J. Appl. Math. & Informatics Vol. 28(2010), No. 1 - 2, pp. 363 - 372
Website: http://www.kcam.biz

THE ALMOST SURE CONVERGENCE FOR THE IDENTICALLY DISTRIBUTED NEGATIVELY ASSOCIATED RANDOM VARIABLES WITH INFINITE MEANS

HYUN-CHULL KIM

ABSTRACT. In this paper we prove the almost sure convergence of partial sums of identically distributed and negatively associated random variables with infinite expectations. Some results in Kruglov[Kruglov, V., 2008 Statist. Probab. Lett. 78(7) 890-895] are considered in the case of negatively associated random variables.

AMS Mathematics Subject Classification : 60F15, 60G09
Key words and phrases : Borel-Cantelli lemma, infinitely often, pairwise negative quadrant dependence, negative association, infinite means.

1. Introduction

For a sequence $\{X_n,\ n \geq 1\}$ of independent and identically distributed random variables with $E|X_1| = \infty$ and a sequence $\{a_n, n \geq 1\}$ of positive numbers satisfying the condition $a_n/n \uparrow$, Feller(1946) proved that if $\sum_{n=1}^{\infty} P(|X_n| > a_n)$ diverges then S_n/a_n converges to zero almost surely, where $S_n = X_1 + \cdots + X_n$ and Feller(1946) proved strong law of large numbers partial sums of independent identically distributed random variables with infinite means and Kruglov(2008) reinforced and generalized Feller's results to the case of pairwise independent random variables.

A finite family $\{X_1, \cdots, X_n\}$ is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, \cdots, n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^{\mathbb{A}}$, g on $\mathbb{R}^{\mathbb{B}}$, $cov\Big(f(X_k, k \in A), g(X_k, k \in B)\Big) \leq 0$. Infinite family is negatively associated if every finite subfamily is negatively associated. $\{X_n,\ n \geq 1\}$ is said to be pairwise negative quadrant dependent(NQD)

Received April 9, 2009. Accepted September 5,2009.

if $P(X_i > x_i, X_j > x_j) \le P(X_i > x_i)P(X_j > x_j)$ $\Big(P(X_i \le x_i, X_j \le x_j) \le P(X_i \le x_i)P(X_j \le x_j)\Big)$ for all $x_i, x_j \in \mathbb{R}$ and $i \ne j$. The concept of negative quadrant dependence was introduced by Lehman(1966), and the concept of negative association was introduced by Joag-Dev and Proschan(1983). Note that NA implies NQD.

In this note we cKruglov's(2008) result to the identically distributed NA random variables.

2. Preliminaries

In this section we introduce some familar results which will be used to prove the main results:

Theorem 2.1(Chung, 1974, p42). *For every nonnegative random variable* ξ

$$\sum_{n=1}^{\infty} P\{\xi \ge n\} \le E\xi \le 1 + \sum_{n=1}^{\infty} P\{\xi \ge n\}. \tag{2.1}$$

Note that the following Theorem means that $E|\xi| < \infty \Leftrightarrow \sum_{n=1}^{\infty} P\{|\xi| \ge n\} < \infty$.

Theorem 2.2(Matula, 1992). *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of negatively associated random variables with finite second moments. If* $\sum_{n=1}^{\infty} Var(X_n) < \infty$, *then* $\sum_{n=1}^{\infty}(X_i - EX_n)$ *converges a.s.*

Theorem 2.3(Matula, 1992). *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of pairwise NQD random variables with the same distribution function. Then*

$$\frac{S_n}{n} \to EX_1 < \infty \ if \ and \ only \ if \ E|X_1| < \infty.$$

Note that Theorem 2.3 still holds under the NA case.

Theorem 2.4(Petrov, 2002). *Let* $A_1, A_2, \cdots$ *be a sequence of events satisfying conditions*

$$\sum_{n=1}^{\infty} P(A_n) = \infty. \tag{2.2}$$

and

$$P(A_k \cap A_j) \le P(A_k)P(A_j),\ k \ne j. \tag{2.3}$$

Then,

$$P\left(\limsup_n A_n\right) = 1. \tag{2.4}$$

where $\limsup_n A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ *and* $P\left(\limsup_n A_n\right) = P(A_n \ i.o)$.

3. Main results

Theorem 3.1. *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of identically distributed NA random variables such that* $EX^- < \infty$ *and* $EX^+ = \infty$ *and* $\{a_n,\ n \geq 1\}$ *be a sequence of positive numbers satisfying* $a_n/n \uparrow$. *If*

$$\sum_{n=1}^{\infty} P(X_n > a_n) = \infty, \tag{3.1}$$

then

$$\frac{S_n}{a_n} \to 0 \ a.s. \tag{3.2}$$

Proof. Let $c = \sup_{n \geq 1} a_n/n$.

Case 1 : If $c < \infty$, then, $\sum_{n=1}^{\infty} P(X_n^+ > ba_n) = \infty$ for any $b > 0$. Indeed, if $\sum_{n=1}^{\infty} P(X_n^+ > ba_n) < \infty$ for some $b > 0$, then by Theorem 2.1 we have

$$\infty = E(X_1^+/2bc) \leq 1 + \sum_{n=1}^{\infty} P(X_1^+ \geq 2bcn) \leq 1 + \sum_{n=1}^{\infty} P(X_1^+ > ba_n) < \infty,$$

which yields a contradiction. Let

$$b > \max(1, EX_1^-/c) \ and \ A_n = \{X_n^+ > 2ba_n\},\ n \geq 1. \tag{3.3}$$

Clearly, (2.2) holds by assumption (3.1) and $\{A_n\}$ satisfies (2.3) since $\{X_n^+,\ n \geq 1\}$ is a sequence of NA random variables. So

$$P\Big(X_1^+ + \cdots + X_n^+ > 2ba_n \ i.o\Big) = 1. \tag{3.4}$$

Since $\{X_n^-,\ n \geq 1\}$ is a sequence of pairwise negative quadrant dependent identically distributed random variables, $X_n^- \geq 0$ and $EX_n^- < \infty$, it follows from Theorem 2.3 that

$$\sum_{i=1}^{n} X_i^-/n \to EX_1^- \ a.s. \tag{3.5}$$

By (3.3), (3.4) and (3.5) we have

$$\limsup_{n\to\infty} \frac{S_n}{a_n} = \limsup_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} X_i^+ - \lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} X_i^- > 2b - b = b > 1 \ a.s$$

and hence $P(S_n > a_n \ i.o.) = 1$. Thus (3.2) holds.

Case 2 : $c = \sup_{n\geq 1} a_n/n = \infty$. By (3.5) we have

$$\sum_{i=1}^{n} \frac{X_i^-}{a_n} \to 0 \ a.s. \tag{3.6}$$

Note that

$$\infty = \sum_{i=1}^{\infty} P(X_n > a_n) \leq \sum_{i=1}^{\infty} P(X_n^+ > a_n) \tag{3.7}$$

and that the events $\{X_n^+ > a_n\}$ satisfy conditions (2,2) and (2.3).

Then by Theorem 2.4 $P(X_n^+ > a_n \ i.o.) = 1$ and hence

$$P\Big(X_1^+ + \cdots + X_n^+ > a_n \ i.o.\Big) = 1. \tag{3.8}$$

Thus $P(S_n > a_n \ i.o.) = 1$ by (3.6) and (3.8), i.e., (3.2) holds. The proof is complete.

Corollary 3.2. *Let $\{X_n, \ n \geq 1\}$ be a sequence of identically distributed NA random variables satisfying $EX_1^+ = \infty$ and $EX_1^- < \infty$. Let $\{a_n, \ n \geq 1\}$ be a sequence of positive numbers satisfying $a_n/n \uparrow$. If*

$$\sum_{n=1}^{\infty} P(|X_n| > a_n) = \infty, \tag{3.9}$$

then (3.2) holds.

Proof. Note that $\sum_{n=1}^{\infty} P(|X_{2n-1}| > a_{2n-1}) + \sum_{n=1}^{\infty} P(|X_{2n}| > a_{2n}) = \infty$. One of these series diverges.

(i) The case that the first one diverges, i.e., $\sum_{n=1}^{\infty} P(|X_{2n-1}| > a_{2n-1}) = \infty$. Because of $a_n/n \uparrow$, $a_{2n-1} \geq 2a_{n-1}$ holds for all $n \geq 2$. From the fact that the random variables X_n are identically distributed we have

$$\infty = \sum_{n=2}^{\infty} P(|X_{2n-1}| > a_{2n-1}) \leq \sum_{n=2}^{\infty} P(|X_{2n-1}| > 2a_{n-1})$$

$$= \sum_{n=2}^{\infty} P(|X_{n-1}| > 2a_{n-1}).$$

Since $\sum_{n=2}^{\infty} P(|X_{n-1}| > 2a_{n-1}) = \infty$, we have

$$\begin{aligned} \infty &= \sum_{n=2}^{\infty} P(X_{n-1}^{+} + X_{n-1}^{-} > 2a_{n-1}) \\ &\le \sum_{n=2}^{\infty} P(X_{n-1}^{+} > a_{n-1}) + \sum_{n=2}^{\infty} P(X_{n-1}^{-} > a_{n-1}). \end{aligned}$$

Since $EX_1^{+} = \infty$ and $EX_1^{-} < \infty$, we have

$$\sum_{n=2}^{\infty} P(X_{n-1}^{+} > a_{n-1}) = \infty. \tag{3.10}$$

It is clear that $\{X_n^{+},\ n \ge 1\}$ are NA and events $\{X_{n-1}^{+} > a_{n-1}\}$ satisfy condition (2.3). Hence, by Theorem 2.4 and (3.10), $P\Big(X_{n-1}^{+} > a_{n-1}\ i.o.\Big) = 1$ which yields $P(|X_{n-1}| > a_{n-1}\ i.o.) = 1$ and hence

$$P\Big(|S_n| > a_n\ i.o.\Big) = 1 \tag{3.11}$$

since $|X_{n-1}| = |S_{n-1} - S_{n-2}| \le |S_{n-1}| + |S_{n-2}|,\ n \ge 3$. Thus $\frac{S_n}{a_n} \not\to 0\ a.s.$.

(ii) The case that $\sum_{n=1}^{\infty} P(|X_{2n}| > a_{2n}) = \infty$. By the similar method as in (i) we obtain (3.11).

Corollary 3.3. *Let $\{X_n,\ n \ge 1\}$ be a sequence of identically distributed non-negative NA random variables with $EX_1 = \infty$ and $\{a_n,\ n \ge 1\}$ be a sequence of positive numbers satisfying $a_n/n \uparrow$. Then (3.1) implies (3.2).*

Theorem 3.4. *Assume that $\{X_n,\ n \ge 1\}$ be a sequence of identically distributed NA random variables satisfying $EX_1^{-} < \infty$ and $EX_1^{+} = \infty$. Let $\{a_n,\ n \ge 1\}$ be a sequence of positive numbers such that $a_n/n \uparrow$. Then*

$$P\left(\lim_{n\to\infty} \frac{|X_1| + \cdots + |X_n|}{a_n} = 0\right) = 1 \ \textit{if and only if} \ \sum_{n=1}^{\infty} P(X_n > a_n) < \infty. \tag{3.12}$$

Proof. Note that $EX_1^{-} < \infty$ and $EX_1^{+} = \infty$ Assume $\sum_{n=1}^{\infty} P(X_n > a_n) < \infty$. The conditions $a_n/n \uparrow$, $EX_1^{-} < \infty$ and $\sum_{n=1}^{\infty} P(X_n > a_n) < \infty$ imply

$$\sum_{n=1}^{\infty} P(X_1^{+} > 2a_n) \le \sum_{n=1}^{\infty} P(X_1 > a_n) + \sum_{n=1}^{\infty} P(X_1^{-} > a_n) < \infty. \tag{3.13}$$

Note that $a_n/n \uparrow \infty$ as $n \to \infty$. Indeed, if $\sup_{n\geq 1} a_n/n = c < \infty$, then we have the following contradiction, i.e.,

$$\infty = E(X_1^+/3c) \leq 1 + \sum_{n=1}^{\infty} P(X_1^+ > 3cn)$$

$$\leq 1 + \sum_{n=1}^{\infty} P(X_1^+ > 2a_n) < \infty.$$

By Theorem 2.3 $\lim_{n\to\infty} (X_1^- + \cdots + X_n^-)/n = EX_1^-$ *a.s.* and hence,

$$\lim_{n\to\infty} (X_1^- + \cdots + X_n^-)/a_n = 0 \ a.s.$$

Considering the equality $|X_1| + \cdots + |X_n| = \left(X_1^+ + \cdots + X_n^+\right) + \left(X_1^- + \cdots + X_n^-\right)$ we need only to prove that $\lim_{n\to\infty} (X_1^+ + \cdots + X_n^+)/a_n = 0$ *a.s.*. Since $a_n/n \uparrow$ we have $n/k \leq a_n/a_k$ for all k, $n(k \leq n)$ and

$$\sum_{n=k}^{\infty} \frac{1}{a_n^2} \leq \frac{k^2}{a_k^2} \sum_{n=k}^{\infty} \frac{1}{n^2} \leq \frac{2k}{a_n^2}. \tag{3.14}$$

Put

$$Y_n = na_n^{-1} X_n^+ I\left[X_n^+ \leq 2a_n\right] + 2nI\left[X_n^+ > 2a_n\right].$$

Then $Y_n^{'}$'s are negatively associated random variables and hence $\{Y_n/n\}$ is a sequence of negatively associated random variables. To show that $\{Y_n/n,\ n \geq 1\}$ satisfies the conditions of Theorem 2.2 it suffices to prove that

$$\sum_{n=1}^{\infty} Var\left(\frac{Y_n}{n}\right) < \infty \tag{3.15}$$

and

$$E\left(\frac{Y_n}{n}\right)^2 < \infty. \tag{3.16}$$

Let $a_0 = 0$. The validity of (3.15) can be proved as follows. We have

$$\sum_{n=1}^{\infty} Var\left(\frac{Y_n}{n}\right) = \sum_{n=1}^{\infty} \frac{E(Y_n - EY_n)^2}{n^2}$$
$$\leq \sum_{n=1}^{\infty} \frac{EY_n^2}{n^2}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{k=1}^{n}E\left((X_1^+)^2 I[a_{k-1}<X_1^+/2\le a_k]\right)\\ &\quad +4\sum_{n=1}^{\infty}P(X_n^+>2a_n)\\ &= I+II(say).\end{aligned}$$

$$\begin{aligned} I &= \sum_{k=1}^{\infty}\sum_{n=k}^{n}\frac{1}{a_n^2}E\left((X_1^+)^2 I[a_{k-1}<X_1^+/2\le a_k]\right)\\ &\le \sum_{k=1}^{\infty}\frac{2k}{a_k^2}E\left((X_1^+)^2 I[a_{k-1}<X_1^+/2\le a_k]\right)\ by\ (3.13)\\ &\le 8\sum_{k=1}^{\infty}kP(a_{k-1}<X_1^+/2<a_k)\\ &= 8\sum_{k=0}^{\infty}P(X_1^+>2a_k).\end{aligned}$$

Hence, $I<\infty$ and $II<\infty$ by (3.13), which yields (3.15).

We also have

$$\begin{aligned} E(\frac{Y_n}{n})^2 &= \frac{1}{a_n^2}\sum_{k=1}^{n}E((X_1^+)^2 I[a_{k-1}<X_1^+/2\le a_k])+4P(X_1^+>2a_n)\\ &\le \sum_{k=1}^{n}EI[a_{k-1}<X_1^+/2\le a_k]+4P(X_1^+>2a_n)\\ &= \sum_{k=1}^{n}P(a_{k-1}<X_1^+/2\le a_k)+4P(X_1^+>2a_n)\\ &= P\Big(X_1^+\le 2a_n)+4P(X_1^+>2a_n\Big)<\infty.\end{aligned}$$

Hence, by Theorem 2.2 it follows from (3.15) and (3.16) that

$$\sum_{n=1}^{\infty}\frac{(Y_n-EY_n)}{n}\ converges\ a.s. \tag{3.17}$$

From the facts that

$$\begin{aligned} E\Big(I[X_n^+>2a_n]-EI[X_n^+>2a_n]\Big)^2 &\le EI^2[X_n^+>2a_n]\\ &= P(X_n^+>2a_n)\\ &< \infty\end{aligned}$$

and

$$\sum_{n=1}^{\infty} E\Big(I[X_n^+ > 2a_n] - EI[X_n^+ > 2a_n]\Big)^2 \;=\; \sum_{n=1}^{\infty} P(X_n^+ > 2a_n) \;<\; \infty,$$

we obtain, by Theorem 2.2

$$\sum_{n=1}^{\infty} \Big(I[X_n^+ > 2a_n] - EI[X_n^+ > 2a_n]\Big) \textit{ converges a.s.} \tag{3.18}$$

Hence, by (3.16) and (3.17) $\sum_{n=1}^{\infty} \frac{1}{a_n} X_n^+ I[X_n^+ \leq 2a_n] - E(X_n^+ I[X_n^+ \leq 2a_n]$ converges and by Kronecker's lemma

$$\frac{1}{a_n}\sum_{k=1}^{n} X_k^+ I[X_k^+ \leq 2a_k] - E(X_k^+ I[X_k^+ \leq 2a_k]) \to 0 \ a.s. \tag{3.19}$$

By (3.13), for any $\epsilon > 0$ there exists an $r \geq 1$ such that

$$\sum_{m=r+1}^{\infty} mP\Big(a_{m-1} < X_1^+/2 \leq a_m\Big) < \epsilon. \tag{3.20}$$

For $n > r$ we have

$$\begin{aligned}
&\frac{1}{a_n}\sum_{k=1}^{n} E(X_k^+ I[X_k^+ \leq 2a_k]) \\
&= \frac{1}{a_n}\sum_{k=1}^{n}\sum_{m=1}^{k} E(X_1^+ I[a_{m-1} < X_1^+/2 \leq a_m]) \\
&= \frac{1}{a_n}\sum_{m=1}^{n} (n-m+1) E(X_1^+ I[a_{m-1} < X_1^+/2 \leq a_m]) \qquad (3.21) \\
&\leq \frac{n}{a_n}\sum_{m=1}^{r} E(X_1^+ I[a_{m-1} < X_1^+/2 \leq a_m]) \\
&\quad + \frac{2n}{a_n}\sum_{m=r+1}^{n} a_m P(a_{m-1} < X_1^+/2 \leq a_m) \\
&\leq \frac{n}{a_n} E(X_1^+ I[X_1^+/2 \leq a_r]) \\
&\quad + 2\sum_{m=r+1}^{\infty} mP(a_{m-1} < X_1^+/2 \leq a_m) \\
&= III + IV.
\end{aligned}$$

The last inequality holds by the fact that $a_m/m \le a_n/n$ for $m \le n$. It remains to show that $III \to 0$ and $IV \to 0$ as $n \to \infty$. By $\lim_{n\to\infty} n/a_n = 0$ $III = 0$ as $n \to \infty$ and by (3.20)

$$IV = 2 \sum_{m=r+1}^{\infty} mP\left(a_{m-1} < \frac{X_1^+}{2} \le a_m\right) < 2\epsilon.$$

Hence

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} E(X_k^+ I[X_k^+ \le 2a_k]) = 0. \tag{3.22}$$

By (3.19) and (3.22) we also have

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} X_k^+ I[X_k^+ \le 2a_k]) = 0 \ a.s. \tag{3.23}$$

By (3.13) and the Borel-Cantelli lemma(Loeve, 1977, p240) for almost every $\omega \in \Omega$ there exists an $n_0(\omega)$ such that $X_n^+(\omega) = X_n^+ I\left[X_n^+ \le 2a_n\right](\omega)$ for all $n > n(\omega)$. Therefore it follows from (3.23) that

$$\lim_{n\to\infty} \frac{X_1^+ + \cdots X_n^+}{a_n} = 0 \ a.s.$$

Thus 'if' part of (3.12) is proved.

To prove 'only if' part of (3.13): By assumption,

$$P\left(\lim_{n\to\infty} (|X_1| + \cdots + |X_n|)/a_n = 0\right) = 1$$

and hence,

$$P\left(\lim_{n\to\infty} |X_n|/a_n = 0\right) = 1.$$

In particular, the latter implies that $P(|X_n| > a_n \ i.o.) = 0$. If $\sum_{n=1}^{\infty} P(X_n > a_n) = \infty$, then $\sum_{n=1}^{\infty} P(|X_n| > a_n) = \infty$ and hence $P(|X_n| > a_n \ i.o.) = 1$ by Theorem 3.1. This contradiction proves the theorem.

Corollary 3.5. *Let $\{X_n,\ n \ge 1\}$ be a sequence of identically distributed NA random variables with $E|X_1| = \infty$ and $\{a_n,\ n \ge 1\}$ be a sequence of positive numbers satisfying $a_n/n \uparrow$. Then*

(3.24)

$$P\left(\lim_{n\to\infty} \frac{(|X_1| + \cdots + |X_n|)}{a_n} = 0\right) = 1 \ if\ and\ only\ if\ \sum_{n=1}^{\infty} P(|X_n| > a_n) < \infty.$$

Proof. To prove (3.24) it suffices to replace X_n by $|X_n|$ in (3.12).

From Theorem 2.1 we obtain the following corollary :

Corollary 3.6. *Let $\{X_n,\ n \geq 1\}$ be a sequence of identically distributed NA random variables such that $EX_1^+ = \infty$ and $EX_1^- < \infty$. Then*

$$\sum_{n=1}^{\infty} P(X_n > n) = \infty \text{ implies } \overline{\lim_{n\to\infty}} \frac{S_n}{n} = \infty \text{ a.s.}$$

where $P(S_n > n\ i.o) = 1$ means $\overline{\lim}_n S_n/n = \infty$ a.s.

REFERENCES

1. Chung, K.L., 1974, *A Course in Probability Theory,* 2nd ed., Academic Press, INC, SanDiego
2. Feller, W., *A limit theorem for random variables with infinite moments*, Amer. J. Math., 66(2), (1946) 257-262
3. Joag-Dev K. and Proschan, F., *Negative association random variables with application*, Ann. Statist., 11, (1983) 286-295
4. Kruglov, V.M., *A strong law of large numbers for pairwise independent identically distributed random variables with infinite means*, Statist. Probab. Letts., 78, (2008) 890-895
5. Lehman, E.L., *Some concepts of dependence*, Ann. Math. Statist., 37, (1966) 1137-1153
6. Loe've, M., *Probability Theory I*, 4th. ed., Springer-Velag, New York (1977)
7. Matula, P., *A note on the almost sure convergence of sums of negatively dependent random variables*, Statist. Prob. Lett., 15, (1992) 209-213
8. Petrov, V.V., *A note on the Borel-Cantelli lemma*, Statist. Prob. Lett., 58, (2002) 283-286

Department of Mathematics Education Daebul University, Jeonnam 526-720, Korea
e-mail : kimhc@mail.daebul.ac.kr