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A CAUCHY-JENSEN FUNCTIONAL INEQUALITY IN BANACH MODULES OVER A C^*-ALGEBRA

ABBAS NAJATI

ABSTRACT. In this paper, we investigate the following functional inequality

$$\left\|f\Big(\frac{x+y}{2}+z\Big)+f\Big(\frac{x+z}{2}+y\Big)+f\Big(\frac{y+z}{2}+x\Big)\right\| \le \|2f(x+y+z)\|$$

in Banach modules over a C^*-algebra, and prove the generalized Hyers–Ulam stability of linear mappings in Banach modules over a C^*-algebra.

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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms: *Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality*

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. D.H. Hyers [7] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

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for all $x, y \in X$ and some $\varepsilon \geq 0$. Then there exists a unique additive mapping $T : X \to Y$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in X$. Th.M. Rassias [26] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1. (Th.M. Rassias) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $L : E \to E'$ which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{2}$$

for all $x \in E$. If $p < 0$ then inequality (1) holds for $x, y \neq 0$ and (2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The above inequality has provided a lot of influence in the development of what is now known as a *generalized Hyers–Ulam–Rassias stability* of functional equations. J.M. Rassias [25] followed the innovative approach of Th.M. Rassias' theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. Găvruta [5] provided a further generalization of Th.M. Rassias' theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and mappings (see [2], [4], [6], [8], [10]–[13], [15]–[24], [27], [28]). We also refer the readers to the books [1], [3], [9], [29]–[32].

Park, Cho and Han [21] investigated the functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| \tag{3}$$

in Banach spaces, and proved the generalized Hyers–Ulam stability of the functional inequality (3) in Banach spaces.

Throughout this paper, let A be a unital C^*-algebra with unitary group $U(A)$, unit e and norm $|.|$. Assume that X is a left Banach A-module with norm $\|.\|_X$ and that Y is a left Banach A-module with norm $\|.\|_Y$. For $a \in A$, let $a^\flat = a, a^*$ or $(a + a^*)/2$. An additive mapping $T : X \to Y$ is called A-linear if $T(ax) = a^\flat T(x)$ for all $a \in A$ and all $x \in X$.

In this paper, we investigate an A-linear mapping associated with the functional inequality

$$\|f(\frac{x+y}{2} + z) + f(\frac{x+z}{2} + y) + f(\frac{y+z}{2} + x)\| \leq \|2f(x+y+z)\| \tag{4}$$

and prove the generalized Hyers–Ulam stability of A-linear mappings in Banach A-modules associated with the functional inequality (4).

For convenience, we use the following abbreviation for a given $a \in A$ and a mapping $f : X \to Y$

$$D_a f(x, y, z) := f\Big(\frac{ax + ay}{2} + az\Big) + f\Big(\frac{ax + az}{2} + ay\Big) + a^\flat f\Big(\frac{y + z}{2} + x\Big)$$

for all $x, y, z \in X$.

2. Functional inequalities in Banach modules over a C^*-algebra

Lemma 1. *Let $f : X \to Y$ be a mapping such that*

$$\|D_a f(x, y, z)\|_Y \le 2\|f(ax + ay + az)\|_Y \tag{5}$$

for all $x, y, z \in X$ and all $a \in U(A)$. Then f is A-linear.

Proof. Letting $x = y = z = 0$ and $a = e \in U(A)$ in (5), we get that $f(0) = 0$. Letting $z = 0, y = -x$ and $a = e \in U(A)$ in (5), we get

$$\Big\|f\Big(\frac{x}{2}\Big) + f\Big(\frac{-x}{2}\Big)\Big\|_Y \le 2\|f(0)\|_Y = 0$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x - y$ and $a = e \in U(A)$ in (5) and using the oddness of f, we get

$$\Big\|f\Big(\frac{x + y}{2}\Big) - f\Big(\frac{x}{2}\Big) - f\Big(\frac{y}{2}\Big)\Big\|_Y \le 2\|f(0)\|_Y = 0$$

for all $x, y \in X$. Thus $f(x + y) = f(x) + f(y)$ for all $x, y \in X$. Hence $f(rx) = rf(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$.

Letting $z = -x$ and $y = 0$ in (5) and using the oddness of f, we get

$$\Big\| - f\Big(\frac{ax}{2}\Big) + a^\flat f\Big(\frac{x}{2}\Big)\Big\|_Y \le 2\|f(0)\|_Y = 0$$

for all $x \in X$ and all $a \in U(A)$. Thus

$$f(ax) = a^\flat f(x) \tag{6}$$

for all $a \in U(A)$ and all $x \in X$. It is clear that (6) holds for $a = 0$.

Now let $a \in A$ $(a \neq 0)$ and m an integer greater than $4|a|$. Then $|\frac{a}{m}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By Theorem 1 of [14], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $\frac{3}{m}a = u_1 + u_2 + u_3$. So $\frac{3}{m}a^\flat = (\frac{3}{m}a)^\flat = u_1^\flat + u_2^\flat + u_3^\flat$. Hence by (6) we have

$$\begin{aligned} f(ax) &= \frac{m}{3} f\Big(\frac{3}{m}ax\Big) = \frac{m}{3} f(u_1 x + u_2 x + u_3 x) = \frac{m}{3}[f(u_1 x) + f(u_2 x) + f(u_3 x)] \\ &= \frac{m}{3}(u_1^\flat + u_2^\flat + u_3^\flat) f(x) = \frac{m}{3} . \frac{3}{m} a^\flat f(x) = a^\flat f(x) \end{aligned}$$

for all $x \in X$. So $f : X \to Y$ is A-linear, as desired.

Now we prove the generalized Hyers–Ulam stability of A-linear mappings in Banach A-modules.

Theorem 2. *Let $r_i > 1$ and θ_i be non-negative real numbers for all $1 \leq i \leq 3$, and let $f : X \to Y$ be a mapping such that*

$$\|D_a f(x,y,z)\|_Y \leq 2\|f(ax+ay+az)\|_Y + \theta_1\|x\|_X^{r_1} + \theta_2\|y\|_X^{r_2} + \theta_3\|z\|_X^{r_3} \tag{7}$$

for all $x, y, z \in X$ and all $a \in U(A)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{3 \times 2^{r_1}\theta_1}{2^{r_1}-2}\|x\|_X^{r_1} + \frac{3 \times 2^{r_2}\theta_2}{2^{r_2}-2}\|x\|_X^{r_2} + \frac{4^{r_3}\theta_3}{2^{r_3}-2}\|x\|_X^{r_3} \tag{8}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ and $a = e \in U(A)$ in (7), we get that $f(0) = 0$. Letting $a = e \in U(A)$, $y = x$ and $z = -2x$ in (7), we get

$$\big\|f(-x) + 2f(\frac{x}{2})\big\|_Y \leq \theta_1\|x\|_X^{r_1} + \theta_2\|x\|_X^{r_2} + 2^{r_3}\theta_3\|x\|_X^{r_3} \tag{9}$$

for all $x \in X$. Letting $a = e \in U(A)$, $y = -x$ and $z = 0$ in (7), we get

$$\big\|f(\frac{-x}{2}) + f(\frac{x}{2})\big\|_Y \leq \theta_1\|x\|_X^{r_1} + \theta_2\|x\|_X^{r_2} \tag{10}$$

for all $x \in X$. It follows from (9) and (10) that

$$\big\|2f(\frac{x}{2}) - f(x)\big\|_Y \leq 3\theta_1\|x\|_X^{r_1} + 3\theta_2\|x\|_X^{r_2} + 2^{r_3}\theta_3\|x\|_X^{r_3}$$

for all $x \in X$. Hence

$$\begin{aligned}\big\|2^n f(\frac{x}{2^n}) - 2^m f(\frac{x}{2^m})\big\|_Y &\leq \sum_{j=m}^{n-1}\big\|2^{j+1}f(\frac{x}{2^{j+1}}) - 2^j f(\frac{x}{2^j})\big\|_Y \\ \leq 3\theta_1\|x\|_X^{r_1}\sum_{j=m}^{n-1}\big(\frac{2}{2^{r_1}}\big)^j &+ 3\theta_2\|x\|_X^{r_2}\sum_{j=m}^{n-1}\big(\frac{2}{2^{r_2}}\big)^j + 2^{r_3}\theta_3\|x\|_X^{r_3}\sum_{j=m}^{n-1}\big(\frac{2}{2^{r_3}}\big)^j\end{aligned} \tag{11}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$. It follows from (11) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (11), we get (8). It follows from (7) that

$$\begin{aligned}\|D_a L(x,y,z)\|_Y &= \lim_{n\to\infty} 2^n\big\|D_a f(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n})\big\|_Y \\ &\leq \lim_{n\to\infty} 2^{n+1}\big\|f(\frac{ax}{2^n} + \frac{ay}{2^n} + \frac{az}{2^n})\big\|_Y \\ &\quad + \lim_{n\to\infty} 2^n\big[\frac{\theta_1}{2^{nr_1}}\|x\|_X^{r_1} + \frac{\theta_2}{2^{nr_2}}\|y\|_X^{r_2} + \frac{\theta_3}{2^{nr_3}}\|z\|_X^{r_3}\big] \\ &= 2\|L(ax+ay+az)\|_Y\end{aligned}$$

for all $x, y, z \in X$ and all $a \in U(A)$. So by Lemma 1, the mapping $L : X \to Y$ is A-linear.

Now, let $T : X \to Y$ be another A-linear mapping satisfying (8). Then we have

$$\|L(x) - T(x)\|_Y = \lim_{n\to\infty} 2^n \|f(\frac{x}{2^n}) - T(\frac{x}{2^n})\|_Y$$
$$\leq \lim_{n\to\infty} 2^n \Big[\frac{3 \times 2^{r_1}\theta_1}{2^{nr_1}(2^{r_1}-2)}\|x\|_X^{r_1} + \frac{3 \times 2^{r_2}\theta_2}{2^{nr_2}(2^{r_2}-2)}\|x\|_X^{r_2} + \frac{4^{r_3}\theta_3}{2^{nr_3}(2^{r_3}-2)}\|x\|_X^{r_3}\Big] = 0$$

for all $x \in X$. So we can conclude that $L(x) = T(x)$ for all $x \in X$. This proves the uniqueness of L. Thus the mapping $L : X \to Y$ is a unique A-linear mapping satisfying (8).

Theorem 3. *Let $0 < r_i < 1$ and θ_i, δ be non-negative real numbers for all $1 \leq i \leq 3$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\begin{aligned}\|D_a f(x,y,z)\|_Y \leq\ & 2\|f(ax+ay+az)\|_Y \\ & + \delta + \theta_1\|x\|_X^{r_1} + \theta_2\|y\|_X^{r_2} + \theta_3\|z\|_X^{r_3}\end{aligned} \tag{12}$$

for all $x, y, z \in X$ and all $a \in U(A)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq 3\delta + \frac{3 \times 2^{r_1}\theta_1}{2 - 2^{r_1}}\|x\|_X^{r_1} + \frac{3 \times 2^{r_2}\theta_2}{2 - 2^{r_2}}\|x\|_X^{r_2} + \frac{4^{r_3}\theta_3}{2 - 2^{r_3}}\|x\|_X^{r_3} \tag{13}$$

for all $x \in X$.

Proof. Similarly to the proof of Theorem 2, it follows from (12) that

$$\|2f(\frac{x}{2}) - f(x)\|_Y \leq 3\delta + 3\theta_1\|x\|_X^{r_1} + 3\theta_2\|x\|_X^{r_2} + 2^{r_3}\theta_3\|x\|_X^{r_3}$$

for all $x \in X$. Hence

$$\begin{aligned}\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\|_Y &\leq \sum_{j=m}^{n-1}\|\frac{1}{2^{j+1}}f(2^{j+1}x) - \frac{1}{2^j}f(2^j x)\|_Y \\ \leq 3\delta \sum_{j=m+1}^{n}(\frac{1}{2})^j &+ 3\theta_1\|x\|_X^{r_1}\sum_{j=m+1}^{n}(\frac{2^{r_1}}{2})^j \\ + 3\theta_2\|x\|_X^{r_2}\sum_{j=m+1}^{n}(\frac{2^{r_2}}{2})^j &+ 2^{r_3}\theta_3\|x\|_X^{r_3}\sum_{j=m+1}^{n}(\frac{2^{r_3}}{2})^j\end{aligned} \tag{14}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$. It follows from (14) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (14), we get (13). The rest of the proof is similar to the proof of Theorem 2.

Theorem 4. *Let r_i and θ be non-negative real numbers such that $\lambda := r_1 + r_2 + r_3 > 1$, and let $f : X \to Y$ be a mapping such that*

$$\|D_a f(x,y,z)\|_Y \le 2\|f(ax+ay+az)\|_Y + \theta . \|x\|_X^{r_1} . \|y\|_X^{r_2} . \|z\|_X^{r_3} \tag{15}$$

for all $x, y, z \in X$ and all $a \in U(A)$ (by letting $\|.\|_X^0 = 1$). Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{2^{(\lambda+r)}\theta}{2^\lambda - 2}\|x\|_X^\lambda \tag{16}$$

for all $x \in X$, where $r := \min\{r_1, r_2, r_3\}$.

Proof. Since $r_1 + r_2 + r_3 > 1$, then $r_j > 0$ for some $1 \le j \le 3$. Without any loss of the generality, we may assume that $r_1 > 0$ and $r = r_3$. Letting $x = y = z = 0$ and $a = e \in U(A)$ in (15), we get that $f(0) = 0$. Letting $a = e \in U(A)$, $z = -y$ and $x = 0$ in (15), we get

$$\|f(\frac{-y}{2}) + f(\frac{y}{2})\|_Y \le 2\|f(0)\|_Y = 0$$

for all $y \in X$. So the mapping f is odd. Letting $a = e \in U(A)$, $y = x$ and $z = -2x$ in (15) and using the oddness of f, we get

$$\|2f(\frac{x}{2}) - f(x)\|_Y \le 2^{r_3}\theta\|x\|_X^\lambda$$

for all $x \in X$. Hence

$$\begin{aligned}\|2^n f(\frac{x}{2^n}) - 2^m f(\frac{x}{2^m})\|_Y &\le \sum_{j=m}^{n-1} \|2^{j+1} f(\frac{x}{2^{j+1}}) - 2^j f(\frac{x}{2^j})\|_Y \\ &\le 2^{r_3}\theta\|x\|_X^\lambda \sum_{j=m}^{n-1} (\frac{2}{2^\lambda})^j\end{aligned} \tag{17}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$. It follows from (17) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (17), we get (16). The rest of the proof is similar to the proof of Theorem 2.

Theorem 5. *Let r_i and θ_i, δ be non-negative real numbers such that $\lambda := r_1 + r_2 + r_3 \in (0, 1)$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\|D_a f(x,y,z)\|_Y \le 2\|f(ax+ay+az)\|_Y + \delta + \theta . \|x\|_X^{r_1} . \|y\|_X^{r_2} . \|z\|_X^{r_3} \tag{18}$$

for all $x, y, z \in X$ and all $a \in U(A)$ (by letting $\|.\|_X^0 = 1$). Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq 3\delta + \frac{2^{(\lambda+r_3)}\theta}{2-2^\lambda}\|x\|_X^\lambda \tag{19}$$

for all $x \in X$.

Proof. Similarly to the proof of Theorem 2, letting $a = e \in U(A)$, $y = x$ and $z = -2x$ in (18), we get

$$\big\|f(-x) + 2f\big(\frac{x}{2}\big)\big\|_Y \leq \delta + 2^{r_3}\theta\|x\|_X^\lambda \tag{20}$$

for all $x \in X$. Without any loss of generality, we may assume that $r_3 > 0$. Letting $a = e \in U(A)$, $y = -x$ and $z = 0$ in (18), we get

$$\big\|f\big(\frac{-x}{2}\big) + f\big(\frac{x}{2}\big)\big\|_Y \leq \delta \tag{21}$$

for all $x \in X$. It follows from (20) and (21) that

$$\big\|2f\big(\frac{x}{2}\big) - f(x)\big\|_Y \leq 3\delta + 2^{r_3}\theta\|x\|_X^\lambda$$

for all $x \in X$. Hence

$$\begin{aligned}\big\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\big\|_Y &\leq \sum_{j=m}^{n-1}\big\|\frac{1}{2^{j+1}}f(2^{j+1}x) - \frac{1}{2^j}f(2^j x)\big\|_Y \\ \leq 3\delta\sum_{j=m+1}^{n}\big(\frac{1}{2}\big)^j &+ 2^{r_3}\theta\|x\|_X^\lambda\sum_{j=m+1}^{n}\big(\frac{2^\lambda}{2}\big)^j\end{aligned} \tag{22}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$. It follows from (22) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (22), we get (19). The rest of the proof is similar to the proof of Theorem 2.

Remark 1. *If X and Y are C^*-algebras, then X and Y are Banach modules over $\mathbb{C}$. In this case, we may assume that $A = \mathbb{C}$ and so $U(A) = \mathbb{S} := \{\mu \in \mathbb{C} : |\mu| = 1\}$ in Theorems 2 - 5. Therefore we conclude that the mapping $f : X \to Y$ is $\mathbb{C}$-linear.*

REFERENCES

1. J. Aczél and J. Dhombres, *Functional Equations in Several Variables,* Cambridge University Press, 1989.
2. C. Baak, D.-H. Boo and Th.M. Rassias, *Generalized additive mapping in Banach modules and isomorphisms between C^*-algebras,* J. Math. Anal. Appl. **314** (2006), 150–161.
3. P. Czerwik, *Functional Equations and Inequalities in Several Variables,* World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.
4. c V.A. Faizev, Th.M. Rassias and P.K. Sahoo, *The space of (ψ,γ)-additive mappings on semigroups,* Trans. Amer. Math. Soc. **354** (11) (2002), 4455–4472.
5. P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings,* J. Math. Anal. Appl. **184** (1994), 431–436.
6. H. Haruki and Th.M. Rassias, *New generalizations of Jensen's functional equation,* Proc. Amer. Math. Soc. **123** (1995), 495–503.
7. D.H. Hyers, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
8. D.H. Hyers and Th.M. Rassias, *Approximate homomorphisms,* Aequationes Mathematicae **44** (1992), 125–153.
9. D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables,* Birkhäuser, Basel, 1998.
10. G. Isac and Th.M. Rassias, *On the Hyers–Ulam stability of ψ-additive mappings,* J. Approx. Theory **72** (1993), 131–137.
11. K. Jun and Y. Lee, *A generalization of the Hyers–Ulam–Rassias stability of the Pexiderized quadratic equations,* J. Math. Anal. Appl. **297** (2004), 70–86.
12. S.-M. Jung and Th.M. Rassias, *Ulam's problem for approximate homomorphisms in connection with Bernoulli's differential equation,* Applied Mathematics and Computation **187** (2007), 223–227.
13. S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis,* Hadronic Press lnc., Palm Harbor, Florida, 2001.
14. R.V. Kadison and G. Pedersen, *Means and convex combinations of unitary operators,* Math. Scand. **57** (1985), 249–266.
15. M.S. Moslehian and Th.M. Rassias, *Stability of functional equations in non-Archimedean spaces,* Applicable Analysis and Discrete Mathematics **1** (2007), 325–334.
16. A. Najati, *Hyers–Ulam stability of an n-Apollonius type quadratic mapping,* Bull.Belgian Math. Soc. Simon-Stevin. **14**, (2007), 755–774.
17. A. Najati, *On the stability of a quartic functional equation,* J. Math. Anal. Appl. **340** (2008), 569–574.
18. A. Najati and M.B. Moghimi, *Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces,* J. Math. Anal. Appl. **337** (2008), 399–415.
19. A. Najati and C. Park, *Hyers–Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation,* J. Math. Anal. Appl. **335** (2007), 763–778.
20. C. Park, *On the stability of the linear mapping in Banach modules,* J. Math. Anal. Appl. **275** (2002), 711–720.
21. C. Park, Y. Cho and M. Han, *Stability of functional inequalities associated with Jordan-von Neumann type additive functional equations,* J. Inequal. Appl. (2007), Art. ID 41820.
22. C.-G. Park and Th.M. Rassias, *On a generalized Trif's mapping in Banach modules over a C^*-algebra,* J. Korean Math. Soc. **43** (2) (2006), 323–356.
23. C. Park and Th.M. Rassias, *Isometric additive mappings in quasi-Banach spaces,* Nonlinear Functional Analysis and Applications, **12** (3) (2007), 377–385.
24. C. Park and Th.M. Rassias, *Homomorphisms and derivations in proper JCQ^*-triples,* J. Math. Anal. Appl. **337** (2008), 1404–1414.

25. J.M. Rassias, *On approximation of approximately linear mappings by linear mappings,* J. Funct. Anal. **46** (1982) 126–130.
26. Th.M. Rassias,*On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc. **72** (1978), 297–300.
27. Th.M. Rassias, *On the stability of functional equations and a problem of Ulam,* Acta Applicandae Mathematicae **62** (1) (2000), 23–130.
28. Th.M. Rassias and J. Tabor, *What is left of Hyers–Ulam stabilirty?* Journal of Natural Geometry, **1** (1992), 65–69.
29. Th.M.Rassias (ed.), *Topics in Mathematical Analysis,* A Volume dedicated to the Memory of A.L. Cauchy, World Scientific Publishing Co., Singapore, New Jersey, London, 1989.
30. Th.M. Rassias and J. Tabor (eds.), *Stability of Mappings of Hyers–Ulam Type,* Hadronic Press Inc., Florida, 1994.
31. Th.M.Rassias (ed.), *Functional Equations and Inequalities,* Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.
32. Th.M. Rassias, *Functional Equations, Inequalities and Applications,* Kluwer Academic Publishers Co., Dordrecht, Boston, London, 2003.
33. S.M. Ulam, *A Collection of the Mathematical Problems,* Interscience Publ. New York, 1960.

Abbas Najati received his MS and Ph.D in Mathematics from University of Tabriz under the direction of Professor M.H. Faroughi. He is currently an Assistance Professor of Department of Mathematics, University of Mohaghegh Ardabili. His research interests focus on Functional Equations, Inequalities, Applications and their related topics.

Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran
e-mail: a.nejati@yahoo.com