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QUOTIENT SUBSTRUCTURES OF R-GROUPS

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ABSTRACT. Throughout this paper, we denote that R is a (right) near-ring and G an R-group. We will derive some properties of substructures and quotient substructures of R and G.

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1. Introduction

A (right) near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $(a+b)c = ac + bc$ for all a, b, c in R. If R has a unity 1, then R is called *unitary*.

A (two sided) *ideal* of R is a subset I of R such that (i) $(I, +)$ is a normal subgroup of $(R, +)$, (ii) $a(I+b) - ab \subset I$ for all $a, b \in R$, (iii) $(I+a)b - ab \subset I$ for all $a, b \in R$, equivalently, $IR \subset I$. If I satisfies (i) and (ii) then it is called a *left ideal* of R. If I satisfies (i) and (iii) then it is called a *right ideal* of R.

We will use the following notations: Given a near-ring R, $R_0 = \{a \in R \mid a0 = 0\}$ which is called the *zero symmetric part* of R, $R_c = \{a \in R \mid a0 = a\}$ which is called the *constant part* of R.

Obviously, we see that R_0 and R_c are subnear-rings of R. Thus a near-ring R is called *zero symmetric*, in case $R = R_0$ also, in case $R = R_c$, R is called *constant*. From the Pierce decomposition theorem, we obtain that $R = R_0 \oplus R_c$ as additive groups. So every element $a \in R$ has a unique representation of the form $a = b + c$, where $b \in R_0$ and $c \in R_c$.

Let $(G, +)$ be a group (not necessarily abelian) and denote the set

$$M(G) := \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G.

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Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if (i) $\theta(a+b) = \theta a + \theta b$, (ii) $\theta(ab) = \theta a \theta b$. We can replace homomorphism by monomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as in rings ([1]).

A group G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a *representation of* R on G, we write that rx (left scalar multiplication by R) instead of $(\theta r)x$ for all $r \in R$ and $x \in G$. If R is unitary and $\theta 1 = 1_G$, then R-group G is called *unitary*. That is, an R-group is an additive group G with a left scalar multiplication satisfying (i) $(a+b)x = ax + bx$, (ii) $(ab)x = a(bx)$, for all a, $b \in R$ and $x \in G$ and (iii) $1x = x$ (if R has a unity 1).

A representation θ of R on G is called $faithful$ if $Ker\theta = \{0\}$, that is, $rx = o$ implies that $r = 0$. In this case, also we say that G is a *faithful R-group* or R *acts faithfully on* G.

For the remainder concepts and results on near-rings, we refer to [5] and [6].

2. Some properties of quotient substructures in R-groups

Let R be a near-ring and let G be an R-group. If there exists x in G such that $G = Rx$, that is,

$$G = \{rx \mid r \in R\},$$

then G is called a *monogenic R-group* and the element x is called a *generator* of G.

We say that a near-ring R has the *insertion of factors property* (briefly, *IFP*) provided that for all a, b, x in R with $ab = 0$ implies $axb = 0$.

For an R-group G, a non-empty subset X of G such that $RX \subset X$ is called an *R-subset* of G, a subgroup T of G such that $RT \subset T$ is called an *R-subgroup* of G, a normal subgroup N of G such that $RN \subset N$ is called a *normal R-subgroup* of G and an *R-ideal* of G is a normal subgroup N of G such that

$$a(N + x) - ax \subset N$$

for all $x \in G$, $a \in R$. Note that every R-ideal of R is a left ideal of R, here, R is considered as an R-group.

Now, we consider the following quotient substructures of R and G, also relations between them.

Let G be an R-group and X and Y be non-empty subsets of G. We can define the following.

$$(X : Y) := \{a \in R \mid aX \subset Y\}.$$

We abbreviate that for $x \in G$, $(\{x\} : Y) =: (x : Y)$. Similarly for $(Y : x)$.

$(X : o)$ is called the *annihilator* of X, denoted it by $Ann(X)$. We note that G is a faithful R-group if $(G : o) = \{0\}$, that is, $Ann(G) = \{0\}$.

Also, a subgroup H of G such that $xa \in H$ for all $x \in H, a \in R$, is an *R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that

$$(x+g)a - ga \in N$$

for all $g \in G, x \in N$ *and* $a \in R$ ([5]).

In the above notation, note that if Y is a subgroup (normal subgroup, R-subgroup, ideal) of G, then so is $(X : Y)$ in R as an R-group. Moreover, we have the following simple statements.

Theorem 2.1. *Let G be an R-group and K_1 and K_2 non-empty subsets of G. Then we have the following conditions:*

(1) *If K_2 is a normal R-subgroup of G, then $(K_1 : K_2)$ is a normal left R-subgroup of R.*
(2) *If K_1 is a R-subset of G and K_2 is an R-subgroup of G, then $(K_1 : K_2)$ is a two-sided R-subgroup of R..*
(3) *If K_1 is an R-subset of G and K_2 is an R-ideal of G, then $(K_1 : K_2)$ is a two-sided ideal of R.*

Proof. (1) and (2) are easily proved by simple calculation.

Now, we prove only (3) : Using the condition (1), $(K_1 : K_2)$ is a normal subgroup of R. Let $a \in (K_1 : K_2)$ and $r \in R$. Then

$$(ar)K_1 = a(rK_1) \subset aK_1 \subset K_2,$$

because K_1 is an R-subset of G, so that $ar \in (K_1 : K_2)$. Whence $(K_1 : K_2)$ is a right ideal of R.

Next, let $r_1, r_2 \in R$ and $a \in (K_1 : K_2)$. Then

$$\{r_1(a + r_2) - r_1 r_2\}k = r_1(ak + r_2 k)r_2 - r_1 r_2 k \in K_2$$

for all $k \in K_1$, since $aK_1 \subset K_2$ and K_2 is an ideal of G. Thus $(K_1 : K_2)$ is a left ideal of R. Therefore $(K_1 : K_2)$ is a two-sided ideal of R.

Corollary 2.2. *Let R be a near-ring and G an R-group.*

(1) *([5]) For any $x \in G$, $Ann(x)$ is a left ideal of R.*
(2) *([5]) For any R-subgroup K of G, $Ann(K)$ is a two-sided ideal of R.*
(3) *For any R-subset K of G, $Ann(K)$ is a two-sided ideal of R.*
(4) *For any non-empty subset K of G, $Ann(K) = \bigcap_{x \in K} Ann(x)$.*

Lemma 2.3. [6]

(1) *Let I be a two-sided ideal of a near-ring R. Then the canonical map $\pi : R \to R/I$ defined by $a \rightsquigarrow a + I$ is a near-ring epimorphism. So R/I is a homomorphic image of R, and $ker\pi = I$.*
(2) *Let the map $\phi : R \to S$ be a near-ring epimorphism. Then $ker\phi$ is a two-sided ideal of R and $R/ker\phi \cong S$.*

Theorem 2.4. *Let R be a near-ring and G an R-group. Then we have the following conditions:*

(1) *$Ann(G)$ is a two-sided ideal of R. Moreover G is a faithful $R/Ann(G)$-group.*

(2) *For any $x \in G$, we get $Rx \cong R/Ann(x)$ as R-groups.*

Proof. (1) By Corollary 2.2 and Theorem 2.1, $Ann(G)$ is a two-sided ideal of R.

We now make G an $R/Ann(G)$-group by defining, for all$r \in R$ *and* $r + Ann(G) \in R/Ann(G)$, the action $(r + Ann(G))x = rx$. If $r + Ann(G) = r' + Ann(G)$, then $-r' + r \in Ann(G)$ hence $(-r' + r)x = o$ for all x in G, that is to say, $rx = r'x$. This tells us that

$$(r + Ann(G))x \;=\; rx \;=\; r'x \;= (r' + Ann(G))x;$$

thus the action of $R/Ann(G)$ on G has been shown to be well defined.

The verification of the structure of an $R/Ann(G)$-group is a routine triviality. Finally, to see that G is a faithful $R/Ann(G)$-group, we note that if $(r + Ann(G))x = 0$ for all $x \in G$, then by the definition of $R/Ann(G)$-group structure, we have $rx = 0$. Hence $r \in Ann(G)$. This says that only the zero element of $R/Ann(G)$ annihilates all of G. Thus G is a faithful $R/Ann(G)$-group.

(2) For any $x \in G$, clearly Rx is an R-subgroup of G. The map $\phi : R \longrightarrow Rx$ defined by $\phi(r) = rx$ is an R-epimorphism, so that from the homomorphism theorems Lemma 2.3, since the kernel of ϕ is $Ann(x)$, we deduce that $Rx \cong R/Ann(x)$ as R-groups.

Corollary 2.5. *Let G be a monogenic R-group with x as a generator. Then we have the isomorphic relation as $G \cong R/Ann(x)$.*

Theorem 2.6. *If R is a near-ring and G an R-group, then $R/Ann(G)$ is embedded in a near-ring $M(G)$.*

Proof. Let $a \in R$. We define $\tau_a : G \longrightarrow G$ by $\tau_a x = ax$ for each $x \in G$. Then τ_a is in $M(G)$. Consider the mapping $\phi : R \longrightarrow M(G)$ defined by $\phi(a) = \tau_a$. Then obviously, we see that

$$\phi(a + b) = \phi(a) + \phi(b) \text{ and } \phi(ab) = \phi(a)\phi(b),$$

that is, ϕ is a near-ring homomorphism from R to $M(G)$.

Next, we must show that $Ker\phi = Ann(G)$: Indeed, if $a \in Ker\phi$, then $\tau_a = O$ (zero map), which implies that $aG = G\tau_a = o$, that is, $a \in Ann(G)$. On the other hand, if $a \in Ann(G)$, then by the definition of $Ann(G)$, $aG = o$ hence $O = \tau_a = \phi(a)$, this implies that $a \in Ker\phi$. Therefore from the homomorphism theorems Lemma 2.3 on $R-$ groups, the image of R is a near-ring isomorphic to $R/Ann(G)$. Consequently, $R/Ann(G)$ is isomorphic to a subnear-ring of $M(G)$.

From Theorem 2.1, and Corollary 2.2, we have the following important conditions.

Theorem 2.7. *Let R be a near-ring. Then the following conditions are equivalent:*

(1) *R has the IFP.*
(2) *For any $a \in R$, $Ann(a)$ is an ideal of R.*
(3) *For any non-empty subset S of R, $Ann(S)$ is an ideal of R.*

Proof. (1) $\Rightarrow$ (2). Let $x \in Ann(a)$. Then $xa = 0$, by definition. Since R has the IFP, $xra = 0$ for each $r \in R$, that is, $xr \in Ann(a)$. Hence $Ann(a)$ is a right ideal of R. On the other hand, by Corollary 2.2 (1), $Ann(a)$ is a left ideal of R. Consequently, $Ann(a)$ is an ideal of R.

(2) $\Rightarrow$ (3). Assume the condition (2). Then because of $Ann(a)$ is an ideal of R, for any $a \in R$, and by Corollary 2.2 (4), $Ann(S) = \bigcap_{a \in S} Ann(a)$, obviously, $Ann(S)$ is an ideal of R.

(3) $\Rightarrow$ (1). Suppose that the condition (3) is true, and $ab = 0$ for a, b in R. Then we see that $a \in Ann(b)$. From the condition (3), $Ann(b)$ is an ideal of R, so that $ax \in Ann(b)$, for any $x \in R$. That is $axb = 0$, for any $x \in R$. Therefore R has the IFP.

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