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FRACTIONAL EULER'S INTEGRAL OF FIRST AND SECOND KINDS. APPLICATION TO FRACTIONAL HERMITE'S POLYNOMIALS AND TO PROBABILITY DENSITY OF FRACTIONAL ORDER

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ABSTRACT. One can construct a theory of probability of fractional order in which the exponential function is replaced by the Mittag-Leffler function. In this framework, it seems of interest to generalize some useful classical mathematical tools, so that they are more suitable in fractional calculus. After a short background on fractional calculus based on modified Riemann Liouville derivative, one summarizes some definitions on probability density of fractional order (for the motive), and then one introduces successively fractional Euler's integrals (first and second kind) and fractional Hermite polynomials. Some properties of the Gaussian density of fractional order are exhibited. The fractional probability so introduced exhibits some relations with quantum probability

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1. Introduction

Fractional calculus is getting more audience among scientists for its interest by itself, but also, and mainly because it provides an alternative to describe hysical phenomena involving coarse-grained space, apart from being of prospect use in the modeling of fractional Brownian motion. There aretwo kinds of fractional calculus: a fractional calculus for differentiable functions, and a fractional calculus for functions which are continuous but non-differentiable. In this framework,recently we have proposed a slight modification of the Riemann-Liouville fractional derivative of which the main features can be summarized as follows [6].

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The fractional derivative of a constant is zero. It is different from Caputo-Djrbashian definition [2,3] in the sense that the latter refers to the derivative of the function under consideration whilst our definition does not.

Our definition provides an expression for Fractional Taylor's series which allows us to recover the series by the means of which the Mittag Leffler function is defined. And with this definition, the Littag-Leffler function $E_\alpha(x^\alpha)$turns to be exactly the solution of the solution of the fractional differential equation

$$y^{(\alpha)}(x) = \lambda y(x)$$

where λ is a real valued constant parameter such that $0 < \lambda < 1$.

In this framework, in quite a natural way, we have been led to generalize probability by introducing the fractional counterpart of probability density, referred to as probability ensity of fractional order. The basic diffusion equation which reads

$$\partial_t p(x,t) = -\partial_x (fp) + (1/2)\partial^2_{xx}(g^2 p)$$

has been generalized in various ways to study porous media, by introducing fractional derivative w.r.t. [1]. For instance, one can find the equation

$$\partial_t p(x,t) = -\partial^\alpha_x (fp) + (1/2)\partial^{2\alpha}_{xx}(g^2 p), \qquad 0 < \alpha < 1.$$

To the best of our knowledge, these equations have been introduced more or less formally, and in an attempt to provide a sound support to their derivation, we have proposed recently a new concept of probability density of fractional order which is fully consistent with the fractional calculus based on the modified Riemann-Liouville as we have introduced it [7].

It appears that this fractional probability density is more or less related with signed measure of probability on the one hand [5], and quantum probability on the other hand, and the purpose of the present paper is to comment on this point. Should this connection be soundly established, then one would be in a position to consider quantum probability with new points of view.

The paper is organized as follows. For the convenience of the reader we think that it is necessary to give a short background on the calculus of fractional order with the modified Riemann-Liouville derivative which we have had to introduce in order to cope with some pitfalls (Section 2), and later we shall bear in mind the essential of probability density of fractional order (Section 3). Then we shall define successively fractional Euler's integral of the first kind (Section 4) and of the second kind (Section 5) and fractional Hermite's polynomials (Section 6). And then, to conclude, we shall show how one can use Mittag-Leffler function to define a somewhat Gaussian probability density of fractional order (Section 7).

The present article can be thought of as the continuation of the Ref [8] in which we introduced Fourier's transformation of fractional order.

2. Background on fractional calculus

2.1. Fractional derivative via fractional difference

Definition 2.1. Let $f : \Re \longrightarrow \Re, x \longrightarrow f(x)$, denote a continuous (but not necessarily differentiable) function, and let denote a constant discretization span. Define the forward operator by the equality (the symbol := means that the left side is defined by the right side)

$$FW(h)f(x) := f(x+h); \tag{1}$$

then the fractional difference of order α, $0 < \alpha < 1$, of $f(x)$ is defined by the expression [5,6].

$$\Delta^{\alpha} f(x) := (FW-1)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f\ [x + (\alpha - k)\, h] \tag{2}$$

and its fractional derivative of order is defined by the limit

$$f^{(\alpha)}(x) = \lim_{h \downarrow 0} \frac{\Delta^{\alpha} f(x)}{h^{\alpha}}. \tag{3}$$

This definition is close to the standard definition of derivative (calculus for beginners), and as a direct result, the -th derivative of a constant is zero.

2.2 Modified fractional Riemann-Liouville derivative (via integral)
An alternative to the Riemann-Liouville definition of fractional derivative

In order to circumvent some drawbacks involved in the classical Riemann-Liouville definition, we have proposed the following alternative to the Riemann-Liouville definition of F-derivative, which is moreover fully supported by the definition 2.1.

Corollary 2.1 *(Riemann-Liouville definition revisited). As a direct result of the definition 2.1, the fractional derivative of a function $f(x)$ can be obtained as follows*

(i) Assume that $f(x)$ is a constant K . Then its fractional derivative of order is

$$D_x^{\alpha} K = \frac{K}{\Gamma(1-\alpha)} x^{-\alpha}, \quad \alpha \leq 0, \tag{4}$$

$$= 0, \quad \alpha > 0 \tag{5}$$

(ii) When $f(x)$ is not a constant, then one will set

$$f(x) = f(0) + (f(x) - f(0)),$$

and its fractional derivative will be defined by the expression

$$f^{(\alpha)}(x) = D_x^{\alpha} f(0) + D_x^{\alpha} (f(x) - f(0)),$$

in which, for negative α , one has

$$D_x^{\alpha} (f(x) - f(0)) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} f(\xi) d\xi, \quad \alpha < 0, \tag{6}$$

whilst for positive α , one will set

$$D_x^{\alpha} (f(x) - f(0)) = D_x^{\alpha} f(x) = D\left(D^{\alpha-1} f(x)\right), \quad 0 < \alpha < 1,$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} \left(f(\xi) - f(0)\right) d\xi, \tag{7}$$

When $n < \alpha \leq n+1$, *one will set*

$$f^{(\alpha)}(x) := \left(f^{(\alpha-n)}(x)\right)^{(n)} \quad , \; n < \alpha \leq n+1 \; , \; n \geq 1. \tag{8}$$

We shall refer to this fractional derivative as to the modified Riemann Liouville derivative.

Remark that the definition expressed by the equations (7) and (8) is different from the expression [2,3], i.e.,

$$f_C^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha} f'(\xi) d\xi$$

which in substance, says that the function under consideration should be differentiable. Both this definition above and the equation (7) provide zero for the fractional derivative of a constant.

For different points of view on fractional derivative, see for instance [10,11,12].

2.3 Fractional Taylor's series for one-variable functions

A generalized Taylor expansion of fractional order which applies to non-differentiable functions reads as follows [6,7].

Proposition 2.1 *Assume that the continuous function* $f : \Re \longrightarrow \Re, x \longrightarrow f(x)$ *has fractional derivative of order* $k\alpha$, *for a given* α, $0 < \alpha < 1$, *and any positive integer* k; *then the following equality holds, which reads*

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(1+\alpha k)} f^{(\alpha k)}(x), \; 0 < \alpha < 1 \tag{9}$$

where $f^{(\alpha k)}(x)$ *is the derivative of order of* $f(x)$ *in the sense that* $D^{\alpha k} = D^{\alpha} D^{\alpha} ... D^{\alpha}$ *k times.*

With the notation $\Gamma(1+\alpha k) =: (\alpha k)!$, one has the formula

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(x), \; 0 < \alpha < 1, \tag{10}$$

which looks like the classical one.

Alternatively, in a more compact form, one can write

$$f(x+h) = E_{\alpha}(h^{\alpha} D_x^{\alpha}) f(x),$$

where D_x is the derivative operator with respect to x and $E_{\alpha}(y)$ denotes the Mittag-Leffler function defined by the expression

$$E_{\alpha}(y) := \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(1+\alpha k)}. \tag{11}$$

Mc-Laurin series of fractional order

Let us make the substitution and into (9), we so obtain the fractional Mc-Laurin series

$$f(x)=\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma((1+\alpha k)}f^{(\alpha k)}(0),\ 0<\alpha<1. \tag{12}$$

A simple rationale to take this formula for granted is to notice that it holds for the Mittag-Leffler function and then to consider those functions which can be approximated by sequences of Mittag-Leffler functions.

2.4 Some useful relations. The equation (9) provides the useful relation

$$d^{\alpha}f\approx\Gamma(1+\alpha)df,\quad 0<\alpha<1, \tag{13}$$

or in a finite difference form $\Delta^{\alpha}f\approx\Gamma(1+\alpha)\Delta f$.

Corollary 2.1 *The following equalities hold, which are*

$$D^{\alpha}x^{\gamma}=\Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha)x^{\gamma-\alpha},\quad \gamma>0, \tag{14}$$

or, what amounts to the same (we set $\alpha=n+\theta$ *)*

$$D^{n+\theta}x^{\gamma}=\Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-n-\theta)x^{\gamma-n-\theta},\quad 0<\theta<1, \tag{15}$$

$$(u(x)v(x))^{(\alpha)}=u^{(\alpha)}(x)v(x)+u(x)v^{(\alpha)}(x), \tag{16}$$

$$(f\left[u(x)\right])^{(\alpha)}=f'_{u}(u)u^{(\alpha)}(x), \tag{17}$$

$$(f\left[u(x)\right])^{(\alpha)}=f_{u}^{(\alpha)}(u)(u'_{x})^{\alpha}. \tag{18}$$

$u(x)$ is non-differentiable in (14) and (15) and differentiable in (16), is non-differentiable in (14), and is differentiable in (15) and non-differentiable in (16).

Corollary 2.2 *Leibniz chain derivative rule for fractional derivative. Assume that $f(x)$ and $x(t)$ are two $\Re\rightarrow\Re$ function which both have derivatives of order α, $0<\alpha<1$, then one has the chain rule*

$$f_{x}^{(\alpha)}(x(t))=\Gamma(2-\alpha)x^{\alpha-1}f_{x}^{(\alpha)}(x)x^{(\alpha)}(t). \tag{19}$$

Proof. One has the equality

$$\frac{d^{\alpha}f(x(t))}{dt^{\alpha}}=\frac{d^{\alpha}f(x)}{dx^{\alpha}}\left(\frac{dx}{dt}\right)^{\alpha},$$

but the derivative

$$\frac{d^{\alpha}x}{dx^{\alpha}}=\frac{1}{(1-\alpha)!}x^{1-\alpha}$$

yields

$$(dx)^{\alpha}=(1-\alpha)!x^{\alpha-1}d^{\alpha}x,$$

therefore the result.

2.5 Fractional integration with respect to $(dx)^{\alpha}$

The integral with respect to $(dx)^{\alpha}$ is defined as the solution of the fractional differential equation

$$dy=f(x)(dx)^{\alpha},\quad x\succeq 0,\quad y(0)=0 \tag{20}$$

which is provided by the following result:

Lemma 2.1 *Let $f(x)$ denote a continuous function, then the solution $y(x)$, $y(0)=0$, of the equation ((18)) is defined by the equality*

$$y=\int_0^x f(\xi)(d\xi)^\alpha=\alpha\int_0^x(x-\xi)^{\alpha-1}f(\xi)d\xi,\quad 0<\alpha\le 1. \tag{21}$$

Proof. On multiplying both sides of (2.18) by $\alpha!$, and on taking account of (2.12), we have the equality $y^{(\alpha)}(x)=\alpha!f(x)$ which provides

$$y(x)=\alpha!D^{-\alpha}f(x),=\frac{\alpha!}{\Gamma(\alpha)}\int_0^x(x-\xi)^{\alpha-1}f(\xi)d\xi. \tag{22}$$

Definition 2.1 Framework lemma 2.1. On assuming that $y(-\infty)=0$, we shall write

$$y(x)=\int_{-\infty}^x f(\xi)(d\xi)^\alpha=\alpha\int_{-\infty}^x(x-\xi)^{\alpha-1}f(\xi)d\xi. \tag{23}$$

For the motivation of this definition, see [6,7].

The fractional integration by part formula reads

$$\int_a^b u^{(\alpha)}(x)v(x)(dx)^\alpha=\alpha!\left[u(x)v(x)\right]_a^b-\int_a^b u(x)v^{(\alpha)}(x)(dx)^\alpha \tag{24}$$

and it can be obtained easily by combining (14) with (19).

Remark that one has the equality

$$D^{-\alpha}f(x)=\frac{1}{\alpha!}\int_0^x f(\xi)(d\xi)^\alpha. \tag{25}$$

All this material is necessary to fully expand the fractional probability calculus outlined below. For other points of view on fractional calculus, see for instance [4,9,10,11,12].

2.6 Transformation of variables in integrals of fractional order

One-dimensional integral

Assume that we make the transformation $x=u(t)$ in the integral

$$I=\frac{1}{\alpha!}\int_a^b f(x)(dx)^\alpha,\quad 0<\alpha<1; \tag{26}$$

then two instances must be taken into account depending upon whether $u(t)$ is differentiable or not. In the first case, one has $dx=u'(t)dt$ and inserting into (2.24) yields

$$I=\int_{u^{-1}(a)}^{u^{-1}(b)} f\left(u(t)\right)\left(u'(t)\right)^\alpha(dt)^\alpha. \tag{27}$$

In contrast, assume now that $u(t)$ is not differentiable but has a fractional derivative of order β, $0<\beta<1$. Then applying the fractional Taylor's series to yields $\beta! du = u^{(\beta)}(t)(dt)^\beta$, therefore (after substituting into (24))

$$I = (\beta!)^{-\alpha} \int_{u^{-1}(a)}^{u^{-1}(b)} f\left(u(t)\right) \left(u^{(\beta)}(t)\right)^\alpha (dt)^{\alpha\beta}. \tag{28}$$

Two-dimensional integral

The same rational holds for the integral

$$J = \int_D f(x,y)(dx)^\alpha (dy)^\beta, \tag{29}$$

with respect to the variable transformation $x = u(t,\tau)$ and $.y = v(t,\tau)$. If the transformation is differentiable, then one has

$$J = \int_D f(u,v) \left| \frac{\partial(u,v)}{\partial(t,\tau)} \right|^\alpha (dt)^\alpha (d\tau)^\beta. \tag{30}$$

Otherwise, assuming that $u(t,\tau)$ and $v(t,\tau)$ have fractional derivatives of order β only, we shall introduce the Jacobian determinant of order defined β by the expression

$$\frac{\partial^\beta(u,v)}{\partial(t,\tau)^\beta} = u_t^{(\beta)} v_\tau^{(\beta)} - u_\tau^{(\beta)} v_t^{(\beta)}, \tag{31}$$

to have

$$J = (\beta!)^{-2\alpha} \int_D f(u,v) \left| \frac{\partial^\beta(u,v)}{\partial(t,\tau)^\beta} \right|^\alpha (dt)^{\alpha\beta} (d\tau)^{\alpha\beta}. \tag{32}$$

In the next section on fractional probability density could be dropped in a first reading, but in fact, it displays the main motivation of the results contained in the paper.

3. Probability density of fractional order

3.1 One dimensional fractional probability density

Main equations

Definition 3.1 Let X denote a real-valued random variable defined on the interval $[a,b]$ and let $p_\alpha(x)$, $p_\alpha(x) \geqslant 0$, denote a positive function also defined on $[a,b]$. X is referred to as a random variable of fractional order α, $0 < \alpha < 1$, with the probability $p_\alpha(x)$, whenever for any (x',x), $a \leqslant x' < X < x \leqslant b$, one has

$$F(x',x) = \Pr\{x' < X \leqslant x\} := \frac{1}{\Gamma(1+\alpha)} \int_{x'}^{x} p_\alpha(\xi)(d\xi)^\alpha \tag{33}$$

with the normalizing condition

$$F(a,b) = 1. \tag{34}$$

Remark that this definition is quite different from Levy's [9], for instance, and others, which are rather related to the Kolmogorov's entropy defined on dynamical systems.

The coefficient $\Gamma^{-1}(1+\alpha)$ has been introduced in (31) to preserve (via (25)), the standard relation $F'(x) = p(x)$, see (3.5); but it could be dropped at first glance, and this will be a matter for further discussion.

According to (20), if we denote by $p(x)$ the corresponding probability density of M, in the usual sense of this term, one has the identit

$$p(x) = \Gamma^{-1}(\alpha)(a-x)^{\alpha-1}p_\alpha(x), \quad x \leqslant a. \tag{35}$$

In other words, the fractional probability density $p_\alpha(x)$ can be thought of as defining a family of probability density functions $p(x)$.

$F(x', x)$ as so defined is a generalization of the (cumulative) distribution $F(x)$, and is introduced, because here one has $F(a,b) + F(b,c) \neq F(a,c)$, $a < b < c$, More precisely, by using the equation (19) one can check that

$$F(a,c) \leqslant F(a,b) + F(b,c), \quad a < b < c. \tag{36}$$

In addition, the relation between $F(x', x)$ and $p_\alpha(x)$ is provided by the equality

$$\frac{\partial^\alpha F(x', x)}{\partial x^\alpha} = p_\alpha(x). \tag{37}$$

Example 3.1 For a uniform random variable on the interval $[a, c]$, one obtains the expressions $p_\alpha(x) = (c-a)^{-\alpha}$, and

$$F(x', x) = (x - x')^\alpha/(c-a)^\alpha,$$

which provides

$$F^{1/\alpha}(a,c) = F^{1/\alpha}(a,b) + F^{1/\alpha}(b,c), \quad a < b < c, \tag{38}$$

or in a like manner

$$\Pr(A \cup B)^{1/\alpha} = \Pr(A)^{1/\alpha} + \Pr(B)^{1/\alpha} - \Pr(A \cap B)^{1/\alpha}. \tag{39}$$

3.2 Characteristic function of fractional order. In probability theory, on assuming that a random variable is completely defined by its moments $\langle X^n \rangle$ (and this is generally the case except in some theoretical pathological cases) one is used to introduce the characteristic function

$$\Phi(u) = \left\langle e^{iuX} \right\rangle, \quad u \in \Re, \tag{40}$$

or, sometimes, the generating function $\Phi(u/i)$. Here, in quite a natural way, we shall consider the moments

$$m_{n\alpha} := \langle X^{n\alpha} \rangle_\alpha = \int_\Re x^{n\alpha} p_\alpha(x)(dx)^\alpha, \tag{41}$$

which suggests to introduce a fractional characteristic function either in the form

$$\langle E_\alpha(iuX^\alpha) \rangle = \sum_{n=0}^{\infty} i^n \frac{u^n}{(n\alpha)!} m_{n\alpha}, \tag{42}$$

or

$$\langle E_\alpha(iu^\alpha X^\alpha) \rangle = \sum_{n=0}^{\infty} i^n \frac{u^{n\alpha}}{(n\alpha)!} m_{n\alpha}. \tag{43}$$

For the sake of physical dimension, we shall rather select the second equation and we shall refer to

$$\Phi_\alpha(u) := \langle E_\alpha(iu^\alpha X^\alpha) \rangle \tag{44}$$

One can show that the corresponding inversion theorem is provided by the expression

$$p_\alpha(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-\infty}^{+\infty} E_\alpha\left(i(-ux)^\alpha\right) \Phi_\alpha(u)(du)^\alpha, \tag{45}$$

where M_α is the period of $E_\alpha(ix^\alpha)$ defined by the equation $E_\alpha\left(i(M_\alpha)^\alpha\right) = 1$.

4. Fractional Euler's integral of the first kind

Definition 4.1 We suggest to generalize the Euler's gamma function $\Gamma(x)$ in the form

$$\Gamma_\alpha(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha\left(-t^\alpha\right) t^{\alpha(x-1)} (dt)^\alpha \ , \quad 0 < \alpha < 1 \tag{46}$$

On making the variable transformation $t = r^2$ in this integral, and using (25), one can re-write $\Gamma_\alpha(x)$ in the form

$$\Gamma_\alpha(x) = \frac{2^\alpha}{\alpha!} \int_0^\infty E_\alpha\left(-r^{2\alpha}\right) t^{\alpha(2x-1)} (dr)^\alpha \ , \quad 0 < \alpha < 1. \tag{47}$$

This being the case an integration by part directly derived from the chain rule (14) provides

$$\Gamma_\alpha(x+1) = (\alpha!) x \Gamma_\alpha(x). \tag{48}$$

Assume that x is an integer n, then, according to (53), one obtains

$$\Gamma_\alpha(n+1) = (\alpha!)^n n! \Gamma_\alpha(1). \tag{49}$$

This being the case, according to (23),one has merely

$$\Gamma_\alpha(1) = \left[D^{-\alpha} E_\alpha\left(-x^\alpha\right)\right]_0^\infty = \left[E_\alpha\left(-x^\alpha\right)\right]_0^\infty = 1,$$

therefore the equality

$$\Gamma_\alpha(n+1) = (\alpha!)^n n!. \tag{50}$$

Explaining (55) yields

$$\Gamma_\alpha(n+1) = n\alpha!(n\alpha! - \alpha!)(n\alpha! - 2\alpha!)...(3\alpha!)(2\alpha!)(\alpha!).$$

in other words everything happens as if $\alpha!$ were substituted for the unit in the definition of the factorial.

5. Fractional Euler's integral of the second kind

Definition 5.1. The fractional Euler's function of second kind is defined by the expression

$$B_\alpha(x,y) = \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{\alpha(x-1)} (1-t)^{\alpha(y-1)} (dt)^\alpha. \tag{51}$$

On making the transformation we can write as well

$$B_\alpha(x,y)=\frac{2^\alpha}{\alpha!}\int_0^1 u^{2\alpha(x-1)}(1-u^2)^{\alpha(y-1)}u^\alpha(du)^\alpha, \tag{52}$$

and the new change of variable $u=\sin\theta$ yields

$$B_\alpha(x,y)=\frac{2^\alpha}{\alpha!}\int_0^{\pi/2}(\sin\theta)^{\alpha(2x-1)}(\cos\theta)^{\alpha(2y-1)}(d\theta)^\alpha. \tag{53}$$

Proposition 5.1 *The following relation holds, which is*

$$B_\alpha(x,y)=\frac{\Gamma_\alpha(x)\Gamma_\alpha(y)}{\Gamma_\alpha(x+y)}. \tag{54}$$

Proof. Using (4.2), one writes

$$\Gamma_\alpha(x)\Gamma_\alpha(y)=\frac{2^{2\alpha}}{(\alpha!)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}E_\alpha\left[-\left(u^2+v^2\right)^\alpha\right]\ u^{2\alpha x-\alpha}v^{2\alpha y-\alpha}(du)^\alpha(dv)^\alpha$$

$$=\frac{2^{2\alpha}}{4(\alpha!)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}E_\alpha\left[-\left(u^2+v^2\right)^\alpha\right]\ |u|^{2\alpha x-\alpha}|v|^{2\alpha y-\alpha}(du)^\alpha(dv)^\alpha. \tag{55}$$

Transforming to polar coordinates with $u=r\cos\theta$ and $v=r\sin\theta$, we obtain

$$\Gamma_\alpha(x)\Gamma_\alpha(y)=\frac{2^{2\alpha}}{4(\alpha!)^2}\int_0^\infty E_\alpha(-r^{2\alpha})r^{2\alpha(x+y)-\alpha}(dr)^\alpha\int_0^{2\pi}\left|\cos^{2\alpha x-\alpha}\theta\sin^{2\alpha y-\alpha}\theta\right|(d\theta)^\alpha$$

$$=\frac{2^{2\alpha}}{(\alpha!)^2}\int_0^\infty E_\alpha(-r^{2\alpha})r^{2\alpha(x+y)-\alpha}(dr)^\alpha\int_0^{\pi/2}\cos^{2\alpha x-\alpha}\theta\sin^{2\alpha y-\alpha}\theta(d\theta)^\alpha$$

$$=\Gamma_\alpha(x+y)B_\alpha(x,y).$$

6. Hermite polynomials of fractional order

6.1 Definition and main properties

Definition 6.1 The Hermite polynomials of order α are defined by the equation

$$\widetilde{H}_n(x)(x)=(-1)^nE_\alpha(x^{2\alpha})\frac{d^{n\alpha}}{dx^{n\alpha}}E_\alpha(-x^{2\alpha}),\quad 0<\alpha<1, \tag{56}$$

where n denotes a positive integer.

First polynomials

$$\widetilde{H}_0(x)=1, \tag{57}$$

$$\widetilde{H}_1(x)=(2x)^\alpha, \tag{58}$$

$$\widetilde{H}_2(x)=(2x)^{2\alpha}-2(\alpha!), \tag{59}$$

$$\widetilde{H}_3(x) = (2x)^{3\alpha} - 2\left((\alpha!) + \frac{(2\alpha)!}{(\alpha)!}\right)(2x)^{\alpha}, \tag{60}$$

$$\widetilde{H}_4(x) = (2x)^{4\alpha} - 2\left((\alpha!) + \frac{(2\alpha)!}{(\alpha)!} + \frac{(3\alpha)!}{(2\alpha)!}\right)(2x)^{2\alpha} + 2^2\,\frac{\alpha!(3\alpha)!}{(2\alpha)!}, \tag{61}$$

$$\widetilde{H}_5(x) = (2x)^{5\alpha} - 2\left(\sum_{k=1}^{4}\frac{(k\alpha)!}{(k\alpha-\alpha)!}\right)(2x)^{3\alpha}$$

$$+2^2\left(\frac{(4\alpha)!}{(3\alpha)!}\left(\alpha! + \frac{(2\alpha)!}{\alpha!}\right) + \frac{\alpha!(3\alpha)!}{(2\alpha)!}\right)(2x)^{\alpha}. \tag{62}$$

Clearly $\widetilde{H}_5(x)$ is a polynomial with respect to the variable x^{α}.

Generating function

Lemma 6.1 *Assume that* $\alpha = 1/(2k+1)$, *then the generating function*

$$\widetilde{G}(x,t) := \sum_{n=0}^{\infty}\frac{t^{n\alpha}}{(n\alpha)!}\widetilde{H}_n(x)$$

is provided by the expression

$$\widetilde{G}(x,t) = E_{\alpha}\left((2xt - t^2)^{\alpha}\right). \tag{63}$$

Proof. Using the definition (61) one can write

$$\begin{aligned}
\widetilde{G}(x,t) &= E_{\alpha}(x^{2\alpha})\sum_{n=0}^{\infty}(-1)^n\frac{t^{n\alpha}}{(n\alpha)!}D^{n\alpha}E_{\alpha}(-x^{2\alpha}) \\
&= E_{\alpha}(x^{2\alpha})\sum_{n=0}^{\infty}\frac{(-t)^{n\alpha}}{(n\alpha)!}D^{n\alpha}E_{\alpha}(-x^{2\alpha}) \\
&= E_{\alpha}(x^{2\alpha})E_{\alpha}(-(x-t)^{2\alpha}) = E_{\alpha}\left((x^2-(x-t)^2)^{\alpha}\right),
\end{aligned}$$

therefore the result.

Note that in order to btain the above expression, we have used the basic property of the Mittag-Leffler function expressed by the equation (48)..

Recursion formulae.

Lemma 6.2 *Assume that* $\alpha = 1(2k+1)$, *then one has the recursion formula*

$$\widetilde{H}_n^{(\alpha)}(x) = 2\frac{(n\alpha)!}{(n\alpha-\alpha)!}\widetilde{H}_{n-1}(x). \tag{64}$$

Proof. The fractional derivative of the generating function provides

$$\frac{\partial^{\alpha}\widetilde{G}(x,t)}{\partial x^{\alpha}} = E_{\alpha}((2xt-t^2)^{\alpha})(2t)^{\alpha} = \sum_{n=0}^{\infty}\frac{t^{n\alpha}}{(n\alpha)!}\frac{d^{\alpha}\widetilde{H}_n(x)}{dx^{\alpha}} \tag{65}$$

On esplaining the Mittag-Leffler function by its series into (69) yields

$$\sum_{j=0}^{\infty} \frac{t^{j\alpha}}{(j\alpha)!} \frac{d^{\alpha}\widetilde{H}_j(x)}{dx^{\alpha}} = (2t^{\alpha}) \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{(n\alpha)!} \widetilde{H}_n(x) \tag{66}$$

and identifying the coefficients of $t^{n\alpha}$ provides the result.

Lemma 6.3 *Assume that* $\alpha = 1(2k+1)$; *then the following recursion formula holds,*

$$\widetilde{H}_{n+1}(x) = (2x)^{\alpha}\widetilde{H}_n(x) - 2\frac{(n\alpha)!}{(n\alpha - \alpha)!}\widetilde{H}_{n-1}(x). \tag{67}$$

Proof. On deriving the definition (61) of the fractional Hermite polynomial, we obtain

$$\begin{aligned} \widetilde{H}_n^{(\alpha)}(x) &= (-1)^n (2x)^{\alpha} E_{\alpha}(x^{2\alpha}) D^{n\alpha} E_{\alpha}(-x^{2\alpha}) - \widetilde{H}_{n+1}(x) \\ &= (2x)^{\alpha}\widetilde{H}_n(x) - \widetilde{H}_{n+1}(x) \end{aligned} \tag{68}$$

This being the case, according to (69) one has as well

$$\widetilde{H}_n^{(\alpha)}(x) = 2\frac{(n\alpha)!}{(n\alpha-\alpha)!}\widetilde{H}_{n-1}(x),.$$

and comparing with (72) provides the result.

Fractional differential equation

Lemma 6.4. *Assume that* $\alpha = 1/(2k+1)$. *Then* $\widetilde{H}_n(x)$ *is solution of the fractional differential equation*

$$\widetilde{H}_n^{(2\alpha)}(x) - (2x)^{\alpha}\widetilde{H}_n^{(\alpha)}(x) + \left(2\frac{(n\alpha+\alpha)!}{(n\alpha)!} - 2^{\alpha}\alpha!\right)\widetilde{H}_n(x) = 0. \tag{69}$$

Proof. We start from the equation (6.9) to write successively

$$\begin{aligned} \widetilde{H}_n^{(2\alpha)}(x) &= 2\frac{(n\alpha)!}{(n\alpha-\alpha)!}\widetilde{H}_{n-1}^{(\alpha)}(x) = D^{\alpha}((2x)^{\alpha}\widetilde{H}_n(x)) - \widetilde{H}_{n+1}^{(\alpha)}(x) \\ &= 2^{\alpha}\alpha!\widetilde{H}_n(x) + (2x)^{\alpha}\widetilde{H}_n^{(\alpha)}(x) - 2\frac{(n\alpha+\alpha)!}{(n\alpha)!}\widetilde{H}_n(x) \end{aligned}$$

therefore the result.

Orthogonality of fractional Hermite polynomials

Lemma 6.5 *The fractional Hermite polynomials are orthogonal on* $\Re$, *provided that* $\alpha = 1/(2k+1)$.

Proof. For $n \neq m$, one has successively

$$\begin{aligned} \left\langle \widetilde{H}_n, \widetilde{H}_m \right\rangle &:= \int_{\Re} E_{\alpha}(-x^{2\alpha})\widetilde{H}_n(x)\widetilde{H}_m(x)(dx)^{\alpha} \\ &:= (-1)^n \int_{\Re} \widetilde{H}_m(x) D^{n\alpha} E_{\alpha}(-x^{2\alpha})(dx)^{\alpha}. \end{aligned}$$

On integrating by parts, one has as well

$$\left\langle \widetilde{H}_n, \widetilde{H}_m \right\rangle = -(-1)^n \int_{\Re} \widetilde{H}_m^{(\alpha)}(x) D^{n\alpha-\alpha} E_\alpha(-x^{2\alpha})(dx)^\alpha$$

and, on taking account of (69),

$$\left\langle \widetilde{H}_n, \widetilde{H}_m \right\rangle = -(-1)^n 2 \frac{(m\alpha)!}{(m\alpha-\alpha)!} \int_{\Re} \widetilde{H}_{m-1}(x) D^{n\alpha-\alpha} E_\alpha(-x^{2\alpha})(dx)^\alpha$$

$$= (-1)^{n+s}\, 2^s \frac{\prod_{k=0}^{s-1}(m\alpha-k\alpha)!}{\prod_{k=1}^{s}(m\alpha-k\alpha)!} \int_{\Re} \widetilde{H}_{m-s}(x) D^{n\alpha-s\alpha} E_\alpha(-x^{2\alpha})(dx)^\alpha, \quad n \geqslant s$$

$$= (-1)^{n+s}\, 2^s \frac{(m\alpha)!}{(m\alpha-s\alpha)!} \int_{\Re} \widetilde{H}_{m-s}(x) D^{n\alpha-s\alpha} E_\alpha(-x^{2\alpha})(dx)^\alpha, \quad n \geqslant s. \quad (70)$$

(i) Assume that $s = n$, and having in mind that $\widetilde{H}_0(x) = 1$, we obtain (with the suitable constant K_m derived from (74))

$$\begin{aligned} \left\langle \widetilde{H}_n, \widetilde{H}_m \right\rangle &= K_m \int_{\Re} D^{(n-m+1)\alpha} E_\alpha(-x^{2\alpha})(dx)^\alpha, \\ &= K_m \alpha! \left[D^{(n-m)\alpha} E_\alpha(-x^{2\alpha}) \right]_{-\infty}^{+\infty} = 0. \end{aligned}$$

(ii) Assume now that , then we eventually have

$$\left\langle \widetilde{H}_n, \widetilde{H}_m \right\rangle = 2^n (n\alpha)! \int_{\Re} E_\alpha(-x^{2\alpha})(dx)^\alpha, \quad (71)$$

7. Gaussian probability density of fractional order

7.1 Fractional Gaussian density and Mittag-Leffler random variable

Definition 7.1. A random variable of fractional order is referred to as a Mittag-Leffler variable, or a ML-variable, when its probability density of fractional order $p_\alpha(x)$ is defined by the expression

$$p_\alpha(x)(dx)^\alpha = \frac{K_\alpha}{\sigma^\alpha} E_\alpha\left(-\frac{(x-\mu)^{2\alpha}}{2^\alpha \sigma^{2\alpha}}\right)(dx)^\alpha, \quad x\text{-}\mu > 0, \quad (72)$$

$$p_\alpha(x-\mu) = p_\alpha(\mu - x),$$

where K_α is a normalizing constant, i.e., such that $2\int_0^\infty p(x)(dx)^\alpha = 1$, and μ together with σ denote real valued parameters of which the meaning will be clarified shortly.

If we had introduced the normalized variable $Y := (X-\mu)/\sigma$, we would have $(dx)^\alpha = \sigma^\alpha (dx)^\alpha$ therefore the fractional probability density

$$q_\alpha(y)(dy)^\alpha = K_\alpha E_\alpha\left(-\frac{y^{2\alpha}}{2^\alpha}\right)(dy)^\alpha. \quad (73)$$

In the following, we shall use the notation $X \propto ML(\mu, \sigma^{2\alpha})$.

Remark. As so defined, the probability density is symmetric with respect to μ, which so clearly appears as being the mean value of the random variable.

7.2 Fractional moments of the Mittag-Leffler random variable

Proposition 7.1 *Assume that X is a fractional random variable $ML(0, \sigma^{2\alpha})$, then its moments of fractional order are provided by the expressions*

$$m_{(2n+1)\alpha} = \left\langle X^{(2n+1)\alpha} \right\rangle_\alpha = 0, \tag{74}$$

$$m_{(2n)\alpha} = \left\langle X^{(2n)\alpha} \right\rangle_\alpha = 2K_\alpha (2n-1)!!(\alpha!)^n \sigma^{2n\alpha} \tag{75}$$

with the notation $(2n-1)!! := 1 \times 3 \times 5 \times ...(2n-3) \times (2n-1)$.

Proof. The equation (93) is easy to be obtained. Next, we have now to calculate the integral

$$\left\langle X^{2n\alpha} \right\rangle_\alpha = \frac{2K_\alpha}{\sigma^\alpha} \int_0^\infty E_\alpha \left(-\frac{x^{2\alpha}}{2^\alpha \sigma^{2\alpha}} \right) x^{2n\alpha} (dx)^\alpha,$$

$$\left\langle X^{2n\alpha} \right\rangle_\alpha = 2K_\alpha \sigma^{2n\alpha} \int_0^\infty E_\alpha \left(-\frac{x^{2\alpha}}{2^\alpha} \right) x^{2n\alpha} (dx)^\alpha = 2K_\alpha \sigma^{2n\alpha} I_{2n}.$$

We use an integrating by parts to calculate I_{2n} , and to this end, we set

$$u(x) := E_\alpha \left(-\frac{x^{2\alpha}}{2^\alpha} \right), \quad u^{(\alpha)}(x) = -E_\alpha \left(-\frac{x^{2\alpha}}{2^\alpha} \right) x^\alpha, \tag{76}$$

$$v(x) := \frac{(x^\alpha)^{2n+1}}{(2n+1)(\alpha!)}, \quad v^{(\alpha)}(x) = (x^\alpha)^{2n}. \tag{77}$$

We then have successively

$$\begin{aligned} \left\langle X^{2n\alpha} \right\rangle_\alpha &= 2K_\alpha \sigma^{2n\alpha} \int_0^\infty u(x) v^{(\alpha)}(x) (dx)^\alpha = -\ 2K_\alpha \sigma^{2n\alpha} \int_0^\infty u^{(\alpha)}(x) v(x) (dx)^\alpha \\ &= \frac{2K_\alpha}{(2n+1)(\alpha!)} \sigma^{2n\alpha} \int_0^\infty E_\alpha \left(-\frac{x^{2\alpha}}{2^\alpha} \right) (x^\alpha)^{2n+2} (dx)^\alpha, \end{aligned}$$

therefore the recursion $I_{2n+2} = (2n+1)(\alpha!) I_{2n}$ which provides $I_{2n} = ((2n-1)!!)\,(\alpha!)^n$.

7.3 Determination of the value of the normalizing constant. This amounts to determine the value of the integral

$$I_\alpha = \int_0^\infty E_\alpha(-x^{2\alpha})(dx)^\alpha \tag{78}$$

and this can be done as follows.

(Step 1) We start from the equality

$$(I_\alpha)^2 = \int_0^\infty \int_0^\infty E_\alpha(-x^{2\alpha})E_\alpha(-y^{2\alpha})(dx)^\alpha(dx)^\alpha$$

$$(I_\alpha)^2 = \int_0^\infty \int_0^\infty E_\alpha\left(-\left(x^2+y^2\right)^\alpha\right)(dx)^\alpha(dx)^\alpha \tag{79}$$

(Step 2) On making the change of variable

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we obtain easily

$$(I_\alpha)^2 = \int_0^{\pi/2}(d\theta)^\alpha \int_0^\infty E_\alpha(-r^{2\alpha})r^\alpha(dr)^\alpha = (\pi/2)^\alpha \int_0^\infty E_\alpha(-r^{2\alpha})r^\alpha(dr)^\alpha. \tag{80}$$

(Step 3) A second change of variable $u = r^2$ provides successively $du = 2rdr$

$$r^\alpha(dr)^\alpha = 2^{-\alpha}(du)^\alpha. \tag{81}$$

(Step 4) Substituting this result into (99) yields

$$(I_\alpha)^2 = \left(\frac{\pi}{2}\right)^\alpha \frac{1}{2^\alpha}\int_0^\infty E_\alpha(-u^\alpha)(du)^\alpha$$

$$= \left(\frac{\pi}{2}\right)^\alpha \frac{\alpha!}{2^\alpha}\left[-E_\alpha(-x^\alpha)\right]_0^\infty = \pi^\alpha 2^{-2\alpha}\alpha!.$$

(Step 5) We then eventually obtain$I_\alpha = \frac{\sqrt{\alpha!}}{2^\alpha}\pi^{\alpha/2}$
Remark that when $\alpha - 1$, one obtains the classical result $I = \sqrt{\pi}/2$

8. Concluding remarks

First remark. Prospect for future research
As we pointed out in the introduction, fractional calculus is getting an increasing audience among scientists, to deal mainly with fractional Brownian motion on the one hand, and physical processes which are defined in coarse-grained space, or in porous media, on the other hand. This is with this purpose in mind, that we have introduced a new concept of probability density of fractional order, and in quite a natural way, we have been so led to generalize the Gaussian probability by using the Mittag Leffler function.

A concernwhich then comes in mind in quite a natural way, is exactly to examine what we can do with this fractional Gaussian density in the modeling of stochastic processes. As we know it, one of the basic stochastic processes in the literature is defined by the partial differential equation $\partial_t p(x,t) = 2^{-1}\sigma^2\partial_{xx}p(x,t)$ of which the solution is

$$p(x,t) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2\sigma^2 t}\right\}..$$

Analogously, we are led to consider probability densities in the form

$$p(xt) = \frac{\left(\sqrt{\alpha!}2^{1-(\alpha/2)}\pi^{\alpha/2}\right)^{-1}}{\sigma^{\alpha} f(t)} E_{\alpha}\left\{-\frac{x^{2\alpha}}{2^{\alpha}\sigma^{2\alpha} f^{\alpha}(t)}\right\}$$

where $f(t)$ would characterize the process under consideration, and to examine whether it would be meaningful in the modeling of fractal processes.

Second remark. Quantum probability and fractional probability
In the case when $\alpha = 1/2$ one has the equality

$$\left(p_{1/2} dx^{1/2}\right)^2 = \left(p_{1/2}\right)^2 dx$$

which suggest that whilst $p_{1/2}(x)$ is a fractional probability, its square is a probabilipy. In other words, the fractional probability would be exactly a quantum probability amplitude.

Third remark. Fractional probability and fractional entropy
This fractional probability appears to be quite consistent with the concept of informational entropy of fractional order which we have introduced in the form

$$H(X) = -\int_{\Re} p(x) \left(Ln_{\alpha} p(x)\right)^{1/\alpha} dx$$

where Ln_{α}x is the inverse of the Mittag-Leffler defined by the equality $x = E_{\alpha}\left(Ln_{\alpha} x\right)$.[9].

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