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ANALYSIS OF THE DISCRETE-TIME GI/G/1/K USING THE REMAINING TIME APPROACH†

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ABSTRACT. The finite buffer GI/G/1/K system is set up by using an unconventional arrangement of the state space, in which the remaining inter-arrival time or service time is chosen as the level. The stationary distributions of resulting Markov chain can be explicitly determined, and the chain is positive recurrent without any restriction. This is an advantage of this method, compared with that using the elapsed time approach [2].

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1. Introduction

Finite buffer queues are very important models in computer and communications systems, and most of the finite buffer queueing models used in practice are special cases of the GI/G/1/K, e.g. GI/Geo/1/K, PH/PH/1/K, etc. Most of the methods for the finite buffer queues, involving direct and iterative methods have their pitfalls. For example, the GTH method, which has finite number of steps, involves determining the inverse of very large matrices; and the iterative methods, e.g. Gauss-Seidel method, GMRES method, the number of iterations before convergence are not predictable.

Matrix-analytic method (MAM) is, in general, a semi-explicit based method for analyzing a class of Markov chains with infinite levels (see Neuts [5, 7]). However, when dealing with finite buffer queues researchers do not fully explore the use of the MAM. The exceptions are the work of Hajek [4] and Naumov [6]. Both fully used MAM to solve the finite buffer problems in traditional manner. But the methods therein still did not exploit the full structure of the queueing systems. Alfa [2] took a different approach using MAM to exploit the full structure of the GI/G/1/K system. With a novel arrangement of the state

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space to represent the arrival process or the service process for the GI/G/1/K as an elapsed time Markov-based distributions, Alfa [2] derived a level-dependent Markov chain, from which the stationary distributions are explicitly determined. The resulting Markov chain is positive recurrent conditionally.

The purpose of this paper is to represent the arrival process or the service process for the GI/G/1/K using the remaining time approach. The resulting Markov chain is a special case of M/G/1 type, and the stationary distributions can be explicitly determined. Further it is proved to be positive recurrent without any restriction.

2. Preliminaries

Discrete phase (PH) type distribution and the stability condition for the M/G/1-type Markov chain are very vital in advancing the ideas in this paper. We first briefly introduce the PH distribution. For a comprehensive exposition we refer to [5, 7].

Consider an $n{+}1$ state absorbing Markov chain, with states $0, 1, 2, ..., n$, where the zeroth state is the absorbing state. Let the transition probability matrix of this Markov chain be

$$\begin{bmatrix} 1 & 0 \\ \boldsymbol{T}^0 & T \end{bmatrix}.$$

The matrix T is an $n \times n$ transition matrix representing transition within the transient states, and it is substochastic with $0 \leq T_{ij} \leq 1$, with the sum of one row of T being strictly less than 1. The column vector $\boldsymbol{T}^0 = 1 - T\mathbf{1}$, where $\mathbf{1}$ is a column vector of ones with appropriate size. Let $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ with α_i being the probability that the Markov chain starts in state i. We have $\alpha_0 + \boldsymbol{\alpha}1 = 1$. For most cases of interest to us, we usually have $\alpha_0 = 0$. Let p_k be the probability that the time to absorption is k, then

$$p_k = \boldsymbol{\alpha} T^{k-1} \boldsymbol{T}^0, \quad k \geq 1 \text{ and } p_0 = \alpha_0.$$

The time to absorption in such a Markov chain is said to have a phase type distribution $(\boldsymbol{\alpha}, T)$. When $n < \infty$, we call it a PH type distribution and when $n = \infty$ we say it an IPH distribution.

Consider a single server queue in discrete time with general interarrival times and general service times. We observe this system at equally spaced time epochs sequentially numbered 0,1,2,...., and assume that observations are carried out only at the beginning of an epoch. Hence all events that occur between epochs n and $n+1$ are assumed to occur at epoch $n+1, n = 0, 1, 2, \ldots$ The basic assumptions are as follows:

- Interarrival times $\mathcal{A}$, of the customers are i.i.d. with distribution

$$a_j = Pr\{\mathcal{A} = i\}, 1 \leq j \leq n \leq \infty \text{ with } \quad \mathbf{a} = [a_1, a_2, \ldots, a_n], \; and \; a^{-1} = E[\mathcal{A}],$$

where for a random variable X, $E[X]$ is the expectation of the random variable X.

- Service times, $\mathcal{S}$, of the customers are i.i.d. with distribution

$$b_j = Pr\{\mathcal{S} = i\}, 1 \leq j \leq m \leq \infty \text{ with } \quad \mathbf{b} = [b_1, b_2, \ldots, b_m], \ and\ b^{-1} = E[\mathcal{S}].$$

It was shown in [1] that both the arrival and service processes can be represented by special cases of discrete time IPH distributions based on the remaining times or elapsed times. In what follows, let the arrival process be represented by the phase type distribution $(\boldsymbol{\alpha}, T)$, and service times by $(\boldsymbol{\beta}, S)$ in terms of the remaining times, where

$$\begin{array}{lllll} \boldsymbol{\alpha} = \mathbf{a}, & T_{i,i-1} = 1, & 2 \leq i \leq n & \text{and} & T_{i,j} = 0, j \neq i-1; \\ \boldsymbol{\beta} = \mathbf{b}, & S_{i,i-1} = 1, & 2 \leq i \leq m & \text{and} & S_{i,j} = 0, j \neq i-1. \end{array}$$

Throughout this paper, we denote by $\mathbf{0}_n$ and $\mathbf{1}_n$ the column vectors of zeros and ones, respectively, of order n(If no confusion occurs, we will drop the subindex). Let A' denote the transpose of the matrix A.

3. Method I

Alfa [2] set up the finite buffer GI/G/1/K system by using an unconventional state space arrangement, i.e., by choosing the elapsed times of arrival or service as the level. The transition matrix associated with the Markov chain has a nice structure and is a special case of G/M/1 type.

In the following, we use the remaining times of arrival as the level to interpret the GI/G/1/K system as a special case of M/G/1 type. At time $t (t \geq 0)$, let N_t be the remaining interarrival time, L_t be the number of customers in the system and J_t be the phase of the service time. Consider the state space

$$\Delta = \{(N_t, 0) \cup (N_t, L_t, J_t), N_t = 1, 2, \ldots, n; L_t = 1, 2, \ldots, K; J_t = 1, 2, \ldots, m\}.$$

We also let $(N_t, L_t, J_t)|_{t \to \infty} = (N, L, J)$. In what follows, we refer to level $i (i \geq 1)$ as the set $\{((i, 0) \cup (i, L, J), L = 1, 2, \ldots, K; J = 1, 2, \ldots, m\}$, and the second and third variables as the sub-levels.

This is indeed a Markov chain. The transition matrix P_{ar} representing the Markov chain for the GI/G/1/K system is

$$P_{ar} = \begin{bmatrix} F_1 & F_2 & F_3 & \cdots & F_{n-1} & F_n \\ V & & & & & \\ & V & & & & \\ & & V & & & \\ & & & \ddots & & \\ & & & & V & 0 \end{bmatrix}, \tag{1}$$

where

$$F_j = a_j \begin{bmatrix} 0 & \beta & & & \\ & S^0\beta & S & & \\ & & \ddots & \ddots & \\ & & & S^0\beta & S \\ & & & & F \end{bmatrix}, \quad V = \begin{bmatrix} 1 & & & & \\ S^0 & S & & & \\ & S^0\beta & S & & \\ & & \ddots & \ddots & \\ & & & S^0\beta & S \end{bmatrix}$$

where $F = \boldsymbol{S^0\beta} + S$ and $\boldsymbol{S}^0 = \mathbf{1} - S\mathbf{1}$, and the matrices F_i, V are of dimensions $(Km+1) \times (Km+1)$.

The Markov chain is level-dependent and is a special case of M/G/1 type. Let the stationary distribution of P_{ar} be $\boldsymbol{x} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n]$, where

$$\boldsymbol{x}_k = [x_{k,0}, \boldsymbol{x}_{k,1}, \ldots, \boldsymbol{x}_{k,K}] \text{ and } \boldsymbol{x}_{k,i} = [x_{k,i,1}, x_{k,i,2}, \ldots, x_{k,i,m}], i = 1, 2, \ldots, K.$$

The entry $x_{k,0}$ is the probability that the remaining interarrival time is k with no customer in service; $x_{k,i,j}$ is the probability that the remaining interarrival time is k, the number of customers in the system is i and the phase of service of the customer in service is j.

If the Markov chain is positive recurrent, then the stationary vector x exists and satisfies

$$\boldsymbol{x}P_{ar} = \boldsymbol{x}, \qquad \boldsymbol{x}\mathbf{1} = 1.$$

Writing $F_j = a_j U$, the equation $\boldsymbol{x}P_{ar} = \boldsymbol{x}$ yields the recursion

$$\boldsymbol{x}_n = a_n \boldsymbol{x}_1 U \quad \text{and} \quad \boldsymbol{x}_k = \boldsymbol{x}_{k+1} V + a_k \boldsymbol{x}_1 U, \quad k = n-1, \ldots, 2, 1. \tag{2}$$

With this, we have

Theorem 1. *If the Markov chain is positive recurrent, then*

$$\boldsymbol{x}_k = \boldsymbol{x}_1 U \hat{a}_k(V), \ k = 1, 2, \ldots, n, \tag{3}$$

where $\hat{a}_k(V) = \sum_{j=0}^{n-k} a_{k+j} V^j$, *and* $\boldsymbol{x}_1$ *is normalized such that*

$$\boldsymbol{x}_1 \mathbf{1} = (\sum_{k=1}^{n} c_k)^{-1}, \qquad c_k = \sum_{i=k}^{n} a_i. \tag{4}$$

Proof. We first prove (3) is true for $k = 1, 2, \ldots, n$ by induction. For $k = n$, it is trivial. Suppose that (3) is true $k = s+1$. Then for $k = s$, we derive from (2) that

$$\boldsymbol{x}_s = \boldsymbol{x}_{s+1} V + a_s \boldsymbol{x}_1 U = \boldsymbol{x}_1 U \hat{a}_{s+1}(V) V + a_s \boldsymbol{x}_1 U = \boldsymbol{x}_1 U \hat{a}_s(V).$$

Thus by induction, (3) is true for $k = 1, 2, \ldots, n$.

For evaluating $\boldsymbol{x}_1$, note that the constraint $\boldsymbol{x}\mathbf{1} = 1$ leads to

$$\boldsymbol{x}_1 U (c_1 I + c_2 V + c_3 V^2 + \cdots + c_n V^{n-1})\mathbf{1} = 1.$$

Consequently (4) follows from the fact that U, V are stochastic.

Remark 1. In practical computations, we can use (2) instead of (3) to calculate $\boldsymbol{x}_k, k = 2, \ldots, n$. In calculating $\boldsymbol{x}_1$ the method has the same computational complexity as that based on the elapsed time approach [2], since the most computationally involved step is evaluating the matrix $\hat{a}_1(V)$. Method I is more appropriate when $n > m$, and when $n < m$ we may use Method II(to be discussed in Section 4). When $n = m$, either method is appropriate.

Remark 2. Let $\boldsymbol{y}_k = \boldsymbol{x}_k\mathbf{1}$, then $\boldsymbol{y} = [y_1, \ldots, y_n]$ satisfies

$$\boldsymbol{y} = \boldsymbol{y}(T + \boldsymbol{T^0}\boldsymbol{\alpha}) \quad \text{and} \quad y\mathbf{1} = 1.$$

This is a good criteria for checking whether $\boldsymbol{x}$ is correctly calculated.

3.1. Positive recurrence. When n or m tends to infinity, we need to consider the positive recurrence of the resulting Markov chain.

Let us first consider the case when $m < \infty$ and $n = \infty$. It is noted that the represented chain (1) is a special case of the M/G/1 system and the associated matrix A is simply the matrix V which is reducible. In this case, the matrix A can be written, possibly after a permutation of its rows and columns, as

$$A = \begin{bmatrix} D^{(0)} & D^{(1)} & D^{(2)} & \cdots & D^{(J)} \\ 0 & C^{(1)} & 0 & \cdots & 0 \\ 0 & 0 & C^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C^{(J)} \end{bmatrix},$$

where the blocks $C^{(i)}$ for $1 \le i \le J$ are irreducible and stochastic and $D^{(0)}$ is substochastic.

Let $\gamma^{(h)}$ be the stationary probability vector of $C^{(h)}$. Then the stationary drifts

$$d^{(h)} = 0 - \boldsymbol{\gamma}^{(h)} C^{(h)}\mathbf{1} < 0, \quad \forall 1 \le h \le J.$$

Consequently in this case that $m < \infty$, the Markov chain is positive recurrent without any restriction.

When $n < \infty$ and $m = \infty$, we simply consider Method II(to be discussed in Section 4) and use the same idea as above. In this case, the resulting chain is also positive recurrent always.

The last case is when $m = n = \infty$. In this case $A = V$ is not a finite TPM. However, the sum of the block rows of A outside the boundaries, i.e., $\widehat{A} = (\boldsymbol{S}^0\boldsymbol{\beta}) + S$. We know that if $E[S]$ is finite then $(\boldsymbol{S}^0\boldsymbol{\beta}) + S$ has a stationary vector [7]. Hence $\widehat{A}$ has a stationary vector $\boldsymbol{\eta}$. An alternative intuitive argument is that at any level $i (i > 1)$ the Markov chain will escape to level $i+1$ with zero probability and escape to level $i-1$ with probability $\boldsymbol{\eta} V \mathbf{1} = 1$. Therefore this Markov chain will be positive recurrent always. This is one advantage of this method, compared with that based on the elapsed time approach [2].

With the stationary distribution of the recurrent Markov chain, one can derive the queue length, waiting time of the GI/G/1/K queue, as what has been done in [2]. Here we omit the details.

3.2. When $n = \infty$. Another advantage with using the remaining time approach is that when $n = \infty$ the Markov chain is a special case of M/G/1 type, and the associated G matrix is simply the matrix V. This thus makes solving for $\boldsymbol{x}$ easy.

Let $\bar{F}_j = \sum_{i=j}^{\infty} F_i V^{i-j}$, then we have

$$\boldsymbol{x}_i = \boldsymbol{x}_1 \bar{F}_i, \quad i = 2, \ldots.$$

This is based on Ramaswami's algorithm [8] for the M/G/1 system which states that

$$\boldsymbol{z}_i = [\boldsymbol{z}_0 \bar{B}_i + \sum_{j=1}^{i-1} \boldsymbol{z}_j \bar{A}_{i+1-j}](I - \bar{A}_1)^{-1},$$

where $\boldsymbol{z} = [\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots]$ is the stationary probability vector of the transition matrix

$$P_{M/G/1} = \begin{bmatrix} B_0 & B_1 & B_2 & B_3 & B_4 & \cdots \\ C_0 & A_1 & A_2 & A_3 & A_4 & \cdots \\ & A_0 & A_1 & A_2 & A_3 & \cdots \\ & & A_0 & A_1 & A_2 & \cdots \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

and $\bar{B}_j = \sum_{i=j}^{\infty} B_i G^{i-j}$, and $\bar{A}_j = \sum_{i=j}^{\infty} A_i G^{i-j}$. The vector $\boldsymbol{z}_0$ is obtained using the results of Neuts [?]. In our case this involves solving

$$\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\pi}} \sum_{j=1}^{\infty} F_j V^{j-1}, \quad \hat{\boldsymbol{\pi}} \mathbf{1} = 1$$

and then obtaining

$$\boldsymbol{x}_1 = c\hat{\boldsymbol{\pi}},$$

where c is a normalization constant.

4. Method II

The second alternative method uses the remaining service times as the level, for the state space of the GI/G/1/K system. At time t, let N_t be the phase of the arrival, L_t be the number of the customers in the system, J_t be the remaining service time of the customer in service. Consider the state space

$$\Xi = \{(0, N_t) \cup (J_t, L_t, N_t),\ J_t = 1, 2, \ldots, m;\ L_t = 1, \ldots, K;\ N_t = 1, \ldots, n\}. \quad (5)$$

It is obvious that Ξ is indeed a Markov chain, and the transition matrix P_{sr} has the form

$$P_{sr} = \begin{bmatrix} T & E & & & & & \\ D_0 & D_1 & D_2 & D_3 & \cdots & D_{m-1} & D_m \\ & H & & & & & \\ & & H & & & & \\ & & & H & & & \\ & & & & \ddots & & \\ & & & & & H & 0 \end{bmatrix},$$

where $E = [\boldsymbol{T}^0\boldsymbol{\alpha}, \hat{\mathbf{0}}]$, $D_0 = [{}^{T}_{\hat{0}'}]$ with $\hat{\mathbf{0}}$ being a zero matrix of dimension $n \times (K-1)n$, and for $j \geq 1$,

$$D_j = b_j \begin{bmatrix} \boldsymbol{T}^0\boldsymbol{\alpha} & & & \\ T & \boldsymbol{T}^0\boldsymbol{\alpha} & & \\ & \ddots & \ddots & \\ & & T & \boldsymbol{T}^0\boldsymbol{\alpha} \end{bmatrix}, \quad H = \begin{bmatrix} T & \boldsymbol{T}^0\boldsymbol{\alpha} & & \\ & \ddots & \ddots & \\ & & T & \boldsymbol{T}^0\boldsymbol{\alpha} \\ & & & D \end{bmatrix},$$

where $D = \boldsymbol{T}^{\mathbf{0}}\boldsymbol{\alpha} + T$ and $\boldsymbol{T}^0 = \mathbf{1} - T\mathbf{1}$, and the matrices $D_i (i \geq 1), H$ are of dimension $(Kn) \times (Kn)$.

Obviously, P_{sr} is also a special case of M/G/1 type, and if the chain is positive recurrent, then the associated stationary distribution exists. Let the stationary distribution of P_{sr} be $\boldsymbol{y} = [\boldsymbol{y}_0, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m]$, where

$$\boldsymbol{y}_0 = [y_{0,1}, y_{0,2}, \ldots, y_{0,n}], \quad \boldsymbol{y}_k = [\boldsymbol{y}_{k,1}, \ldots, \boldsymbol{y}_{k,K}],$$

and $\boldsymbol{y}_{k,i} = [y_{k,i,1}, y_{k,i,2}, \ldots, y_{k,i,n}], i = 1, 2, \ldots, K$.

Theorem 2. *Writing $D_j = b_j F$, $j = 1, 2, \ldots, m$, then*

$$\boldsymbol{y}_k = \boldsymbol{y}_1 F \hat{b}_k(H), \quad k = 2, 3, \ldots, m, \tag{6}$$

where $\hat{b}_k(H) = \sum_{j=0}^{m-k} b_{k+j} H^j$, and $[\boldsymbol{y}_0, \boldsymbol{y}_1]$ is the left eigenvector which corresponds to the eigenvalue of 1 for the stochastic matrix

$$\begin{bmatrix} T & E \\ D_0 & D_{11} \end{bmatrix}.$$

Here $D_{11} = F\hat{b}_1(H)$ records the probability, starting from level 1, of reaching level 1 before level 0, and $[\boldsymbol{y}_0, \boldsymbol{y}_1]$ is normalized with

$$\boldsymbol{y}_0\mathbf{1} + \boldsymbol{y}_1[I + (\sum_{k=2}^{m} \hat{c}_k)F]\mathbf{1} = 1, \tag{7}$$

where $\hat{c}_k = \sum_{v=k}^{m} b_v$.

Proof. The equality $\boldsymbol{y}P_{sr} = \boldsymbol{y}$ yields

$$[\boldsymbol{y}_0 \quad \boldsymbol{y}_1] = [\boldsymbol{y}_0 \quad \boldsymbol{y}_1 \quad \boldsymbol{y}_2] \begin{bmatrix} T & E \\ D_0 & D_1 \\ 0 & H \end{bmatrix}, \tag{8}$$

and

$$\boldsymbol{y}_m = \boldsymbol{y}_1 D_m \quad \text{and} \quad \boldsymbol{y}_k = \boldsymbol{y}_1 D_k + \boldsymbol{y}_{k+1} H, \quad k = m-1, \ldots, 3, 2. \tag{9}$$

From the recursion (9) and induction process, it is easy to prove that (6) is true for $k = 2, 3, \ldots, m$.

Substituting $\boldsymbol{y}_2 = \boldsymbol{y}_1 F \hat{b}_2(H)$ into (8), we obtain

$$[\boldsymbol{y}_0 \quad \boldsymbol{y}_1] = [\boldsymbol{y}_0 \quad \boldsymbol{y}_1] \begin{bmatrix} T & E \\ D_0 & D_{11} \end{bmatrix},$$

where

$$D_{11} = D_1 + F\hat{b}_2(H)H = \sum_{j=1}^{m} b_j F H^{j-1} = F\hat{b}_1(H).$$

Now the constraint $\boldsymbol{y}\mathbf{1} = 1$ leads to

$$\boldsymbol{y}_0 \mathbf{1} + \boldsymbol{y}_1 \mathbf{1} + \sum_{k=2}^{m} \boldsymbol{y}_1 F \hat{b}_k(H) \mathbf{1} = 1.$$

Hence (7) follows from the fact that H is stochastic.

Alfa [2] consider the GI/G/1/K system with the state space as (5), where N_t, L_t are defined as in (5), but J_t is represented using the elapsed time approach. The elapsed time representation of the service times can be seen in [2]. In Model II Alfa [2] only captured the system when it is busy and ignores the idle states. We present a more correct transition matrix P_{se} as follows

$$P_{se} = \begin{bmatrix} T & E & & & & \\ C_1 & B_1 & A_1 & & & \\ C_2 & B_2 & & A_2 & & \\ \vdots & \vdots & & & \ddots & \\ C_{m-1} & B_{m-1} & & & & A_{m-1} \\ C_m & B_m & & & & 0 \end{bmatrix},$$

where $B_j = (1 - \tilde{b}_j)F, A_j = \tilde{b}_j H, C_j = (1 - \tilde{b}_j)D_0$, and F, H, D_0, E are defined in Theorem 2, and $\tilde{b}_j = (1 - \sum_{v=0}^{j} b_v)/(1 - \sum_{v=0}^{j-1} b_v)$. Let the stationary distribution of P_{se} be $\boldsymbol{z} = [\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_m]$. We state the formulae for $\boldsymbol{z}$ without proofs since the proofs are similar to proving Theorem 2.

Theorem 3.

$$\boldsymbol{z}_k = \boldsymbol{z}_{k-1} A_{k-1}, \quad k = 2, 3, \ldots, m,$$

and $[\boldsymbol{z}_0, \boldsymbol{z}_1]$ is the left eigenvector which corresponds to the eigenvalue of 1 for the stochastic matrix

$$\begin{bmatrix} T & E \\ \hat{b}_1(H) D_0 & \hat{b}_1(H) F \end{bmatrix},$$

where $\hat{b}_1(H) = \sum_{j=1}^{m} b_j H^{j-1}$ and $[\boldsymbol{z}_0, \boldsymbol{z}_1]$ is normalized such that

$$\boldsymbol{z}_0\mathbf{1} + (\sum_{k=1}^{m} \hat{c}_k)\boldsymbol{z}_1\mathbf{1} = 1,$$

where $\hat{c}_k = \sum_{v=k}^{m} b_v$.

5. The $GI^X/G/1/K$ system

The idea discussed in Method I-II can be extended to the case where the arrivals are in batch. Let $\gamma_k, k = 1, 2, \ldots$ be the probability when an arrival can occur with batch size k. Set $\Gamma_k = \sum_{j=k}^{\infty} \gamma_j$. Then for this case in Method I, the matrix V in transition matrix P_{ar} remains the same, but the matrix U in $F_j = a_j U$ changes and has the form

$$U = \begin{bmatrix} 0 & H_1 & H_2 & H_3 & \cdots & H_{K-1} & \hat{H}_k \\ & Z_1 & Z_2 & Z_3 & \cdots & Z_{k-1} & \hat{Z}_k \\ & & Z_1 & Z_2 & \cdots & Z_{k-2} & \hat{Z}_{k-1} \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & Z_1 & Z_2 & \hat{Z}_3 \\ & & & & & Z_1 & \hat{Z}_2 \\ & & & & & & \hat{Z}_1 \end{bmatrix},$$

where

$$H_j = \gamma_j \boldsymbol{\beta}, \quad \hat{H}_K = \Gamma_K \boldsymbol{\beta}, \quad Z_1 = \gamma_1(\boldsymbol{S^0\beta}), \quad \hat{Z}_1 = \Gamma_1(\boldsymbol{S^0\beta} + S),$$
$$Z_j = \gamma_j(\boldsymbol{S^0\beta}) + \gamma_{j-1}S, \ j \geq 2, \quad \hat{Z}_J = \Gamma_{J-1}(\boldsymbol{S^0\beta}) + \Gamma_J S, \ J \geq 2.$$

In Method II, the matrices T, E, H of P_{sr} and T, E, A_j, C_j of P_{se} remain the same, and the matrix F in $D_j = b_j F, B_j = (1 - \tilde{b}_j)F$ has the form

$$F = \begin{bmatrix} Z_1 & Z_2 & Z_3 & \cdots & Z_{k-1} & \hat{Z}_k \\ T & Z_1 & Z_2 & \cdots & Z_{k-2} & \hat{Z}_{k-1} \\ & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & T & Z_1 & Z_2 & \hat{Z}_3 \\ & & & T & Z_1 & \hat{Z}_2 \\ & & & & T & \hat{Z}_1 \end{bmatrix},$$

where

$$Z_j = \gamma_j(\boldsymbol{T^0\alpha}), \ 1 \leq j \leq K-1, \quad \hat{Z}_J = \Gamma_J(\boldsymbol{T^0\alpha}), \ 1 \leq J \leq K.$$

Finally, we point out that the idea in this paper can also be extended to the $GI/G^Y/1/K$ system.

6. Conclusion

We have shown that by selecting the remaining interarrival or (or service) times as the level in the state space of the GI/G/1/K system, we can obtain a special structure of the M/G/1-type process. The resulting structure enables us to derive the explicit solution of the associated Markov-based chain and is positive recurrent without any restriction. Hence the remaining time approach is a better choice compared with that based on the elapsed time approach.

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