J. Appl. Math. & Informatics Vol. 28(2010), No. 1 - 2, pp. 473 - 485
Website: http://www.kcam.biz

# REAL WEIGHT FUNCTIONS FOR THE CIRCLE POLYNOMIALS BY THE REGULARIZATION

J. K. LEE*, C. H. LEE† AND D. H. HAN

ABSTRACT. We consider the differential equation

$$(x^2-1)u_{xx}+2xyu_{xy}+(y^2-1)u_{yy}+gxu_x+gyu_y=\lambda_n u, \qquad (*)$$

where $\lambda_n = n(n+g-1)$. We show that the differential equation (*) has a polynomial set as solutions if $g \neq -1,-3,-5,\cdots$. Also, we construct an orthogonalizing distributional weight for $g<1$ and $g \neq 1,0,-1,\cdots$ by regularizing a one-dimensional integral with a singularity on the endpoint of the interval.

AMS Mathematics Subject Classification : 33C50, 35P99
*Key words and phrases* : Differential equation, circle polynomials, real weight function, regularization

## 1. Introduction

In 1967, Krall and Sheffer[6] posed and partially solved the problem: classify all orthogonal polynomials which satisfy a second order differential equation of the form

$$L[u] := Au_{xx}+2Bu_{xy}+Cu_{yy}+Du_x+Eu_y=\lambda_n u, \quad n=0,1,2,\cdots, \qquad (1.1)$$

where $A(x,y),\cdots,E(x,y)$ are polynomials and $\lambda_n$ is the eigenvalue parameter.

Krall and Sheffer found necessary and sufficient conditions for weak orthogonal polynomials (see Definition 2.1) to satisfy the differential equation (1.1) under the assumption that $\lambda_m \neq \lambda_n$ for $m \neq n$. These conditions were expressed in terms of moments $\{\sigma_{mn}\}_{m.n=0}^{\infty}$ of orthogonalizing measure $\sigma$. The moment $\sigma_{mn}$ is defined by

$$\sigma_{mn} := \langle \sigma, x^m y^n \rangle = \iint_{\mathbb{R}^2} x^m y^n \, d\sigma < \infty.$$

Received June 15, 2009. Accepted September 28, 2009. *Corresponding author.
† The third author was supported financially by Hoseo University Research Fund in 2009

Littlejohn[8] gave simpler moment equations for the differential operator $L[\cdot]$:

$$M_1[\sigma] := (A\sigma)_x + (B\sigma)_y - D\sigma = 0, \quad (1.2)$$

$$M_2[\sigma] := (B\sigma)_x + (C\sigma)_y - E\sigma = 0. \quad (1.3)$$

In this paper, we consider the differential equation

$$(x^2-1)u_{xx} + 2xyu_{xy} + (y^2-1)u_{yy} + gxu_x + gyu_y = n(n+g-1)u. \quad (1.4)$$

The orthogonality for polynomial solutions of (1.4) had been known for the special cases: $g = 2$ [2]. For $g > 1$, we know the explicit expression of the weight function (see [6] and [8]). In [7], we know that (1.4) has orthogonal polynomials as solutions if $g \neq 1, 0, -1, \cdots$.

It is quite complicated to find polynomial solutions to the differential equation (1.4). To do this, we introduce a kind of power series based on the matrix-vector notation. By this method, we show that the differential equation (1.4) has a polynomial set as solutions if and only if $g \neq -1, -3, -5, \ldots$.

Although (1.4) has orthogonal polynomial solutions for $g \neq 1, 0, -1, \ldots$ the representation of their orthogonalizing moment functional is not known except for $g > 1$. Our main results are concerned with the construction of their orthogonalizing distributional weight by using the regularization of an one-dimensional integral with a singularity.

## 2. Preliminaries

We consider the set $\mathcal{P}$ of polynomials in $x$ and $y$. The set of all polynomials degree $\leq n$ is denoted by $\mathcal{P}_n$. By a polynomial set (PS), we mean a sequence $\left\{\phi_{n-j,j}(x,y)\right\}_{n=0,j=0}^{\infty\ \ n}$ of polynomials such that $\deg \phi_{n-j,j} = n$ for each $n \geq 0$ and $\left\{\phi_{n-j,j}(x,y)\right\}_{j=0}^{n}$ is linearly independent modulo $\mathcal{P}_{n-1}$. We denote $(\phi_{n0}, \phi_{n-1,1}, \cdots, \phi_{0n})^T$ by an $(n+1)$-dimensional column vector $\Phi_n(x,y)$ and a PS $\left\{\phi_{n-j,j}(x,y)\right\}_{n=0,j=0}^{\infty\ \ n}$ by $\{\Phi_n(x,y)\}_{n=0}^{\infty}$.

A PS $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ is called to be monic if for each $m, n \geq 0$ $P_{m,n}(x,y)$ has the form

$$P_{m,n}(x,y) = x^m y^n + R_{m,n}(x,y), R_{m,n}(x,y) \in \mathcal{P}_{m+n-1}.$$

Any linear functional on $\mathcal{P}$ will be called a moment functional. For a moment functional $\sigma$ and $\pi \in \mathcal{P}$, we denote $\sigma(\pi)$ by $\langle \sigma, \pi \rangle$. The action of $\sigma$ is extended to a matrix $Q(x,y)$ of polynomials through the formula $\langle \sigma, Q \rangle = (\langle \sigma, Q_{ij} \rangle)$. For any moment functional $\sigma$, we define the partial derivatives $\sigma_x$ and $\sigma_y$ of $\sigma$ by the formula

$$\langle \sigma_x, \phi \rangle = -\langle \sigma, \phi_x \rangle, \qquad \langle \sigma_y, \phi \rangle = -\langle \sigma, \phi_y \rangle$$

and define the multiplication by a polynomial $\psi$ by the formula

$$\langle \psi\sigma, \phi \rangle = \langle \sigma, \psi\phi \rangle.$$

Note that $(\sigma_x)_y = (\sigma_y)_x$, $(\psi\sigma)_x = \psi_x\sigma + \psi\sigma_x$ and $(\psi\sigma)_y = \psi_y\sigma + \psi\sigma_y$.

**Definition 2.1.** [4] A PS $\{\Phi_n(x,y)\}_{n=0}^{\infty}$ is a weak orthogonal polynomial system (WOPS) relative to $\sigma$ if there is a nonzero moment functional $\sigma$ such that

$$\langle \sigma, \phi_{mn}\phi_{kl}\rangle = 0 \quad \text{if } m+n \neq k+l.$$

We call $\{\Phi_n\}_{n=0}^{\infty}$ an orthogonal polynomial system ($OPS$) (respectively, a positive-definite OPS) relative to $\sigma$ if there are nonzero (respectively, positive) constants $K_{mn}$ such that

$$\langle \sigma, \phi_{mn}\phi_{kl}\rangle = K_{mn}\delta_{mk}\delta_{nl}.$$

**Definition 2.2.** [4] A moment functional $\sigma$ is quasi-definite (respectively, positive-definite) if there is an OPS (respectively, a positive-definite OPS) relative to $\sigma$.

From Definition 2.1 and 2.2, we see that a PS $\{\Phi_n(x,y)\}_0^{\infty}$ is an OPS (respectively, a positive-definite OPS) relative to $\sigma$ if and only if $\langle \sigma, \Phi_m\Phi_n^T\rangle = H_n\delta_{mn}$ and $H_n := \langle \sigma, \Phi_n\Phi_n^T\rangle$ is a nonsingular (respectively, a positive-definite) diagonal matrix.

For any PS $\{\Phi_n(x,y)\}_0^{\infty}$, there is a unique moment functional $\sigma$, which is called the canonical moment functional of $\{\Phi_n(x,y)\}_0^{\infty}$, defined by the conditions

$$\langle \sigma, 1\rangle = 1, \quad \langle \sigma, \phi_{mn}\rangle = 0, m+n \geq 1.$$

Note that if a PS $\{\Phi_n(x,y)\}_0^{\infty}$ is a WOPS relative to $\sigma$, then $\sigma$ is a nonzero constant multiple of the canonical moment functional of $\{\Phi_n(x,y)\}_0^{\infty}$.

**Theorem 2.1.** [9] *For any PS $\{\Phi_n(x,y)\}_0^{\infty}$, the following statements are equivalent.*

(i) *$\{\Phi_n(x,y)\}_0^{\infty}$ is a WOPS relative to a quasi-definite moment functional $\sigma$.*

(ii) *For $n \geq 0$ and $i = 1,2$ there are matrices $A_{ni}$ of order $(n+1)\times(n+2)$, $B_{ni}$ of order $(n+1)\times(n+1)$and $C_{ni}$ of order $(n+1)\times n$ such that*
  (a) $x_i\Phi_n = A_{ni}\Phi_{n+1} + B_{ni}\Phi_n + C_{ni}\Phi_{n-1} \quad (x_1 = x, x_2 = y)$,
  (b) $\operatorname{rank} C_n = n+1$, *where* $C_n = [C_{n1}, C_{n2}]$.

Now, we return to a PS $\{\Phi_n(x,y)\}_0^{\infty}$ satisfying differential equation (1.1).

**Definition 2.3.** [6] The differential equation (1.1) is admissible if $\lambda_m \neq \lambda_n$ for $m \neq n$.

We know that the differential equation (1.1) is admissible if and only if the differential equation (1.1) has a unique monic PS as solutions. In section 2, we try to find polynomial solutions when (1.4) is not admissible.

**Theorem 2.2.** [4, 6] *If the differential equation* (1.1) *has a WOPS $\{\Phi_n\}_0^{\infty}$ as solutions, then the canonical moment functional $\sigma$ of $\{\Phi_n(x,y)\}_0^{\infty}$ satisfies*

$$M_1[\sigma] := (A\sigma)_x + (B\sigma)_y - D\sigma = 0, \tag{2.1}$$

$$M_2[\sigma] := (B\sigma)_x + (C\sigma)_y - E\sigma = 0. \tag{2.2}$$

We call $M_1[\sigma] = 0$ and $M_2[\sigma] = 0$ the moment equations for the differential equation (1.1).

**Theorem 2.3.** [4] *For any OPS $\{\Phi_n(x,y)\}_0^\infty$ relative to $\sigma$, the following statements are equivalent.*

(i) $\{\Phi_n(x,y)\}_0^\infty$ *satisfy the differential equation* (1.1).
(ii) $\sigma$ *satisfies the moment equations* $M_1[\sigma] = M_2[\sigma] = 0$.
(iii) $\sigma L[\cdot]$ *is formally symmetric on polynomials in the sense that*

$$\langle L[P]\sigma, Q\rangle = \langle L[Q]\sigma, P\rangle \quad \text{for all } P, Q \in \mathcal{P}.$$

## 3. Monic polynomial solutions

In this section, we suggest a systematic method of simultaneously finding a monic PS satisfying the differential equation of the form

$$\begin{aligned} L[u] &= (ax^2 + d_1x + e_1y + f_1)u_{xx} + (2axy + d_2x + e_2y + f_2)u_{xy} \\ &\quad + (ay^2 + d_3x + e_3y + f_3)u_{yy} + (gx + h_1)u_x + (gy + h_2)u_y \\ &= \lambda_n u, \end{aligned} \tag{3.1}$$

where $\lambda_n = an(n-1) + gn$.

Let $\mathcal{H}_n$ be a vector space of homogeneous polynomials of degree $n$ and denote a monic basis $\{x^n, x^{n-1}y, \cdots, y^n\}$ of $\mathcal{H}_n$ by the $(n+1)$-dimensional column vector

$$\mathbf{x}^n = (x^n, x^{n-1}y, \cdots, y^n)^T.$$

Then the multiplication by independent variables as a linear operator from $\mathcal{H}_n$ to $\mathcal{H}_{n+1}$ can be written as

$$x\mathbf{x}^n = M_n^1\mathbf{x}^{n+1}, \quad y\mathbf{x}^n = M_n^2\mathbf{x}^{n+1},$$

and the partial differentiations with respect to $x$ and $y$ (as a linear operator from $\mathcal{H}_n$ to $\mathcal{H}_{n-1}$) has the matrix representations

$$\partial_x\mathbf{x}^n = D_n^1\mathbf{x}^{n-1}, \quad \partial_y\mathbf{x}^n = D_n^2\mathbf{x}^{n-1} \quad (D_0^1 = D_0^2 = 0),$$

where

$$\begin{aligned} M_n^1 &= [I_{n+1}|0], & M_n^2 &= [0|I_{n+1}], \\ D_n^1 &= \Big[\mathrm{Diag}(n, n-1, \cdots, 1)|0\Big]^T, & D_n^2 &= \Big[0|\mathrm{Diag}(1, 2, \cdots, n)\Big]^T \end{aligned} \tag{3.2}$$

and $I_n$ is the $n \times n$ identity matrix.

**Lemma 3.1.** *Let $M_n^i$ and $D_n^i$ $(i = 1,2)$ be matrices defined by* (3.2). *Then we have the followings: for each $n \geq 0$,*

(i) $D_n^1M_{n-1}^1 + D_n^2M_{n-1}^2 = nI_{n+1}$;
(ii) $M_n^1M_{n+1}^2 = M_n^2M_{n+1}^1$;
(iii) $D_n^1D_{n-1}^2 = D_n^2D_{n-1}^1$;
(iv) $D_n^1D_{n-1}^1M_{n-2}^1M_{n-1}^1 + 2D_n^1D_{n-1}^2M_{n-2}^1M_{n-1}^2 + D_n^2D_{n-1}^2M_{n-2}^2M_{n-1}^2 = n(n-1)I_{n+1}$.

*Also, we have*

$$L[\mathbf{x}^k] = \lambda_k \mathbf{x}^k + B_k \mathbf{x}^{k-1} + C_k \mathbf{x}^{k-2}, \tag{3.3}$$

*where $B_k$ and $C_k$ are given by*

$$\begin{aligned} B_k &= D_k^1 D_{k-1}^1 (d_1 M_{k-2}^1 + e_1 M_{k-2}^2) + D_k^1 D_{k-1}^2 (d_2 M_{k-2}^1 + e_2 M_{k-2}^2) \\ &\quad + D_k^2 D_{k-1}^2 (d_3 M_{k-2}^1 + e_3 M_{k-2}^2) + h_1 D_n^1 + h_2 D_n^2, \\ C_k &= f_1 D_k^1 D_{k-1}^1 + f_2 D_k^1 D_{k-1}^2 + f_3 D_k^2 D_{k-1}^2. \end{aligned}$$

*Proof.* (i) follows easily from the fact that

$$\begin{aligned} \left(D_n^1 M_{n-1}^1 + D_n^2 M_{n-1}^2\right) \mathbf{x}^n &= D_n^1 M_{n-1}^1 \mathbf{x}^n + D_n^2 M_{n-1}^2 \mathbf{x}^n \\ &= D_n^1 x \mathbf{x}^{n-1} + D_n^2 y \mathbf{x}^{n-1} = x D_n^1 \mathbf{x}^{n-1} + y D_n^2 \mathbf{x}^{n-1} \\ &= x \partial_x \mathbf{x}^n + y \partial_y \mathbf{x}^n = n \mathbf{x}^n. \end{aligned}$$

(ii) follows from the fact that $x(y\mathbf{x}^n) = y(x\mathbf{x}^n)$ and

$$\begin{aligned} x(y\mathbf{x}^n) &= x M_n^2 \mathbf{x}^{n+1} = M_n^2 x \mathbf{x}^{n+1} = M_n^2 M_{n+1}^1 \mathbf{x}^{n+2}, \\ y(x\mathbf{x}^n) &= y M_n^1 \mathbf{x}^{n+1} = M_n^1 y \mathbf{x}^{n+1} = M_n^1 M_{n+1}^2 \mathbf{x}^{n+1}. \end{aligned}$$

(iii) and (iv) are similarly proved. We leave the proof of (3.3) to the reader since it can be shown by the careful use of (i), (ii), (iii) and (iv).

**3.1. A kind of power series solution.** Now we introduce a method based on the vector notation. It is similar to a power series method in ordinary differential equations. Using this method we find simultaneously a monic PS satisfying the differential equation (3.1).

**Theorem 3.2** *Let the differential equation* (3.1) *have a monic PS* $\{\mathbb{P}_n\}_{n=0}^{\infty}$ *given by*

$$\mathbb{P}_n = \sum_{j=0}^{n} A_j^n \mathbf{x}^j ,$$

*where $A_j^n$ is an $(n+1) \times (j+1)$ matrix for $0 \le j \le n$ and $A_n^n = I_{n+1}$. Then $A_j^n$ $(0 \le j \le n)$ satisfies a recursion formula*

$$(\lambda_n - \lambda_j) A_j^n = A_{j+1}^n B_{j+1} + A_{j+2}^n C_{j+2} \quad (A_{n+1}^n = 0,\, 0 \le j \le n-1). \tag{3.4}$$

*Proof.* Applying Lemma 3.1, we have the following

$$\begin{aligned} L[\mathbb{P}_n] &= \lambda_n \mathbf{x}^n + B_n \mathbf{x}^{n-1} + C_n \mathbf{x}^{n-2} + \sum_{j=0}^{n-1} A_j^n (\lambda_j \mathbf{x}^j + B_j \mathbf{x}^{j-1} + C_j \mathbf{x}^{j-2}) \\ &= \lambda_n \mathbf{x}^n + (\lambda_{n-1} A_{n-1}^n + B_n) \mathbf{x}^{n-1} + (\lambda_{n-2} A_{n-2}^n + A_{n-1}^n B_{n-1} + C_n) \mathbf{x}^{n-2} \end{aligned}$$

$$+\sum_{j=0}^{n-3}(\lambda_j A_j^n + A_{j+1}^n B_{j+1} + A_{j+2}^n C_{j+2})\mathbf{x}^j$$

$$= \lambda_n \left(\mathbf{x}^n + \sum_{j=0}^{n-1} A_j^n \mathbf{x}^j\right).$$

By comparing the coefficients of $\mathbf{x}^j$ $(0 \le j \le n-1)$, we obtain the following recurrence relations for $A_j^n$ :

$$(\lambda_n - \lambda_{n-1})A_{n-1}^n = B_n,$$
$$(\lambda_n - \lambda_{n-2})A_{n-2}^n = A_{n-1}^n B_{n-1} + C_n,$$
$$(\lambda_n - \lambda_j)A_j^n = A_{j+1}^n B_{j+1} + A_{j+2}^n C_{j+2} \quad (0 \le j \le n-3),$$

which is (3.4) if we put $A_{n+1}^n = 0$ and $A_n^n = I_{n+1}$.

**3.2. Monic circle polynomials.** We apply Theorem 3.2 to a specific differential equation

$$(x^2-1)u_{xx} + 2xyu_{xy} + (y^2-1)u_{yy} + gxu_x + gyu_y = \lambda_n u, \tag{3.5}$$

where $\lambda_n = n(n+g-1)$ (The polynomial solutions to the differential equation (3.5) are called the circle polynomials). Then we have

$$(\lambda_n - \lambda_j)A_j^n = A_{j+2}^n C_{j+2} = -A_{j+2}^n \left(D_{j+2}^1 D_{j+1}^1 + D_{j+2}^2 D_{j+1}^2\right)$$

since $B_j = 0$ . Note that

$$C_j \mathbf{x}^{j-2} = -\left(D_j^1 D_{j-1}^1 + D_j^2 D_{j-1}^2\right)\mathbf{x}^{j-2} = (-\partial_x^2 - \partial_y^2)\mathbf{x}^j. \tag{3.6}$$

**Theorem 3.3.** *Let $N$ be a nonnegative integer. Then*

(i) *If the differential equation* (3.5) *is admissible, then it has a unique monic PS as solutions.*
(ii) *If $g = -2N$, then the differential equation* (3.5) *has infinitely many monic PS as solutions.*
(iii) *If $g = -2N-1$, then the differential equation* (3.5) *can not have any PS as solutions.*

*Proof.* If $\mathbb{P}_n = \sum_{j=0}^{n} A_j^n \mathbf{x}^j$ satisfies the differential equation (3.5), then by Theorem 3.2, $A_j^n$ $(0 \le j \le n-1)$ satisfies the following recursion relations

$$(\lambda_n - \lambda_{n-1})A_{n-1}^n = 0, \tag{3.7}$$

$$(\lambda_n - \lambda_j)A_j^n = A_{j+2}^n C_{j+2} \;\; (0 \le j \le n-2). \tag{3.8}$$

Observe that

(a) $\mathbb{P}_n$ becomes a solution to the differential equation (3.5) if we can determine $A_j^n$ $(0 \le j \le n-1)$ so that (3.7) and (3.8) are satisfied ;

(b) $A_j^n$ can be determined if $j \neq -g - n + 1$ since

$$\lambda_n - \lambda_j = (n-j)(n+j+g-1) = 0 \iff j = -n - g + 1.$$

(i) It is obvious since $A_j^n$ is uniquely determined since $\lambda_n - \lambda_j \neq 0$ for $0 \leq j \leq n-1$.

(ii) Let $g = -2N$. Then for $0 \leq j \leq n-1$, we consider four cases:

**Case 1.** $n < N+1 : \lambda_n - \lambda_j = (n-j)(n+j-2N-1) < 0$ since $0 \leq j \leq n-1 < N$ and $n < N+1$. Thus all $A_j^n$ are uniquely determined from (3.7) and (3.8).

**Case 2.** $n = N+1 : \lambda_{N+1} - \lambda_j = 0$ only for $j = N$. Thus all $A_j^{N+1}$ are determined once we fix $A_N^{N+1}$.

**Case 3.** $N+1 < n \leq 2N+1$ : In this case, $A_{n-1}^n = 0$ since $\lambda_n - \lambda_{n-1} = 2n - 2N - 2 > 0$. If $n$ is even (respectively, odd), then $\lambda_n - \lambda_j \neq 0$ for even (respectively, odd) $j$. Hence we can determine uniquely

$$A_{n-2}^n, A_{n-4}^n, \cdots, A_0^n \text{ (respectively, } A_1^n).$$

Using the fact that $A_{n-1}^n = 0$ (since $\lambda_n - \lambda_{n-1} = 2n - 2N - 2 > 0$) and $\lambda_n - \lambda_j \neq 0$ except for $j = 2N + 1 - n$, we can see that

$$A_{n-1}^n = A_{n-3}^n = \cdots = A_{2N+3-n}^n = 0.$$

Once we fix $A_{2N+1-n}^n$, then we can determine successively

$$A_{2N-1-n}^n, A_{2N-3-n}^n, \cdots, A_1^n \text{ (respectively, } A_0^n).$$

**Case 4.** $n > 2N+1$ : All $A_j^n$ are uniquely determined since $\lambda_n - \lambda_j > 0$ for $0 \leq j \leq n-1$.

Therefore (ii) is proved.

(iii) Let $g = -2N - 1$. Then

$$\lambda_n - \lambda_j = (n-j)(n+j-2N-2) = 0 \iff j = 2N + 2 - n,$$

which means that

$$A_{n-1}^n = A_{n-3}^n = \cdots = A_1^n = 0 \text{ if } n \text{ is even;}$$
$$A_{n-1}^n = A_{n-3}^n = \cdots = A_0^n = 0 \text{ if } n \text{ is odd.}$$

**Case 5.** $n \leq N+1$ : We have $\lambda_n - \lambda_j = (n-j)(n+j-2N-2) \leq -1$ since $j \leq n-1 \leq N$ and $n \leq N+1$. Then all $A_j^n$'s are uniquely determined.

**Case 6.** $N+2 \leq n \leq 2N+2$ : Let $n$ be even and $n = 2m$. Then $\lambda_{2m} - \lambda_j \neq 0$ except for $j = 2(N+1-m)$ and we have from (3.8)

$$A_{2m-2}^{2m} = \frac{C_{2m}}{\lambda_{2m} - \lambda_{2m-2}},$$

$$A^{2m}_{2m-4} = \frac{C_{2m}C_{2m-2}}{(\lambda_{2m}-\lambda_{2m-2})(\lambda_{2m}-\lambda_{2m-4})},$$
$$\vdots$$
$$A^{2m}_{2N+4-2m} = \frac{C_{2m}C_{2m-2}\cdots C_{2N+6-2m}}{(\lambda_{2m}-\lambda_{2m-2})(\lambda_{2m}-\lambda_{2m-4})\cdots(\lambda_{2m}-\lambda_{2N+4-2m})},$$
$$0 = A^{2m}_{2N+4-2m}C_{2N+4-2m}.$$

Hence

$$C_{2m}C_{2m-2}\cdots C_{2N+6-2m}C_{2N+4-2m} = 0.$$

However, this can not happen because by (3.6) we can see that

$$C_{2m}C_{2m-2}\cdots C_{2N+6-2m}C_{2N+4-2m}\mathbf{x}^{2N+2-2m} = (-\partial_x^2-\partial_y^2)^{2m-N}\mathbf{x}^{2m} \neq 0.$$

Note that the order of a partial differential operator $(-\partial_x^2-\partial_y^2)^{2m-N}$ is at most $2m$ and

$$(-\partial_x^2-\partial_y^2)^{2m-N}x^{2m} \neq 0.$$

Thus (3.5) can not have a monic PS solution of degree $n$ in this case.

Let $n$ be odd and $n = 2m+1$. Since $\lambda_n - \lambda_j = 0$ for only $j = 2N-2m+1$, we have the similar result

$$C_{2m+1}C_{2m-1}\cdots C_{2N-2m+5}C_{2N-2m+3} = 0.$$

This can not happen since

$$C_{2m+1}C_{2m-1}\cdots C_{2N-2m+3}\mathbf{x}^{2N-2m+1} = (-\partial_x^2-\partial_y^2)^{2m-N}\mathbf{x}^{2m+1} \neq 0.$$

**Case 7.** $n \geq 2N+3$ : We have $\lambda_n - \lambda_j = (n-j)(n+j-2N-2) \geq 1$ since $n-j \geq 1$ and $n+j-2N-2 \geq n-2N-2 \geq 1$. Then all $A^n_j$ are uniquely determined.

Thus (iii) is proved. □

## 4. Real weight for the circle polynomials

In this section, we construct a real weight function for the circle polynomials. Let $\sigma$ be the canonical moment functional of a PS $\{\Phi_n(x,y)\}_{n=0}^{\infty}$ satisfying the differential equation

$$(x^2-1)u_{xx} + 2xyu_{xy} + (y^2-1)u_{yy} + gxu_x + gyu_y = n(n+g-1)u. \quad (4.1)$$

**Theorem 4.1.** [4] *Let $\{\Phi_n(x,y)\}_{n=0}^{\infty}$ be a WOPS relative to a moment functional $\sigma$. Then the following statements are equivalent.*

(i) *$\{\Phi_n(x,y)\}_{n=0}^{\infty}$ satisfy the differential equation (4.1).*

(ii) *$\sigma$ satisfies the moment equations*

$$M_1[\sigma] = ((x^2-1)\sigma)_x + (xy\sigma)_y - gx\sigma = 0, \quad (4.2)$$
$$M_2[\sigma] = (xy\sigma)_x + ((y^2-1)\sigma)_y - gy\sigma = 0 \quad (4.3)$$

To establish the orthogonality of the circle polynomials, we should prove that the canonical moment functional $\sigma$ of the circle polynomials satisfies the moment equations (4.2) and (4.3) and that $\sigma$ is quasi-definite.

We now recall a classical result on the multi-dimensional moment problems [1], which states that for any sequence $\{\mu_\alpha\}_{|\alpha|=0}^{\infty}$ of real numbers, there exists a signed measure $\mu$ on $\mathbb{R}^k$ such that

$$\mu_\alpha = \int_{\mathbb{R}^k} \mathbf{x}^\alpha \, d\mu(\mathbf{x}) \quad (\alpha \in \mathbb{N}_0^k;\ \mathbf{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k), \tag{4.4}$$

where $\mathbb{N}_0^k := \Big\{(\alpha_1, \alpha_2, \cdots, \alpha_k) | \alpha_i \in \mathbb{N}_0,\ 1 \le i \le k\Big\}$, $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ and $|\alpha| = \sum_{i=1}^{k} \alpha_i$. From this fact, the orthogonality for any OPS $\{\Phi_n(\mathbf{x})\}$ in $k$ variables can be rewritten as the following integral form

$$\int_{\mathbb{R}^k} \phi_\alpha(\mathbf{x}) \phi_\beta(\mathbf{x}) \, d\mu(\mathbf{x}) = K_\alpha \delta_{\alpha,\beta}, \quad \alpha, \beta \in \mathbb{N}_0^k$$

where $K_\alpha \neq 0$ is a nonzero constant and $\mu$ is a signed measure on $\mathbb{R}^k$. The measure $\mu$ will be called a *real weight function* of a PS $\{\Phi_n(\mathbf{x})\}$.

As regarding to the circle polynomials, it is well known (see [5, 6, 8]) that if $g > 1$, the differential equation (4.1) has a positive-definite OPS as solutions and, by solving the moment equation (4.2) and (4.3) in the classical or distributional sense, we can find a real (distributional) weight

$$w(x, y) = (1 - x^2 - y^2)^{\frac{g-3}{2}} \quad (x^2 + y^2 < 1)$$

whose action on a polynomial $\varphi(x, y)$ is given by the integral

$$\Big\langle (1 - x^2 - y^2)^{\frac{g-3}{2}}, \varphi(x, y) \Big\rangle := \iint_D \varphi(x, y)(1 - x^2 - y^2)^{\frac{g-3}{2}} \, dxdy. \tag{4.5}$$

Although there is a quasi-definite (not a positive-definite) OPS satisfying the differential equation (4.1) for the case $g < 1$ and $g \neq 0, -1, \cdots$ and the existence of orthogonalizing weight for them is guaranteed, we still do not know the integral representation of the orthogonalizing moment functional $\sigma$ satisfying the moment equations (4.2) and (4.3).

In the following, we construct an orthogonalizing real (distributional) weight by regularizing the integral (4.5).

Let $\lambda = \dfrac{g-3}{2}$ and $-n - 1 < \operatorname{Re} \lambda < -n$. The regularization of the integral

$$\langle (1 - x)_{[0,1]}^\lambda, \varphi \rangle := \int_0^1 (1 - x)^\lambda \varphi(x) \, dx$$

is defined by the formula (see [3])

$$\int_0^1 (1-x)^\lambda \varphi(x)\,dx = \int_0^1 (1-x)^\lambda \left[\varphi(x) - \sum_{k=0}^{n-1} \frac{(-1)^k \varphi^{(k)}(1)}{k!}(1-x)^k\right] dx + \sum_{k=0}^{n-1} \frac{(-1)^k \varphi^{(k)}(1)}{k!(k+\lambda+1)}.$$

It is not hard to see that

$$\begin{aligned} &(1-x)(1-x)^\lambda_{[0,1]} = (1-x)^{\lambda+1}_{[0,1]}, \\ &\frac{d}{dx}(1-x)^\lambda_{[0,1]} = -\lambda(1-x)^{\lambda-1}_{[0,1]} + \delta(x), \\ &\frac{d}{dx}\left(\psi(x)(1-x)^\lambda_{[0,1]}\right) = \psi'(x)(1-x)^\lambda_{[0,1]} + \psi(x)\frac{d}{dx}(1-x)^\lambda_{[0,1]}. \end{aligned}$$

Since (4.5) can be written in the polar coordinates as the following form

$$\begin{aligned} \left\langle (1-x^2-y^2)^\lambda, \varphi(x,y)\right\rangle &= \int_0^1 \int_{-\pi}^{\pi} r(1-r^2)^\lambda \varphi(r\cos\theta, r\sin\theta)\,d\theta dr \\ &= \int_0^1 (1-r)^\lambda \Phi(r)\,dr, \end{aligned}$$

it is natural to define the regularization of the integral (4.5) for $-n-1 < \operatorname{Re}\lambda < -n$ by

$$\langle (1-x^2-y^2)^\lambda_+, \varphi(x,y)\rangle = \langle (1-r)^\lambda_{[0,1]}, \Phi(r)\rangle, \tag{4.6}$$

where

$$\Phi(r) = r(1+r)^\lambda \int_{-\pi}^{\pi} \varphi(r\cos\theta, r\sin\theta)\,d\theta.$$

**Lemma 4.2.** *Let $-n-1 < \operatorname{Re}\lambda < -n$ and let the regularization of $w(x,y) = (1-x^2-y^2)^\lambda$ defined by (4.6). Then we have*

$$\frac{\partial}{\partial x}(1-x^2-y^2)^\lambda_+ = -2\lambda x(1-x^2-y^2)^{\lambda-1}_+, \tag{4.7}$$

$$\frac{\partial}{\partial y}(1-x^2-y^2)^\lambda_+ = -2\lambda y(1-x^2-y^2)^{\lambda-1}_+, \tag{4.8}$$

$$(1-x^2-y^2)(1-x^2-y^2)^\lambda_+ = (1-x^2-y^2)^{\lambda+1}_+. \tag{4.9}$$

*Proof.* We prove (4.7) only since (4.8) is proved similarly. In polar coordinates, we have the following

$$
\begin{aligned}
&-\left\langle \frac{\partial}{\partial x}(1-x^2-y^2)_+^\lambda, \varphi(x,y)\right\rangle = \langle (1-x^2-y^2)_+^\lambda, \varphi_x(x,y)\rangle\\
&= \left\langle (1-r)_{[0,1]}^\lambda, r(1+r)^\lambda \int_{-\pi}^{\pi}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\varphi(r\cos\theta, r\sin\theta)\,d\theta\right\rangle\\
&= \left\langle (1-r)_{[0,1]}^\lambda, (1+r)^\lambda \frac{\partial}{\partial r}\int_{-\pi}^{\pi} r\cos\theta\varphi(r\cos\theta, r\sin\theta)\,d\theta\right\rangle.
\end{aligned}
$$

If we put $\Psi(r) = r\int_{-\pi}^{\pi}\varphi(r\cos\theta, r\sin\theta)\cos\theta d\theta$, then $\Psi(0) = 0$ and we can see the following

$$
\begin{aligned}
&\left\langle (1-r)_{[0,1]}^\lambda, (1+r)^\lambda\Psi'(r)\right\rangle\\
&\quad= \left\langle (1-r)_{[0,1]}^\lambda, ((1+r)^\lambda\Psi(r))' - \lambda(1+r)^{\lambda-1}\Psi(r)\right\rangle\\
&\quad= -\left\langle \frac{d}{dr}(1-r)_{[0,1]}^\lambda, (1+r)^\lambda\Psi(r)\right\rangle - \lambda\left\langle (1-r)_{[0,1]}^\lambda, (1+r)^{\lambda-1}\Psi(r)\right\rangle\\
&\quad= -\left\langle -\lambda(1-r)_{[0,1]}^{\lambda-1} + \delta(r), (1+r)^\lambda\Psi(r)\right\rangle - \lambda\left\langle (1-r)_{[0,1]}^\lambda, (1+r)^{\lambda-1}\Psi(r)\right\rangle\\
&\quad= 2\lambda\left\langle (1-x^2-y^2)_+^{\lambda-1}, x\varphi(x,y)\right\rangle,
\end{aligned}
$$

which implies (4.7).

Next, if we put $\Phi(r) = \int_{-\pi}^{\pi}\varphi(r\cos\theta, r\sin\theta)\,d\theta$, then we have the following

$$
\begin{aligned}
&\langle (1-x^2-y^2)(1-x^2-y^2)_+^\lambda, \varphi(x,y)\rangle\\
&\qquad= \left\langle (1-x^2-y^2)_+^\lambda, (1-x^2-y^2)\varphi(x,y)\right\rangle\\
&\qquad= \left\langle (1-r)_{[0,1]}^\lambda, r(1+r)^\lambda(1-r^2)\Phi(r)\right\rangle\\
&\qquad= \left\langle (1-r)_{[0,1]}^{\lambda+1}, r(1+r)^{\lambda+1}\Phi(r)\right\rangle,
\end{aligned}
$$

which implies (4.9).

Now we are ready to state our main theorem:

**Theorem 4.3.** *Let $g < 1$ and $g \neq 1, 0, -1, \cdots$. Then $\tau_g := (1-x^2-y^2)_+^{\frac{g-3}{2}}$ is a distributional weight for the circle polynomials satisfying the differential equation* (4.1).

To prove Theorem 4.3, we need the following:

**Lemma 4.4.** *If $g \neq 1$, then the moments equations* (4.2) *and* (4.3) *are equivalent to the modified moment equations*

$$\tilde{M}_1[\sigma] = ((1-x^2-y^2)\sigma)_x - (1-g)x\sigma = 0, \tag{4.10}$$

$$\tilde{M}_2[\sigma] = ((1-x^2-y^2)\sigma)_y - (1-g)y\sigma = 0. \tag{4.11}$$

*Proof.* In the first, we will show that

$$x\sigma_y = y\sigma_x. \tag{4.12}$$

Differentiating (4.11) with respect to $x$, we see that

$$(1-g)y\sigma_x = (((1-x^2-y^2)\sigma)_x)_y = ((1-g)x\sigma)_y = (1-g)x\sigma_y$$

which implies (4.12) since $g \neq 1$.

Using (4.12), we can obtain the following

$$((x^2-1)\sigma)_x + (xy\sigma)_y = ((x^2+y^2-1)\sigma - y^2\sigma)_x + x\sigma + xy\sigma_y = gx\sigma,$$

which is (4.2). Similarly, we can derive (4.3). Thus we have showed that (4.10) and (4.11) imply (4.2) and (4.3).

Conversely, (4.2) and (4.3) imply (4.10) and (4.11). We leave the proof to the reader since it is a simple calculation.

*Proof of Theorem 4.3.* We know that the moment equations (4.2) and (4.3) has a unique solution since the differential equation (4.1) is admissible. By Lemma 4.4, it is sufficient to prove that $\tau_g := (1-x^2-y^2)_+^{\frac{g-3}{2}}$ satisfies (4.10) and (4.11).

Let $-n-1 < \mathrm{Re}\,\dfrac{g-3}{2} < -n$. Since by Lemma 4.2 we can see that

$$\begin{aligned}\Big((1-x^2-y^2)\tau_g\Big)_x &= \frac{\partial}{\partial x}(1-x^2-y^2)_+^{\frac{g-1}{2}} = (1-g)x\tau_g,\\ \Big((1-x^2-y^2)\tau_g\Big)_y &= \frac{\partial}{\partial y}(1-x^2-y^2)_+^{\frac{g-1}{2}} = (1-g)y\tau_g.\end{aligned}$$

Thus our assertion is proved.

**Remark 4.1.** If $g = -2N\,(N \geq 0)$, then the differential equation (4.1) has infinitely many PS as solutions. On the other hand, since the distribution $(1-x^2-y^2)_+^{-\frac{2N+3}{2}}$ is one of the solutions to the moment equations (4.2) and (4.2), we expect that there exists at least one WOPS satisfying the differential equation (4.1).

For example, the monic polynomial solutions $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ of the differential equation (4.1) with $g=0$ are uniquely determined except for the case that $n=1$. If we take

$$P_{10} = x+\alpha, \quad P_{01} = y+\beta \quad (\alpha, \beta \text{ arbitrary real number})$$

and require that $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ is a WOPS relative to the distribution $(1-x^2-y^2)_+^{-\frac{3}{2}}$, we can see that $\alpha = \beta = 0$.

## REFERENCES

1. C. Berg, J. P. R. Christensen and C. U. Jensen, *A remark on the multidimensional moment problem,* Math. Anal. **243** (1979), 163-169.
2. A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions,* Vol. II, MacGraw Hill, New York, 1953.
3. I. M. Gel'fand and G. E. Shilov, *Generalized functions Vol. 1,* Academic Press, N. Y. and London, 1964.
4. Y. J. Kim, K. H. Kwon and J. K. Lee, *Orthogonal polynomials in two variables and differential equations*, J. Comp. Appl. Math. **82** (1997), 239- 260.
5. T. H. Koornwinder, *Two-variable analogues of the classical orthogonal polynomials*, Theory and Applications of Special Functions, R. Askey Ed. , Academic Press, New York, 1995.
6. H. L. Krall and I. M. Sheffer, *Orthogonal polynomials in two variables*, Ann. Mat. Pura Appl. seri **4** (1967), 325-376.
7. K. H. Kwon, J. K. Lee and L. L. Littlejohn, *Orthogonal polynomial eigenfunctions of second order differential equations*, Trans. Amer. Math. Soc. 353 (2001), 3629-3647.
8. L. L. Littlejohn, *Orthogonal polynomial solutions to ordinary and differential equations*, Lecture Notes in Math. **1329** (1988), 98-124.
9. Y. Xu, *On multivariate orthogonal polynomials*, SIAM J. Math. Anal. **24** no. 3 (1993), 783-794.

**J. K. Lee**,
Department of Mathematics, SunMoon University, Kalsan-Ri TangJeong-Myun Asan-Si ChoongNam, Korea, 336-840
e-mail:jklee@@sunmoon.ac.kr

**C. H. Lee**,
Department of Mathematics, Hoseo University, Sechul-Ri Baebang-Eup Asan-Si ChoongNam, Korea, 336-795
e-mail:chlee@@hoseo.edu

**D. H. Han**,
Department of Mathematics, SunMoon University, Kalsan-Ri TangJeong-Myun Asan-Si ChoongNam, Korea, 336-840
e-mail:dhhan@@sunmoon.ac.kr