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# FINITE TIME BLOW-UP SOLUTION FOR A CLASS OF NONLINEAR HYPERBOLIC EQUATIONS

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ABSTRACT. In this work, We consider an initial boundary value problem for a class of nonlinear hyperbolic equations. We establish a blow-up result for certain solutions with a dissipative term.

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## 1. Introduction

Let $\Omega$ be a bounded domain of $R^n$ with a sufficiently smooth boundary $\partial\Omega$. We consider the following initial boundary value problem

$$u_{tt} - \Delta u_t - div(|\nabla u|^{\alpha-2}\nabla u) = h(u), \quad x \in \Omega, \quad t \geq 0, \tag{1}$$

$$u|_{\partial\Omega} = 0, \quad x \in \partial\Omega, \quad t \geq 0, \tag{2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{3}$$

where $\alpha \geq 2$, and $h$ is a continuous function.

Research of global nonexistence and finite time blow-up solutions for nonlinear evolution equations has attracted a great deal of people. The obtained results show that global existence and nonexistence depends on the degree of nonlinearity of source term, the dimension $n$ and the size of initial data's. There are many results that have been considered by many authors; see [1-10]. Levine has a survey article [5] with many relevant references. In [2] Levine studied the initial value problem for the following abstract wave equation with dissipation

$$Pu_{tt} + Au_t + Bu = G(u)$$

in a Hilbert space, where $P, A$ and $B$ are positive linear operators defined on some dense subspace of the Hilbert space and $G$ is a gradient operator. He introduced convexity method and showed, if the energy is initially negative then the

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solution can not be global. Later Kalantarov and Ladyzhenskaya [6] improved this method to accommodate more general cases.

In our study, we establish a blow-up result for solutions with negative initial energy. The proof of our technique is similar to the one in [7]. Moreover we find the lifespan for the solution. Let us begin by the following lemma in [6, Lemma 2.1] that we will use in the proof of our main result.

**Lemma 1**. *If a function $\psi(t) \in C^2$ which is positive and satisfy the inequality*

$$\psi''(t)\psi(t) - (1+\gamma)(\psi'(t))^2 \geq -2C_1\psi(t)\psi'(t) - C_2(\psi(t))^2 \tag{4}$$

*for some real numbers $\gamma > 0$, $C_1, C_2 \geq 0$ then the following hold:*
*i) If*

$$\psi(0) > 0, \ \ \psi'(0) > -\gamma_1\gamma^{-1}\psi(0), \ \ C_1 + C_2 > 0 \tag{5}$$

*where*

$$\gamma_1 = -C_1 + \sqrt{C_1^2 + \gamma C_2}, \quad \gamma_2 = -C_1 - \sqrt{C_1^2 + \gamma C_2}. \tag{6}$$

*Then for the real number,*

$$t_2 = \frac{1}{2\sqrt{C_1^2 + C_2^2}} ln \frac{\gamma_1\psi(0) + \gamma\psi'(0)}{\gamma_2\psi(0) + \gamma\psi'(0)} \tag{7}$$

*there exist a positive real number $t_1 < t_2$ such that as $t \to t_1$, $\psi(t) \to +\infty$.*
*ii) If $\psi(0) > 0$, $\psi'(0) > 0$ and $C_1 = C_2 = 0$ then for all real number*

$$t_2 = \frac{\psi(0)}{\gamma\psi'(0)}, \tag{8}$$

*there exists a positive real number $t_1 \leq t_2$ such that as $t \to t_1$, $\psi(t) \to +\infty$.*

## 2. Finite time blow-up

The source term $h(u)$ in (1) with primitive $H(u) = \int_0^u h(\xi)d\xi$ satisfy

$$uh(u) \geq 2(2\rho + 1)H(u) \tag{9}$$

for some real numbers $\rho \geq 0$, and for all $u \in \mathbb{R}$. We define the energy functional associated with a solution $u(x,t)$ of (1)-(3) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha - \int_\Omega H(u)dx. \tag{10}$$

Furthermore, by means of the divergence theorem, it is easily seen that the following holds.

**Lemma 2**. *For the problem (1)-(3), the energy functional $E(t)$ is a non-increasing function on $(0,t)$ and*

$$E(t) = E(0) - \int_0^t \|\nabla u_\tau\|_2^2 d\tau \leq E(0). \tag{11}$$

*Proof.* Multiplying the equation (1) by $u_t$ and integrating over the domain $\Omega$. Then by divergence theorem and using (2)-(3), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2 dx + \int_\Omega |\nabla u_t|^2 dx + \frac{1}{\alpha}\frac{d}{dt}\int_\Omega |\nabla u|^\alpha dx = \int_\Omega (\frac{d}{dt}\int_0^u h(\xi)d\xi)dx$$

and by using the definition of primitive function $H(u)$ we find

$$\frac{d}{dt}(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha - \int_\Omega H(u)dx) = -\|\nabla u_t\|_2^2. \quad (12)$$

Utilizing (10) in (12) gives $E^{'}(t) = -\|\nabla u_t\|_2^2$ and the integration on $[0,t)$ carry out the inequality (11).

The main result reads as follows.

**Theorem 1.** *Let $u$ be the solution of problem (1)-(3). Assume that the following conditions are valid:*

$$u_0(x) \in W_0^{1,\alpha}(\Omega), \quad u_1(x) \in L_2(\Omega)$$

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{\alpha}\|\nabla u_0\|_2^2 - \int_\Omega H(u_0)dx < 0 \quad (13)$$

*and*

$$2\int_\Omega u_0 u_1 dx + \|\nabla u_0\|_2^2 \geq 2\rho^{-1}(1+\rho)(\|u_0\|_2^2 + \|\nabla u_0\|_2^2) \quad (14)$$

*where $\rho > 0$. Then the solution blows up in finite time and the lifespan $T$ of the solution can be bounded above as*

$$T \leq \frac{\|u_0\|_2^2 + \|\nabla u_0\|_2^2}{\rho(2\int_\Omega u_0 u_1 dx + \|\nabla u_0\|_2^2)}.$$

*Proof.* To prove, it suffices to show the function

$$F(t) = \|u\|_2^2 + \int_0^t \|\nabla u\|_2^2 d\tau + \|\nabla u_0\|_2^2 \quad (15)$$

satisfies the hypotheses of the Lemma 1.To get this purpose, let us compute $F^{'}(t)$ and $F^{''}(t)$. We observe that

$$\|\nabla u\|_2^2 = \int_0^t \frac{d}{d\tau}\|\nabla u\|_2^2 d\tau + \|\nabla u_0\|_2^2$$
$$= 2\int_0^t \int_\Omega \nabla u \nabla u_\tau dx d\tau + \|\nabla u_0\|_2^2. \quad (16)$$

Utilizing (16) in (15) one obtain

$$F^{'}(t) = 2\int_\Omega u u_t dx + 2\int_0^t \int_\Omega \nabla u \nabla u_\tau dx d\tau + \|\nabla u_0\|_2^2 \quad (17)$$

and

$$F^{''}(t) = 2\|u_t\|_2^2 + 2\int_\Omega u u_{tt} dx + 2\int_\Omega \nabla u \nabla u_t dx. \quad (18)$$

Consequently, due to the equation (1),

$$\int_\Omega uu_{tt}dx = \int_\Omega u\Delta u_t dx + \int_\Omega udiv(|\nabla u|^{\alpha-2}\nabla u)dx + \int_\Omega uh(u)dx. \tag{19}$$

Thus, from (18) and (19),we obtain

$$F''(t) = 2\|u_t\|_2^2 - 2\|\nabla u\|_\alpha^\alpha + 2\int_\Omega uh(u)dx \tag{20}$$

and from assumption (9),(20) reduced into

$$F''(t) \geq 2\|u_t\|_2^2 - 2\|\nabla u\|_\alpha^\alpha + 4(2\rho+1)\int_\Omega H(u)dx. \tag{21}$$

Hence using the definition of energy functional $E(t)$ in (10) and (11) a lower bound for $F''(t)$ can be found:

$$F''(t) \geq 4(\rho+1)(\frac{1}{2}\|u_t\|_2^2 + \int_0^t \|\nabla u_\tau\|_2^2 d\tau) + 4\rho\int_0^t \|\nabla u_\tau\|_2^2 d\tau$$
$$+(\frac{4(2\rho+1)}{\alpha} - 2)\|\nabla u\|_\alpha^\alpha - 4(2\rho+1)E(0). \tag{22}$$

Using Hölder's inequality

$$\|\nabla u\|_\alpha^\alpha \geq |\Omega|^{1-\frac{2}{\alpha}}(\|\nabla u\|_2^2)^{\frac{\alpha}{2}}.$$

in (22), we obtain

$$F''(t) \geq 4(\rho+1)(\|u_t\|_2^2 + \int_0^t \|\nabla u_\tau\|_2^2 d\tau)$$
$$+(\frac{4(2\rho+1)}{\alpha} - 2)|\Omega|^{1-\frac{2}{\alpha}}(\|\nabla u\|_2^2)^{\frac{\alpha}{2}} - 4(2\rho+1)E(0)$$
$$\geq 4(\rho+1)(\|u_t\|_2^2 + \int_0^t \|\nabla u_\tau\|_2^2 d\tau), \tag{23}$$

where assumption (13) and $\rho \geq \frac{\alpha-2}{4} \geq 0$ have been used. Hence $F''(t) \geq 0$ for all $t \geq 0$, also by assumption (14) $F'(0) \geq 0$. Thus $F'(t) \geq 0$ for all $t \geq 0$. Whence

$$F(0) = \|u_0\|_2^2 + \|\nabla u_0\|_2^2 > 0.$$

Now we search for a lower bound for the functional

$$y(t) = F''(t)F(t) - (1+\rho)(F'(t))^2. \tag{24}$$

By inserting (15), (17) and (23) in (24), we obtain

$$y(t) \geq 4(\rho+1)(\|u_t\|_2^2 + \int_0^t \|\nabla u_\tau\|_2^2 d\tau)(\|u\|_2^2 + \int_0^t \|\nabla u\|_2^2 d\tau + \|\nabla u_0\|_2^2)$$
$$-(\rho+1)(2\int_\Omega uu_t dx + 2\int_0^t\int_\Omega \nabla u\nabla u_\tau dxd\tau + \|\nabla u_0\|_2^2)^2$$
$$\geq 4(\rho+1)\{(\|u_t\|_2^2 + \int_0^t \|\nabla u_\tau\|_2^2 d\tau)(\|u\|_2^2 + \int_0^t \|\nabla u\|_2^2 d\tau + \|\nabla u_0\|_2^2)$$

$$-(\int_{\Omega} uu_t dx + \int_0^t \int_{\Omega} \nabla u \nabla u_\tau dx d\tau)^2$$
$$-(\int_{\Omega} uu_t dx + \int_0^t \int_{\Omega} \nabla u \nabla u_\tau dx d\tau)\|\nabla u_0\|_2^2 - (\frac{1}{2}\|\nabla u_0\|_2^2)^2\}$$
$$\geq 4(\rho+1)\{(\|u_t\|_2^2 + \int_0^t \|\nabla u_\tau\|_2^2 d\tau)(\|u\|_2^2 + \int_0^t \|\nabla u\|_2^2 d\tau + \|\nabla u_0\|_2^2)$$
$$-(\int_{\Omega} uu_t dx + \int_0^t \int_{\Omega} \nabla u \nabla u_\tau dx d\tau)^2 - \frac{1}{2}\|\nabla u_0\|_2^2 F'(t)\}. \tag{25}$$

Since from the Cauchy-Schwartz inequality we have

$$(\int_{\Omega} uu_t dx + \int_0^t \int_{\Omega} \nabla u \nabla u_\tau dx d\tau)^2$$
$$\leq [\|u\|_2\|u_t\|_2 + (\int_0^t \|\nabla u\|_2^2 d\tau)^{\frac{1}{2}}(\int_0^t \|\nabla u_\tau\|_2^2 d\tau)^{\frac{1}{2}}]^2$$
$$\leq (\|u\|_2^2 + \int_0^t \|\nabla u\|_2^2 d\tau)(\|u_2\|_2^2 + \int_0^t \|\nabla u_\tau\|_2^2 d\tau). \tag{26}$$

Exploiting (25) and (26), we obtain

$$\begin{aligned} y(t) &\geq -2(\rho+1)\|\nabla u_0\|_2^2 F'(t) \\ &\geq -2(\rho+1)(\int_0^t \|\nabla u\|_2^2 d\tau + \|u\|_2^2 + \|\nabla u_0\|_2^2)F'(t) \\ &= -2(\rho+1)F(t)F'(t). \end{aligned}$$

Thus, the hypotheses of Lemma 1 are fulfilled with $C_1 = 1 + \rho$ and $C_2 = 0$. From the assumption (14) and the application of condition (8) in Lemma 1, we remark that the lifespan $T$ is estimated by

$$0 < T \leq \frac{\|u_0\|_2^2 + \|\nabla u_0\|_2^2}{\rho(2\int_{\Omega} u_0 u_1 dx + \|\nabla u_0\|_2^2)} \leq \frac{1}{2(\rho+1)}$$

which shows that $F(t)$ becomes infinite in a finite time $T$, and this completes the proof.

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