(\lambda) := x_\lambda = A^{-1}f(\lambda)$ is analytic on $\mathbb{C} \setminus F$ and consequently that $x = (T - \lambda)x(\lambda) = Y_S(F)$.

Corollary 8. *Every subadmissible operator with Dunford's property (C) is admissible.*

Proof. Let $T \in L(X)$ be a subadmissible operator with Dunford's property (C). Assume that $S \in L(Y)$ is an admissible extension of T. Then, by Theorem 7 $X_T(F) = E_T(F)$ for all closed $F \subseteq \mathbb{C}$. Since T has property (C), $E_T(F) = X_T(F)$ is closed, and hence T is admissible.

An operator $T \in L(X)$ has property $(\beta)_\epsilon$ if for every open set $U \subseteq \mathbb{C}$, whenever $f_n : U \to X$ is a sequence of $\mathbb{C}^\infty$ X-valued functions for which $(T - \lambda)f_n(\lambda) \to 0$ uniformly on compact subsets of U, it follows that $f_n \to 0$ in the same topology. It is clear that if T has property $(\beta)_\epsilon$ then T has property (β). It is well known [2] that the property $(\beta)_\epsilon$ characterizes those operators with some generalized scalar extension.

Corollary 9. *Every operator with property $(\beta)_\epsilon$ is admissible.*

Proof. Let $T \in L(X)$ be an operator with property $(\beta)_\epsilon$. Then T has property (β) and hence T has property (C). It follows from Corollary 4.5 [2] that T has a generalized scalar extension. If $S \in L(Y)$ is a generalized scalar extension of T then $E_S(F) = X_S(F)$ for all closed $F \subseteq \mathbb{C}$. By Theorem 7, we have $E_T(F) = X_T(F)$ for all closed $F \subseteq \mathbb{C}$. Since T has property (C), $X_T(F)$ is closed and hence T is admissible.

In [14], M. Putinar shows that all hyponormal operators are similar to a subscalar operator, that is, subscalar is similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces. Thus we have the following corollary.

Corollary 10. *Every hyponormal operator on a Hilbert space is admissible.*

In [2], Eschmeier and Putinar have shown that an operator $T \in L(X)$ has property $(\beta)_\epsilon$ if and only if T is subscalar. Thus we have the following corollary.

Corollary 11. *Every subscalar operator on a Banach space is admissible.*

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Jong-Kwang Yoo,
Department of Liberal Arts and Science, Chodang University, Muan 534-701, Korea
email : jkyoo@chodang.ac.kr

Yong Il Kim,
Department of Internet Software, Honam University, Kwangju 506-714, Korea
e-mail : yikim@honam.ac.kr