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REFLECTION OF ROOT LATTICES FOR GENERALIZED KAC-MOODY ALGEBRAS

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ABSTRACT. In this paper we determine all elements in the root lattice of symmetrizable generalized Kac-Moody algebras whose reflections preserve the root systems. Also we discuss elements in the root lattices whose reflection preserve the root lattices.

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1. Introduction

R. Bocherds (1988) initiated the study of generalized Kac-Moody algebras (GKM algebras). The main difference between Kac-Moody algebras and GKM algebras is that GKM algebras can have simple roots of non positive norm (called imaginary roots). In [1], C. Bennet determined all imaginary roots whose reflection preserve the root systems for Kac-Moody algebras.

In [4], Zhao Kaiming determined all elements in the root lattices of symmetrizable Kac-Moody algebras whose reflection preserve the root systems and those elements whose reflection preserve root latties.

In this article we consider symmetrizable generalized Kac-Moody algebras. We enlarge the concept of reflection of root systems to the root lattices and find elements in the root lattice whose reflection preserve the root system. Also we determine all elements in the root lattices of symmetrizable generalized Kac-Moody algebras whose reflections preserve the root systems.

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2. Preliminaries

In this section we recall below some preliminaries regarding generalized Kac-Moody algebras and its root systems. The main difference between Kac-Moody algebras and GKM algebras is that GKM algebras can have simple roots of non positive norm (called imaginary roots).

Let $I = \{1, 2, 3, \cdots, n\}$ be a finite index set, and let $A = (a_{ij})_{i,j \in I}$ be a real $n \times n$ matrix satisfying the following conditions:

(R1) Either $a_{ij} = 2$ or $a_{ij} \leq 0$ for $i \in I$.

(R2) $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} \in \mathbb{Z}$ if $a_{ij} = 2$.

(R3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

We call such a matrix a generalized generalized Cartan matrix(abbreviated as GGCM). A GGCM is called indecomposable if it cannot be reduced to a block diagonal from by shuffling rows and columns. We consider the elements of GGCM as elements of $\mathbb{Z}$ only. Let $A = (a_{ij})_{i,j=1}^{n}$ be an indecomposable GGCM. A GGCM is called symmetrizable if there exists a diagonal matrix $D = diag(\epsilon_1, \cdots, \epsilon_n)$ with $\epsilon_i \in R$ and $\epsilon_i > 0$, $\forall i$, such that DA is symmetric.

For any GGCM $A = (a_{ij})_{ij \in I}$, we have a triple $\left(\hbar, \prod, \prod^{\vee}\right)$ where $\prod = \{\alpha_i\}_{i \in I}$ and $\prod^{\vee} = \left\{\alpha_i^{\vee}\right\}_{i \in I}$ satisfying the following:

(R1) $\hbar$ is a finite dimensional (complex) vector space such that $dim\ \hbar = 2n - rank\ A$.

(R2) $\prod = \left\{\alpha_i\right\}_{i \in I} \subset \hbar^*$ is linearly independent and $\prod^{\vee} = \left\{\alpha_i^{\vee}\right\}_{i \in I} \subset \hbar$ is linearly independent where $\hbar^* = Hom_{\mathbb{C}}(\hbar, \mathbb{C})$.

(R3) $< \alpha_j, \alpha_i^{\vee} >= a_{ij}$ where $< \cdot, \cdot >$ denotes a duality pairing between $\hbar$ and $\hbar^*$.

The above triple is called a realization of A.

Definition. Let GGCM $A = (a_{ij})_{i,j \in I}$ be symmerizable. The generalized Kac-Moody algebra (abbreviated as GKM algebra) $g(A)$ associated to a symmetrizable GGCM $A = (a_{ij})_{i,j \in I}$ is the Lie algebra (over $\mathbb{C}$) generated by the above vector space $\hbar$ and the elements $e_i, f_i (i \in I)$ satisfying the following relations:

(F1) $\left[h, h'\right] = 0$ for $h, h' \in \hbar$.

(F2) $\left[h, e_i\right] =< \alpha_i, h > e_i$, $[h, f_i] = - < \alpha_i, h > f$, $[e_i, f_i] = \delta_{ij} \alpha_i^{\vee}$ for $i, j \in I$

(F3) $\left(ad\ e_i\right)^{1 - a_{ij}} e_j = 0$, $(ad\ f_i)^{1 - a_{ij}} f_j = 0$ if $a_{ij} = 2$ and $j \neq i$

(F4) $\left[e_i, e_j\right] = 0$, $\left[f_i, f_j\right] = 0$ if $a_{ii}, a_{ij} \leq 0$ and $a_{ij} = 0$

The elements of $\prod$ $\left(\text{respectively } \prod^{\vee}\right)$ are called the simple roots (respectively simple coroots) of g.

We have the root space decomposition of $g(A)$ with respect to the Cartan subalgebra $\hbar$. $g(A) = \hbar \oplus \sum_{\alpha \in \Delta_+} \oplus g_\alpha \oplus \sum_{\alpha \in \Delta_-} \oplus g_\alpha$ where $\Delta_+ \Big(\subset Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \Big)$ is the set of positive roots, $\Delta_-(= -\Delta_+)$ the set of negative roots, and g_α is the root space corresponding to a root $\alpha \in \Delta$ (= set of roots) and $\Delta = \Delta_+ \cup \Delta_-$. We recall $\Delta = \Delta_+ \cup \Delta_-$ the root system of $g(A)$, and $Q = Q_+ \cup Q_-$ the root lattice.

We put $I^{re} = \Big\{ i \in I | a_{ii} = 2 \Big\}$, $I^{im} = \Big\{ i \in I | a_{ii} \leq 0 \Big\}$ and $\prod^{re} = \Big\{ \alpha_i \in \prod | i \in I^{re} \Big\}$, the set of real simple roots; $\prod^{im} = \Big\{ \alpha_i \in \prod | i \in I^{im} \Big\}$, the set of imaginary simple roots.

For $i \in I^{re}$, let r_i be the simple reflection of $\hbar^*$ given by

$$r_i(\lambda) = \lambda - \Big\langle \lambda, \alpha_i^\vee \Big\rangle \alpha_i (\lambda \in \hbar^\vee).$$

The *Weyl group* W of $g(A)$ is the subgroup of $GL(\hbar^*)$ generated by the $r_i's (i \in I^{re})$. Note that $\Big(W, \{ r_i | i \in I^{re} \} \Big)$ is a coxeter system.

For a real root $\alpha = w(\alpha_i) \Big(w \in W, \alpha_i \in \prod^{re} \Big)$ we also define the reflection r_α of $\hbar^*$ with respect to α by $r_\alpha(\lambda) = \lambda - < \lambda, \alpha^\vee > \alpha(\lambda \in \hbar^*)$, where $\alpha^\vee = w(\alpha_i^\vee) \in \hbar$ is the dual real root of α. It can be proved $r_\alpha = w r_i w^{-1} \in W$.

Let $\Delta^{re} = W \prod^{re}$ (the set of real roots), $\Delta^{im} = \Delta \backslash \Delta^{re}$ (the set of imaginary roots). For an element $\alpha = \sum_{i \in I} k_i \alpha_i \in Q_+ \backslash \{0\}$, we define supp (α) to be the subdiagram of the Dynkin diagram of $A = (a_{ij})_{i,j \in I}$ corresponding to the subset $\{ i \in I | \ k_i \geq 1 \}$ of I. As in Kac-Moody algebra case the set of all imaginary root in generalized Kac-Moody case is described as follows:

$$\Delta_+^{im} = \Delta^{im} \cap \Delta_+ = \bigcup_{w \in W} w(K),$$

where
$K = \Big\{ \alpha \in Q_+ \backslash \{0\} | < \alpha, \alpha_i^\vee > \leq 0 \text{ if } a_{ij} = 2 \text{ and } \textit{supp } (\alpha) \textit{ is connected} \Big\}$
$\backslash \bigcup_{j \geq 2} j \prod^{im}$. In particular, the set Δ_+^{im} of positive imaginary roots is W-stable (Kac, 1990). Since we have been assuming that the GGCM $A = (a_{ij})_{i,j \in I}$ is symmetrizable, there exists a non-degenerate, symmetric, W-invariant bilinear form $(\cdot, \cdot)$ on $g(A)$. Note that the restriction of this bilinear from $(\cdot, \cdot)$ to the Cartan subalgebra $\hbar$ is also non-degenerate, so that it induces through the linear isomorphism $\nu : \hbar \mapsto \hbar^*$ on $\hbar^*$ a non-degenerate, symmetric, W-invariant bilinear from which we again denoted by $(\cdot, \cdot)$. In particular, we have $(\alpha_i, \alpha_j) = \delta_i a_{ij} (1 \leq i, j \leq n)$. We remark that a root $\alpha \in \Delta$ is imaginary if and only if $(\alpha, \alpha) \leq 0$.

Since the reflections generating *Weyl group* are defined only with α_i's with $i \in I^{re}$, many properties on reflections and *Weyl group* for Kac-Moody algebras are generalized to generalized Kac-Moody algebras. We state some properties on *Weyl group* and their proofs can be found in [2, 5, 8].

Lemma 2.1. [5] *If A is a GGCM and $\alpha \in \Delta(A)$, then supp α is connected.*

Proposition 2.2. [2] *W preserves Δ, and in fact dim $g_\alpha =$ dim $g_{w\alpha}$ for $w \in W$ and $\alpha \in \Delta$.*

Proposition 2.3. [2] *For all $i \in I^{re}$, the reflection r_i permutes the elements of $\Delta_+ \backslash \{\alpha_i\}$.*

Proposition 2.4. [2] *For all $\alpha \in \Delta^{re}$, dim $g_\alpha = 1$ and in fact $\alpha \in \Big\{W \cdot \alpha_i | \alpha_i \in \prod^{re}\Big\}$. Furthermore:*

$$\begin{aligned} W\Delta^{re} = \Delta^{re},\ W\Delta^{im} = \Delta^{im} \\ \Delta^{re} = -\Delta^{re},\ \Delta^{im} = -\Delta^{im} \\ W\Big(\Delta^{re} \cap \Delta_+\Big) = \Delta^{re} \cap \Delta_+ \\ W\Big(\Delta^{im} \cap \Delta_+\Big) = \Delta^{im} \cap \Delta_+ \end{aligned}$$

Given $w \in W$ define the length of w, denoted $l(w)$, to be the smallest positive integer k such that w can be written as the product of k of the reflections $r_i, i \in I^{re}$. An expression $w = r_{i_1} r_{i_2} \cdots r_{i_k},\ i_j \in I$ such that α_{i_j} is real, is called reduced if $k = l(w)$. By convention $l(1) = 0$.

Proposition 2.5. [2] *Let $w \in W$, $i \in I^{re}$, and suppose that $w\alpha_i = \alpha_j$ for some $j \in J$. Then $wr_iw^{-1} = r_j$.*

Proposition 2.6. [2] *Let $r_{i_1} r_{i_2} \cdots r_{i_j}$, where $i_k \in I^{re}$, be a reduced expression of $w \in W$. Then*

$$r_{i_1} r_{i_2} \cdots r_{i_{j-1}} \alpha_{i_j} \in \Delta_+.$$

Lemma 2.7. [8] *For $\alpha \in \Delta$ and $i \in I^{re}$, one has the following:*

$$\begin{aligned} (\alpha, \alpha_i) > 0 \Rightarrow \alpha - \alpha_i \in \Delta_+, \\ (\alpha, \alpha_i) < 0 \Rightarrow \alpha + \alpha_i \in \Delta_+. \\ \alpha + \alpha_i \notin \Delta \Rightarrow (\alpha, \alpha_i) \geq 0, \\ \alpha - \alpha_i \notin \Delta \Rightarrow (\alpha, \alpha_i) \leq 0. \end{aligned}$$

3. Reflections preserving root systems, preserving root lattices

In this section, we enlarge the notion of special imaginary root of Kac-Moody algebras to Generalized Kac-Moody Algebras and show some properties related to special imaginary roots of Kac-Moody can be generalized to GKM algebras. In particular we examine the question of existence of special imaginary roots and we find two large classes of Generalized Kac-Moody algebras having special imaginary roots. Also we give the necessary and sufficient conditions for elements in the root lattices of symmetrizable generalized Kac-Moody algebras so that their reflections preserve the root system in Proposition 3.2. In Proposition 3.3 we also prove the necessary and sufficient condition for imaginary root preserve root lattices.

First, we recall the notion of reflection r was defined to simple roots $\alpha_i \in \prod = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ for Kac-Moody algebras and we enlarge this notion to the elements in root lattice $Q = \sum \mathbb{Z}\alpha_i$ for GKM algebras as follows:

For each $\alpha \in \hbar^*$, $(\alpha|\alpha) \neq 0$, we define the reflection r_α of α as

$$r_\alpha(\lambda) = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha, \quad for\ \lambda \in \hbar^*.$$

It is clear that for simple real root α reflection r_α, preserves root system and root lattice as well. Now we define special imaginary root for generalized Kac-Moody algebra case as follows:

Definition. [4] An imaginary root α is called a special imaginary root if α satisfies the following conditions:

(s1) $(\alpha|\alpha) \neq 0$

(s2) $r_\alpha(\Delta) = \Delta$,

$r_\alpha(\Delta^{re}) = \Delta^{re}$,

$r_\alpha(\Delta^{im}) = \Delta^{im}$

(s3) r_α preserves root multiplicifies.

The following lemma will be used in the proof of Proposition 3.2.

Lemma 3.1. *Suppose $r_{i_1} r_{i_2} \cdots r_{i_s} r_1$ is not reduced. Then there exists $1 \leq k \leq s$ such that $r_{i_1} r_{i_2} \cdots r_{i_s} r_1 = r_{i_1} r_{i_2} \cdots \widehat{r_{i_k}} \cdots r_{i_s}$, where the $\widehat{}$ means to delete the corresponding element.*

Proof. The proof can be found in [2].

We call $\alpha = \sum_{i=1}^{n} k_i\alpha_i \in Q$ primitive if $(k_1, k_2, \cdots, k_n) = 1$. It is clear that there are finitely many primitive special imaginary roots in $-C^\vee$ for any generalized Kac-Moody algebra $g(A)$.

The following theorem gives the necessary and sufficient condition for $\alpha \in \Delta^{re}$ to satisfy $r_\alpha \in W$. Since the Weyl group is generated only by real simple roots there come no imaginary simple roots and the proof of Proposition 3.2 is similar to the Kac-Moody algebra case which has no imaginary simple roots.

Proposition 3.2. *Let $\alpha \in Q$ be primitive, and $(\alpha|\alpha) \neq 0$. Then $r_\alpha \in W$ if and only if $\alpha \in \Delta^{re}$.*

Proof. Since any real root α can be represented by $w(\alpha_i)$ for some $\alpha_i \in \prod^{re}$ and $w \in W$ one can prove the necessary condition easily by definition of reflection. To prove the sufficient condition we use induction on the length $l(r_\alpha)$ of r_α. We know that $l(r_\alpha)$ is odd.

If $l(r_\alpha) = 1$, we know that $r_\alpha = r_{i_1}$. So $\alpha = \pm\alpha_{i_1} \in \Delta^{re}$. Suppose that $\alpha \in \Delta^{re}$ if $l(r_\alpha) < s$. Now we consider the case that $l(r_\alpha) = s$ and $s \geq 3$. Let $r_\alpha = r_{i_1} r_{i_2} \cdots r_{i_s}$ be a reduced expression of r_α. Because $r_\alpha = r_\alpha^{-1}$, i.e.,

$$r_\alpha = r_{i_1} r_{i_2} \cdots r_{i_s} = r_{i_s} r_{i_{s-1}} \cdots r_{i_1},$$

$r_{i_1} r_{i_2} \cdots r_{i_s} r_{i_1}$ is not reduced. By Lemma 3.1, there exists $1 \leq k \leq s$ such that $r_{i_1} r_{i_2} \cdots r_{i_s} r_{i_1} = r_{i_1} r_{i_2} \cdots \hat{r_{i_k}} \cdots r_{i_s}$, where the $\hat{}$ means to delete the corresponding element.

If $k \neq 1$, we get $r_\alpha = r_{i_1} r_{i_2} \cdots r_{i_{s-1}} r_{i_1}$. By Proposition 2.5. and with simple computation gives that $r_{r_{i_1}(\alpha)} = r_{i_2} \cdots r_{i_{s-1}}$, which lies in W. Also. it is easy to see that $l(r_{r_{i_1}(\alpha)}) \leq s - 2$. We use inductive hypothesis on $l(r_{r_{i_1}(\alpha)})$ to get that $r_{i_1}(\alpha) \in \Delta^{re}$. Hence $\alpha \in \Delta^{re}$.

If $k = 1$, we know that $r_\alpha = r_{r_{i_1}} r_{i_2} \cdots r_{i_s} = r_{i_2} \cdots r_{i_s} r_{i_1}$. So $r_\alpha r_{i_1} = r_{i_1} r_\alpha$, i.e., $r_{r_{i_1}}(\alpha) = r_\alpha$. Hence $r_{i_1}(\alpha) = \pm\alpha$. In case $r_{i_1}(\alpha) = -\alpha$, we get $\alpha = \alpha_{i_1} \in \Delta^{re}$. In case $r_{i_1}(\alpha) = \alpha$ we consider $r_\alpha = r_{i_2} \cdots r_{i_s} r_{i_1}$ instead. Similarly we get $\alpha \in \Delta^{re}$ or $r_{i_2}(\alpha) = \alpha$. Continuing this method we can get that either $\alpha \in \Delta^{re}$ or $r_{i_j}(\alpha) = \alpha$ for all $j = 1, 2, \cdots, s$. If $\alpha \in \Delta^{re}$, the theorem is proved. If $r_{i_j}(\alpha) = \alpha$ for all $j = 1, 2, \cdots, s$, we get $-\alpha = r_\alpha(\alpha) = r_{i_1} r_{i_2} \cdots r_{i_s}$. This is impossible, therefore we get the theorem. □

In Proposition 3.3 we discuss reflections preserving the root system for a GKM algebras.

Proposition 3.3. *Let* $\alpha = \sum_{i=1}^{n} k_i \alpha_i \in Q \backslash \Delta$ *primitive, and* $(\alpha|\alpha) \neq 0$. *Then* $r_\alpha \Delta = \Delta$ *if and only if* α *is* W*-equivalent to* $\alpha_i - \alpha_j$ *where the permutation of* i *and* j *in the Dynkin diagram* $D(A)$ *is a diagram automorphism of* $D(A)$.

Proof. First, lets assume that $\alpha = \alpha_1 - \alpha_2$ where the permutation of 1 and 2 in the Dynkin diagram $D(A)$ is a diagram automorphism of $D(A)$. So $(\alpha_1|\alpha_1) = (\alpha_2|\alpha_2)$, i.e., $a_{12} = a_{21}$. And for any $k \neq 1$ or 2, we have $a_{1k} = a_{2k}$, $a_{k1} = a_{k2}$. Then $(\alpha|\alpha) = 2(\alpha_1|\alpha_1) - 2(\alpha_1|\alpha_2) = 2(\alpha|\alpha_1)$ and $(\alpha|\alpha) = 2(\alpha|\alpha_2)$. Therefore $r_\alpha(\alpha_k) = \alpha_k$ if $k \neq 1, 2$, $r_\alpha(\alpha_1) = \alpha_2$, $r_\alpha(\alpha_2) = \alpha_1$. Hence $r_\alpha \prod = \prod$, Therefore, $r_\alpha \Delta = \Delta$.

Conversely, lets suppose $r_\alpha \Delta = \Delta$. Set $Q^0 = Q \backslash (Q_+ \cup Q_-)$. If $W\alpha \cap Q^0 = \emptyset$ we know that $\alpha \in \Delta$, a contradiction. So $W\alpha \cap Q^0 \neq \emptyset$. By the action of W we can assume that $\alpha \in Q^0$ and $\alpha = \sum_{i \in I} k_i \alpha_i - \sum_{j \in J} k_j \alpha_j$, where $I, J \subset \{1, 2, \cdots, n\}$, $I \cap J = \emptyset$, $I \neq \emptyset$, $J \neq \emptyset$ and $I \cup J = supp\ \alpha$.

If $(\alpha|\alpha_p) \neq 0$ for some $p \notin I \cup J$, we get $r_\alpha(\alpha_p) = \alpha_p - (2(\alpha|\alpha_p)/(\alpha|\alpha))\alpha$. This expression has both positive coefficients and negative coefficients, thus it can not be in the root system, which contradicts to the hypothesis. Hence $(\alpha|\alpha_p) = 0$ for all $p \notin I \cup J$. If $|I| > 1$, $(\alpha|\alpha_i) = 0$ for all $i \in I$. If $|J| > 1$, we get $(\alpha|\alpha_j) = 0$ for all $j \in J$. If $|I| > 1$, $|J| > 1$, we get $(\alpha|\alpha) = 0$, a contradiction. So we must have

$|I| = 1$ or $|J| = 1$. If $|I| = 1$, $|J| > 1$, we assume that $\alpha = k_1\alpha_1 - \sum_{j\in J} k_j\alpha_j$. So $(\alpha|\alpha) = k_1(\alpha|\alpha_1)$. Now we want to show that r_α is the diagram automorphism of $D(A)$ interchanging 1 and 2 and fixing the other vertices.

By hypothesis $r_\alpha(\alpha_1) \in \Delta$. On the other hand

$$\begin{aligned} r_\alpha(\alpha_1) = & \quad \alpha_1 - (2(\alpha|\alpha_1)/(\alpha|\alpha)) \left(k_1\alpha_1 - \sum_{j\in J} k_j\alpha_j \right) \\ = & \quad -\alpha_1 + \frac{2}{k_1} \sum_{i\neq j} k_j\alpha_j \end{aligned}$$

In this expression both negative and positive coefficients arise therefore $r_\alpha \notin \Delta$, which contradicts to the hypothesis $r_\alpha(\alpha_1) = \Delta$. So it is impossible that $|I| = 1$, $|J| > 1$. Similarly, it is impossible that $|I| > 1$, $|J| = 1$. Hence $|I| = |J| = 1$. We write $\alpha = k_1\alpha_1 - k_2\alpha_2$. Similarly we have

$$r_\alpha(\alpha_1) = \alpha_1 - \Big(2(\alpha|\alpha_1)/(\alpha|\alpha)\Big)\Big(k_1\alpha_1 - k_2\alpha_2\Big) \in \Delta$$

which implies $2k_1(\alpha|\alpha_1)/(\alpha|\alpha) = 1$. This gives $r_\alpha(\alpha_1) = \Big(2(\alpha|\alpha_1)/(\alpha|\alpha)\Big)k_2\alpha_2 \in \Delta$ and so $2k_2(\alpha|\alpha_1)/(\alpha|\alpha) = 1$.

Therefore $k_1 = k_2 = 1$. We get

$$r_\alpha(\alpha_p) = \begin{cases} \alpha_p, & p \neq 1, 2 \\ \alpha_2, & p = 1, \\ \alpha_1, & p = 2. \end{cases}$$

Therefore r_α is a diagram automorphism of $D(A)$ interchanging 1 and 2 and fixing the other vertices.

Corollary 3.4. *Let $\alpha \in Q$ be primitive, and $(\alpha|\alpha) \neq 0$. Then $r_\alpha\Delta = \Delta$ if and only if one of the following conditions holds: (a) $\alpha \in \Delta^{re}$ (b) α is a special imaginary root (c) α is W-equivalent to $\alpha_i - \alpha_j$ where the permutation of i and j in the Dynkin diagram $D(A)$ is a diagram automorphism of $D(A)$.*

Proof. The proof is immediate from Proposition 3.2.and Proposition 3.3.

We discuss elements in the root lattices of symmetrizable GKM algebras, whose reflections preserve the root lattice. It is not as easy to determine all reflections preserving the root lattices as it is to determine all reflections preserving the root systems.

We prove necessary and sufficient condition for imaginary root to preserve root lattices.

Proposition 3.5. *Let $\alpha = \sum_{i=1}^{n} k_i\alpha_i \in Q_+\backslash\{0\}$ primitive, and $(\alpha, \alpha) \neq 0$. Then $r_\alpha(Q) = Q$ if and only if $2(\alpha_i, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all $i = 1, 2, \cdots, n$.*

Proof. By investigating $r_\alpha(\alpha_i) \in Q$ for all $i = 1, 2, \cdots, n$, we can easily get the theorem.

Proposition 3.6. *Let* $\alpha = \sum_{i=1}^{n} k_i\alpha_i \in Q_+\backslash\{0\}$ *be primitive satisfing that all* i*'s with* $k_i > 0$ *are in* I^{re}, $(\alpha, \alpha) \neq 0$, $\alpha \in W \cdot C^\vee$ *where* $C^\vee = \Big\{\beta \in Q_+\backslash\{0\} | (\beta, \alpha_i) \geq 0, \forall i \in I^{re}\Big\}$. *Then* $r_\alpha(Q) = Q$ *if and only if* $r_\alpha\Big(\Delta_{supp(\alpha)}\Big) = \Delta_{supp(\alpha)}$ *and* $2(\alpha_i, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ *for all* $i \notin supp(\alpha)$.

Proof. The necessary condition follows from Proposition 3.5.
Next we want to prove sufficient condition. By the action of W we may assume that $\alpha \in W \cdot C^\vee$.

Denote *supp* α by J, the submatrix of A corresponding to J by A_J and the subdiagram of D by D_J. Set $(\alpha_i, \alpha) = x_i$. By hypothesis we know $x_i \geq 0$ for $i \in J^{re}$. From hypothesis and Proposition 3.5 we have $2x_j / \sum_{i \in J} k_i x_i \in \mathbb{Z}$, for all $j = 1, 2, \cdots, n$, which implies at most two $x_i (i \in J)$ are not zero.

If only one $x_i (i \in J)$ is not zero, we may assume $x_1 > 0$. Since $2x_j / \sum_{i \in J} k_i x_i \in \mathbb{Z}$ either $k_1 = 1$ or $k_1 = 2$. In case $k_1 = 1$ we have $\alpha = \alpha_1 + \sum_{i=2}^{n} k_i\alpha_i$. Then $r_\alpha(\alpha_1) = -\alpha_1 - \sum_{i=2}^{n} 2k_i\alpha_i$, $r_\alpha(\alpha_i) = -\alpha_i$ for $i \in J\backslash\{1\}$. Because $(2\alpha, \alpha_1) > 0$, we know $2\alpha - \alpha_1 \in \Delta$. But $2\alpha - \alpha_1 = \alpha_1 + \sum_{i=2}^{n} 2k_i\alpha_i = r_\alpha(-\alpha_1)$. So $r_\alpha(\alpha_1)$ is a real root. Therefore $\{r_\alpha(\alpha_i) | i \in J\}$ is also a root basis of $g(A_J)$. Hence $r_\alpha(\Delta_J) = \Delta_J$. In case $k_1 = 2$ we can prove using the same argument.

If two of $x_i (i \in J)$ are not zero, we may assume $x_1 > 0$, $x_2 > 0$. So similar to the case above we have $k_1 = k_2 = 1$. Further $r_\alpha(\alpha_i) = \alpha_i$ for $i \in J\backslash\{1, 2\}$, $r_\alpha(\alpha_1) = -\alpha_1 - \sum_{i=2}^{n} 2k_i\alpha_i$, $r_\alpha(\alpha_2) = -\alpha_2 - \sum_{i=2}^{n} 2k_i\alpha_i$. Because $(2\alpha, \alpha_1) > 0$, we know $2\alpha - \alpha_1 \in \Delta$. But $2\alpha - \alpha_1 = \alpha_1 + \sum_{i=2}^{n} 2k_i\alpha_i$. So $r_\alpha(\alpha_1)$ is a real root. Similarly $r_\alpha(\alpha_2)$ is a real root. Therefore $\{r_\alpha(\alpha_i) | i \in J\}$ is also a root basis of $g(A_J)$. Hence $r_\alpha(\Delta_J) = \Delta_J$.

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