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A SPECTRALLY ARBITRARY COMPLEX SIGN PATTERN

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ABSTRACT. A spectrally arbitrary complex sign pattern A is a complex sign pattern of order n such that for every monic nth degree polynomial $f(x)$ with coefficients from $\mathbb{C}$, there is a matrix in the qualitative class of A having the characteristic polynomial $f(x)$. In this paper, we show a necessary condition for a spectrally arbitrary complex sign pattern and introduce a minimal spectrally arbitrary complex sign pattern A_n all of whose superpatterns are also spectrally arbitrary for $n \geq 2$. Furthermore, we study the minimum number of nonzero parts in a spectrally arbitrary complex sign pattern.

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1. Introduction

The sign of a real number a, denoted by sgn(a), is defined to be $1, -1$ or 0, according to $a > 0, a < 0$ or $a = 0$. A real sign pattern matrix is a matrix whose entries are from the set $\{1, -1, 0\}$. The sign pattern of a real matrix B, denoted by sgn(B), is the $(1, -1, 0)$-matrix obtained from B by replacing each entry by its sign. For a real sign pattern A of order n, the set of all real matrices with the same sign pattern as A is called the qualitative class of A, and is denoted by $Q(A)$. A real sign pattern $B = [b_{jk}]$ is a superpattern of a real sign pattern $A = [a_{jk}]$ if $b_{jk} = a_{jk}$ whenever $a_{jk} \neq 0$. And A is a subpattern of B if B is a superpattern of A.

The sign of a complex number z, denoted by csgn(z), is defined as csgn$(z) =$ sgn$(a)+i\cdot$ sgn(b) (if $z = a+i\cdot b$ and a, b are real numbers). A complex sign pattern matrix A is a matrix $A = A_1 + i\cdot A_2$, where A_1 and A_2 are real sign patterns. The sign pattern of a complex matrix A, denoted by csgn(A), is the matrix obtained from A by replacing each entry by its sign. Namely, csgn$A =$ sgn$A_1 + i\cdot$sgnA_2 (if $A = A_1 + i \cdot A_2$, A_1 and A_2 are real matrices). The qualitative class of the

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complex sign pattern A, denoted by $Q_s(A)$, is similarly defined as the following:

$$Q_s(A) = \{B \mid \text{csgn } (B) = A\}.$$

A complex sign pattern $B = B_1 + i \cdot B_2$ is a superpattern of a complex sign pattern $A = A_1 + i \cdot A_2$ if the real sign patterns B_1 and B_2 are superpatterns of A_1 and A_2, respectively. The complex sign pattern A is called a subpattern of B, if B is a superpattern of A. Note that each complex sign pattern is a superpattern and a subpattern of itself. The complex sign pattern $A = A_1 + i \cdot A_2$ is a proper subpattern of $B = B_1 + i \cdot B_2$ if A is a subpattern of B and $A \neq B$. If, for any given monic nth degree polynomial $g(x)$ with coefficients from $\mathbb{C}$, there is a complex matrix in $Q_s(A)$ having the characteristic polynomial $g(x)$, then A is a spectrally arbitrary complex sign pattern. A complex sign pattern A is minimal spectrally arbitrary if A is spectrally arbitrary and any proper subpattern of A is not spectrally arbitrary. A complex sign pattern A of order n is potentially nilpotent (or allows nilpotence)if there exists a complex matrix $B \in Q_s(A)$ with the characteristic polynomial x^n. Note that each spectrally arbitrary complex sign pattern must allow nilpotence.

A digraph is a directed graph. A walk in a digraph of length k is a sequence $v_1, v_2, \ldots v_{k+1}$ of vertices such that there is an arc in the digraph from v_j to v_{j+1} for $j = 1, 2, \cdots, k$. The walk is closed if $v_{k+1} = v_1$, and cycle is a closed walk in which $v_1, v_2, \ldots v_k$ are distinct. A two-colored digraph D is a digraph whose arcs are colored red and blue. The two-colored digraph allows loops and both a red arc and a blue arc from j to k for all pairs (j, k) of vertices.

For an $n \times n$ complex matrix B, the characteristic polynomial of B is $P_B(x) = x^n - (\lambda_{11}(B) + i \cdot \lambda_{21}(B))x^{n-1} + \cdots + (-1)^{n-1}(\lambda_{1n-1}(B) + i \cdot \lambda_{2n-1}(B))x + (-1)^n(\lambda_{1n}(B) + i \cdot \lambda_{2n}(B))$,where $\lambda_{1k}(B) + i \cdot \lambda_{2k}(B)$ is the sum of the $k \times k$ principal minors and $\lambda_{jk}(B)$ is a real number for $k = 1, 2, \cdots, n$ and $j = 1, 2$. A complex sign pattern A is λ_{jk}-sign-arbitrary if there exist matrices A_+, A_0 and $A_- \in Q_s(A)$ such that $\lambda_{jk}(A_+) > 0$, $\lambda_{jk}(A_0) = 0$ and $\lambda_{jk}(A_-) < 0$. For a spectrally arbitrary complex sign pattern A of order n, it is necessary that A be λ_{jk}-sign-arbitrary for all $k = 1, 2, \cdots, n$ and $j = 1, 2$.

2.Necessary condition for a spectrally arbitrary complex sign pattern

Graph theoretical methods are usefull in the study of spectrally arbitrary complex sign pattern. In this section, we study the spectrally arbitrary complex sign patterns using two-colored digraph.

Let $A = A_1 + i \cdot A_2$ be an order n complex sign pattern with $A_1 = [a_{jk}]$ and $A_2 = [b_{jk}]$. The associated two-colored digraph of A, denoted by $D(A)$, is a two-colored digraph with n vertices which has a red arc from j to k if and only if $a_{jk} \neq 0$ and a blue arc from j to k if and only if $b_{jk} \neq 0$ for $j, k = 1, 2, \cdots, n$. The associated signed two-colored digraph $S(A)$ of A is obtained from $D(A)$ by assigning the sign a_{jk} and b_{jk} to the red arc(j, k) and the blue arc(j, k) in $D(A)$, respectively. Here, we establish a map Φ from the set of all complex sign pattern matrices of order n, denoted by C_n, to the set of all signed two-colored digraph with n vertices, denoted by S_n :

$$\begin{array}{rccc} \Phi : & C_n & \longrightarrow & S_n \\ & A & \longmapsto & S(A). \end{array}$$

Obviously, Φ is a isomorphic map and the sets C_n and S_n are isomorphic equivalent. In a two-colored digraph D, a simple cycle of length k (or simple k-cycle) is a sequence $v_1, v_2, \ldots, v_k$ of distinct vertices such that there is a red arc or a blue arc in D from v_j to v_{j+1} for $j = 1, 2, \ldots, k$ with $v_{k+1} = v_1$. A composite k-cycle consists of some simple cycles whose total length is k and whose index sets are mutually disjoint. A cycle (simple or composite) in $D(A)$ just corresponds to a nonzero term in the determinant expansion of the principal submatrix of A associated with the indices of the cycle. The sign of a cycle in a signed digraph is the product of the signs of the arcs of the cycle.

Theorem 2.1 *If a complex sign pattern A of order n is spectrally arbitrary, then the associated signed two-colored digraph $S(A)$ of A has at least two k-cycles with even blue arcs and opposite signs and two k-cycles with odd blue arcs and opposite signs for $k = 1, 2, \cdots, n$.*

Proof. Since A is a spectrally arbitrary complex sign pattern, A satisfies $\lambda_{jk}(A)$-sign-arbitrary for $j = 1, 2$ and $k = 1, 2, \cdots, n$. $\lambda_{1k}(A)$-sign-arbitrary implies that there are at least two real nonzero terms with opposite signs for $k = 1, 2, \cdots, n$. The two nonzero terms correspond to two k-cycles with even blue arcs and opposite signs in $S(A)$. Similarly, $\lambda_{2k}(A)$-sign-arbitrary implies that there exist at least two nonzero imaginary terms with reverse signs corresponding to two k-cycles with odd blue arcs and opposite signs in $S(A)$. The results hold.

Corollary 2.2 *For any spectrally arbitrary complex sign pattern A, the associated two-colored digraph $D(A)$ of A has at least four k-cycles for $k = 1, 2, \cdots, n$.*

From the necessary condition of the theorem 2.1, we can easily verify that some complex sign patterns which *don't* satisfy the condition are not spectrally arbitrary.

3. A minimal spectrally arbitrary complex sign pattern

In [1] J.J. McDonald described a method for establishing that a ray pattern and all of its superpatterns are spectrally arbitrary. The method using the Implicit Function Theorem can be easily extended to complex sign patterns. The following Lemma shows the method for complex sign patterns.

Lemma 3.1 *Suppose $A = A_1 + i \cdot A_2$ be a complex sign pattern of order n and $A_1 = [a_{jk}]$ and $A_2 = [b_{jk}]$ be real sign patterns. The number of the nonzero entries of A_1 and A_2 is at least $2n$.*

1. Find a nilpotent matrix in $Q_s(A)$.

2. Change $2n$ of the nonzero entries in A_1 and A_2 (denoted $r_1, r_2, \ldots, r_{2n}$) in this nilpotent matrix to variables $t_1, t_2, \ldots, t_{2n}$.

3. Express the characteristic polynomial of the resulting matrix as :

$$x^n-(f_1(t_1,\ldots,t_{2n})+i\cdot g_1(t_1,\ldots,t_{2n}))x^{n-1}+\cdots+(-1)^{n-1}(f_{n-1}(t_1,\ldots,t_{2n})$$
$$+i\cdot g_{n-1}(t_1,\ldots,t_{2n}))x+(-1)^n(f_n(t_1,\ldots,t_{2n})+i\cdot g_n(t_1,\ldots,t_{2n})).$$

4. Find the Jacobi matrix $J=\frac{\partial(f_1,g_1,\cdots,f_n,g_n)}{\partial(t_1,t_2,\cdots,t_{2n})}$.

5. If the determinant of J, *evaluated at* $(t_1,t_2,\ldots,t_{2n})=(r_1,r_2,\ldots,r_{2n})$ *is nonzero, then every superpattern of* A *is spectrally arbitrary.*

Let n be a positive integer with $n\geq 2$ and let

$$A_n=\begin{pmatrix} 1+i & -1 & & & \\ 1+i & & -1 & & \\ \vdots & & & \ddots & \\ 1+i & & & & -1 \\ i & & & & -1-i \end{pmatrix}_{n\times n} \qquad (3.1).$$

It is easy to check that the associated signed two-colored digraph $S(A_n)$ of A_n satisfies the necessary condition in Theorem 2.1. By performing suitable similarities via positive diagonal matrices, we may assume that $B_n\in Q_s(A_n)$ has the following form :

$$B_n=\begin{pmatrix} a_1+i\cdot b_1 & -1 & & & \\ a_2+i\cdot b_2 & & -1 & & \\ \vdots & & & \ddots & \\ a_{n-1}+i\cdot b_{n-1} & & & & -1 \\ i\cdot b_n & & & & -a_n-i\cdot a_0 \end{pmatrix}_{n\times n} \qquad (3.2),$$

where a_j and b_k are positive real numbers for $j=0,1,\cdots,n$ and $k=1,2,\cdots,n$.

We first give a definition and a result on the zeros of nonzero real polynomials (of finite degree). If $f(t)$ is a nonzero real polynomial, we set

$$Z_f=\{a\in R\mid f(a)=0\}.$$

If Z_f is nonempty, the maximum of Z_f is denoted by $\max(Z_f)$. If Z_f is empty, we define $\max(Z_f)=-\infty$.

For convenience, the degree of a polynomial $f(t)$ is denoted by $\partial(f)$.

Lemma 3.2 *Let* $f_j(t)$ *and* $g_j(t)$ *be nonzero real polynomials for* $j=1,2,\cdots,n$. *Suppose* $f_j(t)$ *and* $g_j(t)$ *satisfy the following conditions :*

(1) $f_j(t)=tf_{j-1}(t)-g_{j-1}(t)$, $g_j(t)=tg_{j-1}(t)+f_{j-1}(t)$ *for* $j=2,3,\cdots,n$;

(2) $f_1(t)$ *and* $g_1(t)$ *have positive leading coefficients;*

(3) Z_{f_1} *is nonempty and* $\max(Z_{f_1})\geq 0$;

(4) $\max(Z_{g_1})<\max(Z_{f_1})$;

(5) $\partial(g_1)\leq\partial(f_1)$.

Then $\max(Z_{f_1})<\max(Z_{f_2})<\cdots<\max(Z_{f_n})$, Z_{f_j} *is nonempty and* $\max(Z_{g_j})<\max(Z_{f_j})$ *for* $j=1,2,\cdots,n$.

Proof. Suppose $t_{f_1} = \max(Z_{f_1})$. Since $\max(Z_{g_1}) < \max(Z_{f_1})$ and since the leading coefficient of $g_1(t)$ is positive, we have $g_1(t_{f_1}) > 0$. Thus,

$$f_2(t_{f_1}) = t_{f_1} f_1(t_{f_1}) - g_1(t_{f_1}) = -g_1(t_{f_1}) < 0.$$

Since the leading coefficient of $f_1(t)$ is positive and $\partial(g_1) \leq \partial(f_1)$, $f_2(t) = tf_1(t) - g_1(t)$ has positive leading coefficient. Therefore, there exists a real number t' with $t' > t_{f_1}$ such that $f_2(t') > 0$. From the Intermediate Value Theorem, there exists a real number a with $t_{f_1} < a < t'$ such that $f_2(a) = 0$. Namely, Z_{f_2} is nonempty and $\max(Z_{f_1}) < \max(Z_{f_2})$. Since the leading coefficients of $f_1(t)$ and $g_1(t)$ are positive and $\max(Z_{g_1}) < \max(Z_{f_1})$, it follows that $g_2(t) = tg_1(t) + f_1(t) > 0$ for any real number $t > \max(Z_{f_1}) \geq 0$. Then $\max(Z_{g_2}) \leq \max(Z_{f_1}) < \max(Z_{f_2})$. If $n = 2$, then the results hold. If $n > 2$, it is obvious that f_2 and g_2 have positive leading coefficients and $\partial(g_2) \leq \partial(f_2)$. So f_2 and g_2 have all the conditions of f_1 and g_1, respectively. Thus, by repeating the above proof, we have that $\max(Z_{f_1}) < \max(Z_{f_2}) < \cdots < \max(Z_{f_n})$, Z_{f_j} is nonempty and $\max(Z_{g_j}) < \max(Z_{f_j})$ for $j = 1, 2, \cdots, n$. The results hold. □

Theorem 3.3 *For each $n \geq 2$, the complex sign pattern A_n having the form (3.1) is minimal spectrally arbitrary and all of its superpatterns are spectrally arbitrary.*

Proof. Let $B_n \in Q_s(A_n)$ be of the form (3.2). For convenience, we set $a_0 = 1$ and $a_n = t$. The characteristic polynomial of B_n is

$$P_{B_n}(x) = x^n - (f_1 + i \cdot g_1)x^{n-1} + \cdots + (-1)^{n-1}(f_{n-1} + i \cdot g_{n-1})x + (-1)^n(f_n + i \cdot g_n)$$

where

$$\begin{aligned} f_1 &= a_1 - t, \\ g_1 &= b_1 - 1, \\ f_j &= a_j - ta_{j-1} + b_{j-1} \ (2 \leq j \leq n-1), \\ g_j &= b_j - tb_{j-1} - a_{j-1} \ (2 \leq j \leq n), \\ f_n &= -ta_{n-1} + b_{n-1}. \end{aligned}$$

In order to show that there exists a nilpotent matrix in $Q_s(A_n)$, it is sufficient to determine the existence of positive numbers $a_1, \cdots, a_{n-1}, t, b_1, \cdots, b_n$ satisfying the following equations (obtained by setting $f_j's$ and $g_j's$ to be zero for all $j = 1, 2, \cdots, n$):

$$\begin{aligned} a_1 &= t, \\ b_1 &= 1, \\ a_j &= ta_{j-1} - b_{j-1} \quad (2 \leq j \leq n-1), \\ b_j &= tb_{j-1} + a_{j-1} \quad (2 \leq j \leq n), \\ 0 &= ta_{n-1} - b_{n-1}. \end{aligned}$$

Let $h(t) = ta_{n-1} - b_{n-1}$. Consider $a_1, b_1, \cdots, a_{n-1}, b_{n-1}, h(t)$ and b_n as the functions of t. It is easy to verify that the real polynomials $a_1, b_1, \cdots, a_{n-1}, b_{n-1}, h(t)$

and b_n satisfy all the conditions in Lemma 3.2. Therefore, the Lemma 3.2 implies that $0 = \max(Z_{a_1}) < \cdots < \max(Z_{a_{n-1}}) < \max(Z_h)$, Z_h is nonempty and $\max(Z_{b_j}) < \max(Z_h)$ for $j = 1, 2, \cdots n$. Suppose $t_h = \max(Z_h)$. Since $a_1, \cdots, a_{n-1}, b_1, \cdots b_{n-1}$ and b_n have positive leading coefficients and since $t_h > \max(Z_{a_j})$ and $t_h > \max(Z_{b_k})$, it follows that $a_j(t_h) > 0$ for $j = 1, 2, \cdots, n-1$, and $b_k(t_h) > 0$ for $k = 1, 2, \cdots n$. Therefore, there exists a nilpotent matrix having the form (3.2) in $Q_s(A_n)$ where $a_j = a_j(t_h), b_j = b_j(t_h)$ $b_n = b_n(t_h)$ and $a_n = t_h$ are positive real numbers for $j = 1, 2, \cdots, n-1$.

Consider the Jacobian matrix J given by

$$J = \frac{\partial(f_1, g_1, \cdots, f_n, g_n)}{\partial(a_1, b_1, \cdots, a_n, b_n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a_n & 1 & 1 & 0 & 0 & \cdots & 0 & -a_1 & 0 \\ -1 & -a_n & 0 & 1 & 0 & \cdots & 0 & -b_1 & 0 \\ 0 & 0 & -a_n & 1 & 1 & \cdots & 0 & -a_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -b_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -a_n & -b_{n-1} & 1 \end{pmatrix} \quad (3.3).$$

Adding the first column to the $(2n-1)$th column, the $(2n-1)$th column of the resulting matrix is $(0, 0, -2a_1, -2b_1, -a_2, -b_2, \cdots, -a_{n-1}, -b_{n-1})^T$, by $a_1 = a_n$ and $b_1 = 1$. Adding $(k+1)a_k$ multiples of the $(2k+1)$th column and $(k+1)b_k$ multiples of the $(2k+2)$th column to the $(2n-1)$th column for $k = 1, 2, \cdots, n-2$, the $(2n-1)$th column of the resulting matrix is $(0, \cdots, 0, -na_{n-1}, -nb_{n-1})^T$, by $a_j = a_n a_{j-1} - b_{j-1}$ and $b_j = a_n b_{j-1} + a_{j-1}$ for $j = 2, \cdots, n-1$. Thus, $\det(J) = -na_{n-1}$ which is nonzero when $a_{n-1} = a_{n-1}(t_h)$. From Lemma 3.1, every superpattern of A_n is spectrally arbitrary.

If one of the nonzero entries in column $2, 3, \cdots, n-1$ of A_n is replaced by zero, each matrix in $Q_s(A_n)$ is necessarily singular, a contradiction. If the $(n-1, n)$ entry of A_n is changed to zero, then every matrix in $Q_s(A_n)$ is nonsingular, a contradiction. If the $(j,1)$ entry of A_n is replaced by i or 1 for $j = 1, 2, \cdots, n-1$, then A_n is not λ_{1j+1}-sign-arbitrary or λ_{2j}-sign-arbitrary. If the (n,1) entry of A_n is changed to zero, then A_n is not S_{2n}-sign-arbitrary. If the (n, n) entry of A_n is changed to $-i$ or -1, then A_n is not λ_{11}-sign-arbitrary or λ_{21}-sign-arbitrary. If more than one nonzero real or complex parts are replaced by zeros, contradiction can be easily found. Therefore, A_n is a minimal spectrally arbitrary complex sign pattern for $n \geq 2$. □

In [1], the authors prove that any 2×2 ray patterns are not spectrally arbitrary. Here, from theorem 3.3, we know that there exist 2×2 spectrally arbitrary complex sign patterns.

4. The minimum number of nonzero parts in a spectrally arbitrary complex sign pattern

For any digraph D, let $G(D)$ denote the underlying multigraph of D, i.e., the graph obtained from D by ignoring the direction of each arc. The following Lemma is well known in the cases of real sign patterns and ray patterns. Here, we extend it to complex sign patterns.

Lemma 4.1 *Let $A = A_1 + i \cdot A_2$ be an $n \times n$ irreducible complex sign pattern and let $B \in Q_s(A)$. If T is a subdigraph of the associated two-colored digraph $D(A)$ of A such that $|T| = n-1$ and $G(D(T))$ is a tree, then A has a realization that is positive diagonally similar to B such that each real and complex part corresponding to an arc in T has magnitude* 1.

A set $S \subseteq R$ is algebraically independent if, for all $s_1, s_2, \cdots, s_n \in S$ and each nonzero polynomial $P(x_1, x_2, \cdots, x_n)$ with rational coefficients, $P(s_1, s_2, \cdots, s_n) \neq 0$. Let $Q(S)$ denote the extension field of S over Q, the field of rational numbers. Let the transcendental degree of S be

$$tr.d.S = \text{Sup}\{|T| : T \subseteq S, T \, \text{is algebraically independent}\}(\text{see}[5]).$$

Theorem 4.2 *For $n \geq 2$, an $n \times n$ irreducible spectrally arbitrary complex sign pattern has at least $3n-1$ nonzero real and complex parts. Namely, the associated two-colored digraph has at least $3n-1$ arcs.*

Proof. Let A be an $n \times n$ irreducible spectrally arbitrary complex sign pattern with m nonzero real and complex parts. Choose a set $V = \{\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n\} \subseteq R$ that $tr.d.V = 2n$. By Lemma 4.1, A has a realization $B = [a_{jk}] + i \cdot [b_{jk}]$ with characteristic polynomial

$$P_B(x) = x^n - (\alpha_1 + i \cdot \beta_1)x^{n-1} + \cdots + (-1)^{n-1}(\alpha_{n-1} + i \cdot \beta_{n-1})x + (-1)^n(\alpha_n + i \cdot \beta_n)$$

and $n-1$ real and complex parts with magnitude 1. Since α_j and β_j are polynomials in the real and complex parts of B for $1 \leq j \leq n$, it follows that

$$Q(\{\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n\}) \subseteq Q(\{a_{jk}, b_{jk} : 1 \leq j, k \leq n\}).$$

Then $2n = tr.d.Q(\{\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n\}) \leq tr.d.Q(\{a_{jk}, b_{jk} : 1 \leq j, k \leq n\}) \leq m - (n-1)$.

Thus $m \geq 3n - 1$.

In [5], the authors have proved that every irreducible spectrally arbitrary real sign pattern of order n has at least $2n-1$ nonzero entries. In [1], it is shown that every $n \times n$ irreducible spectrally arbitrary ray pattern must have at least $3n-1$ nonzero entries. And a well known conjecture in [5] is that for $n \geq 2$, an $n \times n$ real sign pattern that is spectrally arbitrary has at least $2n$ nonzero entries. Here, we extend the conjecture to complex sign patterns.

Conjecture 4.3 *For $n \geq 2$, an $n \times n$ spectrally arbitrary complex sign pattern has at least $2n$ nonzero entries and at least $3n$ nonzero real and complex parts.*

The minimal spectrally arbitrary complex sign pattern A_n having the form (3.1) has $2n$ nonzero entries and $3n$ nonzero real and complex parts. Namely, A_n verifies the conjecture 4.3 for $n \geq 2$.

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