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C^1 HERMITE INTERPOLATION WITH MPH QUARTICS USING THE SPEED REPARAMETRIZATION METHOD

GWANG-IL KIM

ABSTRACT. In this paper, we propose a new method to obtain C^1 MPH quartic Hermite interpolants generically for any C^1 Hermite data, by using the speed raparametrization method introduced in [16]. We show that, by this method, without extraordinary processes ($C^{\frac{1}{2}}$ Hermite interpolation introduced in [13]) for non-admissible cases, we are always able to find C^1 Hermite interpolants for any C^1 Hermite data generically, whether it is admissible or not.

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1. Introduction

In 1999, Pythagorean hodograph (PH) curves were introduced by Farouki and Sakkalis [7], which have their roots in the rational parametrization of curves and surfaces. They have been widely studied for applications [4, 9, 10] in the fields of Computer Aided Geometric Design (CAGD). Also, there have been several researches from the formal representation of them [3, 7, 12, 15] to several interpolation schemes using them [1, 5, 6, 8, 11, 14, 16].

Minkowsi Pythagorean hodograph (MPH) curves was introduced by Moon in [17]. They also have their roots in the rational parametrization of curves and surfaces, for example, the rational parametrization of the offset given by a polynomial spine curve and a polynomial distance function. They are also use to compute the medial axis transform (MAT) of a domain [2].

The characterization and classification of MPH curves were studied in [3, 13, 15]. The interpolation schemes using MPH curves were done in [2, 13, 18]

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respectively; G^1 Hermite interpolation was studied with MPH cubics in [2], and C^1 Hermite interpolation was done with MPH quartics in [13]. Especially in [13], for an admissible C^1 Hermite data, it is shown that there generically exist MPH quartic interpolants satisfying the given data. Moreover, for an arbitrary C^1 Hermite data, it is also shown that there generically exist $C^{\frac{1}{2}}$ interpolants which consist of two MPH quartics.

In this paper, using the speed reparametrization method introduced in [16], we propose another method to solve C^1 Hermite interpolation problems with MPH quartics, whether the given C^1 Hermite data is admissible for interpolation or not: We show that a C^1 Hermite data is to be admissible, if MPH velocities in the data are suitably small. In addition, we show that, according to the speed reparametrization method, in solving an interpolation problem for an arbitrary C^1 Hermite data, it is sufficient to solve an interpolation problem only for an admissible C^1 Hermite data with suitably small velocities in the same directions of ones in the original data.

2. Preliminaries

In this section, we introduce some fundamental definitions and theorems for our main result.

Definition 1. A polynomial curve $\alpha(t) = \big(x(t), y(t), z(t)\big)$ in $\mathbb{R}^{2,1}$ is a *Minkowski Pythagorean hodograph (MPH)* curve if there is a polynomial $\sigma(t)$ such that

$$x'(t)^2 + y'(t)^2 - z'(t)^2 = \sigma(t)^2, \quad \text{equivalently} \quad \|\alpha'(t)\|_*^2 = \sigma(t)^2,$$

where $\|\cdot\|_*$ denotes the Minkowski norm in $\mathbb{R}^{2,1}$. The plane curve $\delta_\alpha(t) = (x(t), y(t))$, the projection of $\alpha(t)$ into $\mathbb{R}^2$, is called the *spine curve* of $\alpha(t)$.

Throughout this paper, considering MPH curves in $\mathbb{R}^{2,1}$, we stand on two fundamental cornerstones: one is the complex representation of plane curves, and the other is the fundamental theorem of algebra. By the *complex representation* of plane curves, a planar real polynomial curve $(x(t)\,,\, y(t))$ can be identified with a complex polynomial $x(t) + y(t)\,i$ over the complex number field $\mathbb{C}$. By the fundamental theorem of algebra, the complex polynomial can be rewritten in the completely factorized form $\mathbf{k}\,(t - c_1)\cdots(t - c_n)$, where $\mathbf{k}\,, c_1, \cdots, c_n \in \mathbb{C}$. This means that a planar real polynomial curve can be completely characterized by some characteristic complex numbers; one is the leading coefficient of the complex representation of the curve and the others are the roots of the polynomial.

In the following, we show that a MPH curve in $\mathbb{R}^{2,1}$ can be characterized with the complex roots of the hodograph of the spine curve.

Definition 2. Two numbers $z_1, z_2 \in \mathbb{C}$ are said to be `semiequal` if $z_1 = z_2$ or $z_1 = \overline{z_2}$, (denote by $z_1 \approx z_2$). If z_1, z_2 are not semiequal, they are said to be *distinct up to conjugate*.

Definition 3. For a real polynomial $h(t)$ such that $h(t) \geq 0$ with $^{\forall}t \in \mathbb{R}$, by $[h(t)]$, we denote the set of all polynomial curves $\gamma(t) = \Big(x(t), y(t)\Big)$ satisfying $x(t)^2 + y(t)^2 = h(t)$, i.e., $\|\gamma(t)\|^2 = h(t)$, where $\|\cdot\|$ is the complex norm in $\mathbb{C}$.

A polynomial $f(t)$ is called a *component* of $[h(t)]$ if there exists a polynomial $g(t)$ such that the curve $(f(t), g(t))$ or $(g(t), f(t)) \in [h(t)]$.

Let $\alpha(t) = \Big(x(t), y(t), z(t)\Big)$ be a polynomial curve in $\mathbb{R}^{2,1}$. Then, by Definition 1, we can easily obtain that $\alpha(t)$ is a MPH curve if and only if $z'(t)$ is a component of $[x'(t)^2 + y'(t)^2]$. This implies that the component functions of the hodograph of $\alpha(t)$ are strongly related to the speed function of the spine curve: the speed function of the spine curve of $\alpha(t)$, the speed function of $\alpha(t)$ in $\mathbb{R}^{2,1}$ and $z'(t)$ are a Pythagorean triple, i.e., $\|(x'(t), y'(t))\|^2 = x'(t)^2 + y'(t)^2 = z'(t)^2 + \|\alpha'(t)\|_*^2$.

By the following theorem, we can characterize a MPH curve in $\mathbb{R}^{2,1}$ with the complex roots of the hodograph of its spine curve, and also we can explain how many MPH curves can be obtained for the speed function of a spine curve.

Theorem 1 ([13]). *Suppose $\alpha(t) = \Big(x(t), y(t), z(t)\Big)$ is a polynomial curve in $\mathbb{R}^{2,1}$. Let $r_i (1 \leq i \leq k_1)$ and $c_j (1 \leq j \leq k_2)$ be respectively a real root and a non-real complex root of the complex representation of the hodograph of the spine curve of $\alpha(t)$; $\delta_\alpha{}'(t) = \mathbf{k}\prod_{i=1}^{k_1}(t - r_i)\prod_{j=1}^{k_2}(t - c_j)$ for a complex number $\mathbf{k}$. Then $\alpha(t)$ is a MPH curve if and only if $z'(t)$ is the real (or imaginary) part of another complex polynomial $\xi(t) = \tilde{\mathbf{k}}\prod_{i=1}^{k_1}(t - r_i)\prod_{j=1}^{k_2}(t - c_j^*)$, where $\tilde{\mathbf{k}}$ is a complex number with $\|\mathbf{k}\| = \|\tilde{\mathbf{k}}\|$ and c_j^* is semiequal to c_j for $j = 1, \ldots, k_2$.*

Example 1. Consider a quadratic curve $\alpha(t) = \Big(x(t), y(t), z(t)\Big)$ in $\mathbb{R}^{2,1}$ with $x(t) = t^2 - t$ and $y(t) = t - 1$. Then the hodograph of the spine curve of $\alpha(t)$, i.e. $\delta_\alpha{}'(t)$, has a complex root $\frac{1-i}{2}$ in the complex representation. Assume that $\alpha(t)$ is a MPH curve. Then, by Theorem 1, we have $z'(t) = Re\Big(2 \cdot e^{\theta\, i} \cdot (t - d)\Big)$ or $z'(t) = Im\Big(2 \cdot e^{\theta\, i} \cdot (t - d)\Big)$, where $\theta \in \mathbb{R}$ and d is semiequal to $\dfrac{1-i}{2}$. That's, for a given real number θ, there are four possibilities; $z(t) = \cos\theta \cdot (t^2 - t) - \sin\theta \cdot t + h_0$, $z(t) = \sin\theta \cdot (t^2 - t) + \cos\theta \cdot t + h_0$, $z(t) = \cos\theta \cdot (t^2 - t) + \sin\theta \cdot t + h_0$ and $z(t) = \sin\theta \cdot (t^2 - t) - \cos\theta \cdot t + h_0$, where h_0 is an arbitrary real constant.

3. MPH quartics revisited and speed reparametrization

C^1 Hermite interpolation with MPH quartics was studied in [13]. In the paper, it is shown that, for a C^1 Hermite data, if the data is admissible (see

Definition 5 in this section), then MPH quartic interpolants satisfying the data generically exist, and if else, using the $C^{\frac{1}{2}}$ interpolation method, double MPH quartic interpolants can be obtained.

In this section, for our main result in the next section, we summarize C^1 Hermite interpolation with MPH quartics when the given C^1 Hermite data is admissible, and prove that there always exists a speed reparmetrization between any two C^1 Hermite data with exactly same data only except speeds at both end points in $\mathbb{R}^{2,1}$.

Let $H_C^1 = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0, \mathbf{D}_1\right\}$ be a C^1 Hermite data in $\mathbb{R}^{2,1}$, and let $\alpha(t)$ be a MPH interpolant satisfying H_C^1. Then we have

$$\alpha(0) = \mathbf{P}_0,\ \alpha(1) = \mathbf{P}_1,\ \alpha'(0) = \mathbf{D}_0,\ \alpha'(1) = \mathbf{D}_1. \tag{1}$$

If $\alpha(t)$ is a MPH quartic satisfying H_C^1, then there exist four polynomials $x(t), y(t), z(t), w(t)$ of maximal degree 4 such that $x'(t)^2+y'(t)^2 = z'(t)^2+w'(t)^2$, and for $\alpha(t) = \Big(x(t), y(t), z(t)\Big)$, Eq. (1) hold. Recall the fact that a MPH curve in $\mathbb{R}^{2,1}$ can be characterized with the complex roots of the hodograph of its spine curve. This fact implies that we can re-state a C^1 Hermite interpolation problem in $\mathbb{R}^{2,1}$, using the complex representation, into two *coupled* C^1 Hermite interpolation problems $\mathbb{R}^2$ as follows.

First, suppose that $\mathbf{P}_j = (x_j, y_j, z_j)$, $\mathbf{D}_j = (d_j^x, d_j^y, e_j)$ where $\mathbf{D}_j$ are space-like $\Big($i.e., $\|\mathbf{D}_j\|_*^2 \geq 0$ for $j = 0, 1\Big)$. Note that MPH property is invariant under parallel translation. Here, for computational convenience, we assume that $\mathbf{P}_0 = (x_0, y_0, z_0) = (0, 0, 0)$. Next, we define four complex numbers defined by $\mathbf{s}_j = x_j + y_j\, i$ and $\mathbf{d}_j = d_j^x + d_j^y\, i$ for $j = 0$, 1. Here, assume that the hodograph of the spine curve $s(t) = x(t) + y(t)\, i$ is given by

$$s'(t) = \mathbf{k}(t - c_1)(t - c_2)(t - c_3)$$

and let $\tilde{s}(t) = z(t) + w(t)\, i$. Then, by Theorem 1, the hodograph of $\tilde{s}(t)$ is given by

$$\tilde{s}\,'(t) = \tilde{\mathbf{k}}(t - c_1^*)(t - c_2^*)(t - c_3^*)$$

with $\mathbf{k} = e^{\theta\, i}\cdot\tilde{\mathbf{k}}$ and c_j^* is semiequal to c_j for each $j = 1, 2, 3$, and moreover, $z'(t) = Re(\tilde{s}'(t))$ or $Im(\tilde{s}'(t))$ (in fact, since $Im(\tilde{s}'(t)) = Re(-i\cdot\tilde{s}'(t))$ and $\|i\cdot\mathbf{k}\| = \|\mathbf{k}\|$, it is sufficient to state $z'(t) = Re(\tilde{s}'(t))$). Thus, consequently from Eq. (1), we have the following constraints;

$$\begin{aligned}
\mathbf{s}_1 &= \mathbf{k}\left(\frac{1}{4} - \frac{1}{3}S_1 + \frac{1}{2}S_2 - S_3\right), && (2)\\
\mathbf{d}_0 &= -\mathbf{k}S_3, && (3)\\
\mathbf{d}_1 &= \mathbf{k}(1 - S_1 + S_2 - S_3), && (4)\\
\tilde{s}(1) &= e^{\theta\, i}\cdot\mathbf{k}\left(\frac{1}{4} - \frac{1}{3}S_1^* + \frac{1}{2}S_2^* - S_3^*\right) + \tilde{s}(0), && (5)
\end{aligned}$$

$$\begin{aligned} \tilde{\mathbf{d}}_0 &= -e^{\theta i} \cdot \mathbf{k} S_3^*, & (6) \\ \tilde{\mathbf{d}}_1 &= e^{\theta i} \cdot \mathbf{k}(1 - S_1^* + S_2^* - S_3^*), & (7) \end{aligned}$$

where $\theta \in \mathbb{R}$, $\tilde{\mathbf{d}}_j = e_j \pm i\sqrt{\|\mathbf{d}_j\|^2 - e_j^2}$ and S_j (S_j^*) is the j-th symmetric polynomial over $\{c_j\}$ ($\{c_j^*\}$), respectively for $j = 0, 1$. Next, by solving Eq. (2) - Eq. (7) together with respect to four complex numbers $\mathbf{k}$, c_1, c_2, c_3 and one real number θ, we might finally obtain the target interpolants satisfying H_C^1, if they exist. By the way, in the system of constraint equations, depending on the choices of c_j^*, there are four possible cases as follows:

(i) $c_j^* = c_j$ for $j = 1, 2, 3$, (*ii*) $c_j^* = \bar{c}_j$ for $j = 1, 2, 3$,
(iii) $c_1^* = \bar{c}_1$ and $c_j^* = c_j$ for $j = 2, 3$, (*iv*) $c_j^* = \bar{c}_j$ for $j = 1, 2$ and $c_3^* = c_3$.

In [13], it is shown that, while Case (i) and Case (ii) are singular, Case (iii) and Case (iv) are generic (in fact, Case (iv) is covered by Case (iii)). Moreover, in the generic cases, by the solvability of a special quadratic equation in a real variable, the existence of target interpolants is entirely determined. For the given C^1 Hermite data H_C^1, the characteristic quadratic equation is expressed as follows: Let $\mathbf{a}_j = \dfrac{\mathbf{d}_j}{\tilde{\mathbf{d}}_j}$, $\theta_j = \arg(\mathbf{a}_j)$ for $j = 0$, 1 and let $\eta = \dfrac{\theta_1 - \theta_0}{2}$. Then the equation is given by

$$m_2 r^2 + m_1 r + m_0 = 0, \tag{7}$$

where r denote $\left\|\dfrac{1 - c_1}{c_1}\right\|$ $\left(\text{or } -\|\dfrac{1 - c_1}{c_1}\|\right)$, and

$$\begin{aligned} 2m_0 &= -6z_1 + Re\Big(\mathbf{a}_0^{-1}(6\mathbf{s}_1 + \mathbf{d}_1 e^{-2i\eta} - \mathbf{d}_1)\Big), \\ 2m_1 &= 12 z_1 \cos(\eta) + Re\Big(\mathbf{a}_0^{-1}(\mathbf{d}_1 e^{-i\eta} - 12\mathbf{s}_1 e^{-i\eta} - \mathbf{d}_1 e^{-3i\eta})\Big) \\ &\quad + 2Im(\tilde{\mathbf{d}}_0)\sin(\eta), \\ 2m_2 &= -6z_1 + Re\Big(\mathbf{a}_0^{-1}(\mathbf{d}_0 - \mathbf{d}_0 e^{-2i\eta} + 6\mathbf{s}_1 e^{-2i\eta}\Big), \end{aligned}$$

where $z_1 = Re(\tilde{s}(1) - \tilde{s}(0)) = z(1) - z(0) = z(1)$. (See Appendix in [13] for derivation).

If Eq. () is solvable for the real variable r, using Eq. (2) - (7), we can obtain $\mathbf{k}$, c_1, c_2, c_3 and θ, i.e. equivalently the interpolants satisfying H_C^1.

Definition 4. For a C^1 Hermite data $H_C^1 = \Big\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0^*, \mathbf{D}_1^*\Big\}$, the C^1 Hermite data given by $\Big\{\mathbf{0}, \mathbf{P}_1 - \mathbf{P}_0, \mathbf{D}_0^*, \mathbf{D}_1^*\Big\}$ is called the normalization of H_C^1, in symbol, $\mathcal{N}[H_C^1]$.

Definition 5. A C^1 Hermite data H_C^1 is said to be *admissible* if the discriminant $\Delta = m_1^2 - 4m_0m_2$ of Eq. () for $\mathcal{N}[H_C^1]$ satisfies $\Delta \geq 0$.

Example 2. Consider a C^1 Hermite data $H_C^1 = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0^*, \mathbf{D}_1^*\right\}$ given by

$$\mathbf{P}_0 = (1,1,0.5), \mathbf{P}_1 = (2,2,0.5), \mathbf{D}_0^* = (4,4,4) \text{ and } \mathbf{D}_1^* = (1,3,2).$$

Then, the discriminant Δ for $\mathcal{N}(H_C^1)$ is negative for any possible $\tilde{\mathbf{d}}_j$ from $\mathcal{N}(H_C^1)$. Thus, H_C^1 is not admissible. Whereas, if $\mathbf{P}_0 = (1,1,0.5)$, $\mathbf{P}_1 = (2,2,1)$, $\mathbf{D}_0^* = (4,4,2)$ and $\mathbf{D}_1^* = (1,3,1)$, the Hermite data is always admissible, since Δ for $\mathcal{N}(H_C^1)$ is not negative for each possible $\tilde{\mathbf{d}}_j$ from $\mathcal{N}(H_C^1)$. Moreover, in this case, according to the choices for $\tilde{\mathbf{d}}_j$, we can obtain maximally eight possible interpolants satisfying H_C^1, by using the solutions of Eq. ().

Remark 1. Note that MPH property is invariant under parallel translation. Thus, for an admissible C^1 Hermite data $H_C^1 = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0^*, \mathbf{D}_1^*\right\}$, we automatically the interpolants satisfying H_C^1 from those satisfying $\mathcal{N}(H_C^1)$, only by parallel-translating: Let $\alpha(t)$ be an interpolant satisfying $\mathcal{N}(H_C^1)$. Then $\alpha(t) + \mathbf{P}_0$ is obviously that satisfying H_C^1.

Theorem 2 ([13]). *If a C^1 Hermite data $H_C^1 = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0^*, \mathbf{D}_1^*\right\}$ is admissible, then there generically exist eight MPH quartic interpolants satisfying H_C^1.*

Proof. For details, see Section 3 in [13].

Now, we consider the speed reparametrization of MPH curves. The speed reparametrization is defined by handling specially the current speed of the time variable as follows:

Definition 6. Let $\alpha(\tilde{t})$ be a curve with $\alpha(0) = p_0$, $\alpha(1) = p_1$, $\alpha'(0) = \tilde{v_0}$ and $\alpha'(1) = \tilde{v_1}$. For two vectors v_0 and v_1, which are parallel to $\tilde{v_0}$ and $\tilde{v_1}$ without direction reversion respectively, a *speed reparametrization* of $\alpha(\tilde{t})$ is defined by a monotone increasing real valued function $\tilde{t} = \phi(t)$ with $\alpha(\phi(0)) = p_0$, $\alpha(\phi(1)) = p_1$, $\phi'(0) \cdot \alpha'(\phi(0)) = v_0$ and $\phi'(1) \cdot \alpha'(\phi(1)) = v_1$.

Remark 2. Let $H_C^1 = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0^*, \mathbf{D}_1^*\right\}$ and $\widetilde{H_C^1} = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0, \mathbf{D}_1\right\}$ be two C^1 Hermite data in $\mathbb{R}^{2,1}$ with $\mathbf{D}_0 = \epsilon_0 \mathbf{D}_0^*$ and $\mathbf{D}_1 = \epsilon_1 \mathbf{D}_1^*$. If a monotone increasing function $\tilde{t} = \phi(t)$ satisfies $\phi(0) = 0$, $\phi(1) = 1$, $\phi'(0) = \epsilon_0$ and $\phi'(1) = \epsilon_1$, then $\phi(t)$ is called a speed reparametrization from H_C^1 to $\widetilde{H_C^1}$.

For the interpolant satisfying H_C^1, the curve $\widetilde{\alpha}(t) = \alpha(\phi(t))$ is called the speed-reparametrized interpolant satisfying $\widetilde{H_C^1}$. Also, note that, throughout this paper, we assume that ϵ_0 and ϵ_1 are *positive*, i.e., we exclude the direction reversion of velocity vectors in the speed reparametrization.

Note that the speed reparametrization works, independently of the shape of curves, as a speed fitting for two selected Hermite data without changing the direction of time variable. This means that the speed reparametrization for MPH Hermite interpolants is also to be possible naturally:

Theorem 3. *Let* $H_C^1 = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0^*, \mathbf{D}_1^*\right\}$ *and* $\widetilde{H_C^1} = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0, \mathbf{D}_1\right\}$ *be two* C^1 *Hermite data in* $\mathbb{R}^{2,1}$ *with* $\mathbf{D}_0 = \epsilon_0 \mathbf{D}_0^*$ *and* $\mathbf{D}_1 = \epsilon_1 \mathbf{D}_1^*$ *for* $\epsilon_0, \epsilon_1 \in \mathbb{R}^+$. *There exist a speed reparametrization* $\tilde{t} = \phi(t)$ *from* H_C^1 *to* $\widetilde{H_C^1}$.

Proof. Whether the given data are in an Euclidean space or in a Minkowski space, it is sufficient only to show that there exists a monotone increasing function $\tilde{t} = \phi(t)$ such that $\phi(0) = 0$, $\phi(1) = 1$, $\phi'(0) = \epsilon_0$ and $\phi'(1) = \epsilon_1$. By Theorem **17** in [16], we are always able to obtain such a function.

Remark 3. The previous theorem implies that, for any two C^1 Hermite data in $\mathbb{R}^{2,1}$ given as in the theorem, if we find an interpolant satisfying one of two data, we can simultaneously obtain another interpolant satisfying the other data: That's, let $\alpha(\tilde{t})$ be an interpolant satisfying $\widetilde{H_C^1}$. Then the curve $\alpha(\phi(t))$ is also an interpolant satisfying H_C^1, if by $\tilde{t} = \phi(t)$ is a speed reparametrization from $\widetilde{H_C^1}$ to H_C^1.

4. Interpolation with MPH quartics using speed reparametrization

As stated in the previous section, the interpolation with MPH quartics is not always possible for arbitrary C^1 Hermite data; only for admissible Hermite data, it is possible. Especially, for non-admissible data, we need another special technique called $C^{\frac{1}{2}}$ *Hermite interpolation* [13].

In this section, as our main result, we propose a new method to obtain C^1 MPH quartic Hermite interpolants generically for any arbitrary C^1 Hermite data, by using the speed raparametrization method introduced in [16]. We will see that, by this method, without extraordinary processes related to $C^{\frac{1}{2}}$ Hermite interpolation in non-admissible cases, we are always able to find C^1 Hermite interpolants for any C^1 Hermite data generically, whether it is admissible or not.

Theorem 4. *Let* $H_C^1 = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0^*, \mathbf{D}_1^*\right\}$ *be a* C^1 *Hermite data in* $\mathbb{R}^{2,1}$ *with* $\|\mathbf{P}_1 - \mathbf{P}_0\|_*^2 \geq 0$. *If* $\|\mathbf{d}_0\|$ *and* $\|\mathbf{d}_1\|$ *are sufficiently small, there generically exist MPH quartic interpolants satisfying* H_C^1.

Proof. Recall that MPH property is invariant under parallel translation. Thus, without loss of generality, we can assume that $\mathbf{P}_0 = (0, 0, 0)$. This means that, to complete this theorem, it is sufficient to show the generic solvability of the quadratic equation Eq. () for $\mathcal{N}(H_C^1)$, equivalently the given data is generically admissible.

Thus we show that, if $\|\mathbf{d}_0\|$ and $\|\mathbf{d}_1\|$ are sufficiently small, the discriminant Δ for $\mathcal{N}(H_C^1)$ is generically non-negative. First, using $\mathbf{a}_1 = \mathbf{a}_0 e^{2\eta i}$ and $\tilde{\mathbf{d}}_0 = \dfrac{\mathbf{d}_0}{\mathbf{a}_0}$, we rewrite the coefficients of Eq. () follows:

$$m_0 = -3z_1 + \left(3Re(\frac{\mathbf{s}_1}{\mathbf{a}_0}) - \frac{1}{2}Re(\frac{\mathbf{d}_1}{\mathbf{a}_0} - \frac{\mathbf{d}_1}{\mathbf{a}_1})\right),$$

$$\begin{aligned} m_1 &= 6z_1\cos(\eta) - \left((6Re(\frac{\mathbf{s}_1}{\mathbf{a}_0}e^{-\eta i}) - \frac{1}{2}Re((\frac{\mathbf{d}_1}{\mathbf{a}_0} - \frac{\mathbf{d}_1}{\mathbf{a}_1})e^{-\eta i}) - Im(\frac{\mathbf{d}_0}{\mathbf{a}_0})\sin(\eta)\right), \\ m_2 &= -3z_1 + \left(3Re(\frac{\mathbf{s}_1}{\mathbf{a}_1}) + \frac{1}{2}Re(\frac{\mathbf{d}_0}{\mathbf{a}_0} - \frac{\mathbf{d}_0}{\mathbf{a}_1})\right). \end{aligned}$$

Here, assume that $\|\mathbf{d}_0\|$ and $\|\mathbf{d}_1\|$ are sufficiently small. Then, for $\xi_0 = -6z_1 + 6\left(\frac{\mathbf{s}_1}{\mathbf{a}_0}\right)$ and $\xi_1 = -6z_1 + 6\left(\frac{\mathbf{s}_1}{\mathbf{a}_1}\right)$, we get

$$m_0 \approx \frac{1}{2}Re(\xi_0), \quad m_1 \approx -\frac{1}{2}Re(\xi_0 e^{-\eta i} + \xi_1 e^{\eta i}), \quad m_2 \approx \frac{1}{2}Re(\xi_1),$$

and the discriminant Δ of Eq. () for $\mathcal{N}(H_C^1)$ is to be computed as follows;

$$\begin{aligned} \Delta &= m_1^2 - 4m_0m_2 \\ &\approx \left(3z_1Re(e^{-\eta i} + e^{\eta i}) - 3Re(\frac{\mathbf{s}_1}{\mathbf{a}_0}e^{-\eta i} + \frac{\mathbf{s}_1}{\mathbf{a}_1}e^{\eta i})\right)^2 \\ &\quad -4\left(-3z_1 + 3Re(\frac{\mathbf{s}_1}{\mathbf{a}_0})\right)\left(-3z_1 + 3Re(\frac{\mathbf{s}_1}{\mathbf{a}_1})\right). \end{aligned}$$

Let $u = Re\left(\frac{\mathbf{s}_1}{\mathbf{a}_0}\right)$ and $v = Im\left(\frac{\mathbf{s}_1}{\mathbf{a}_0}\right)$. Then, we have

$$\begin{aligned} \frac{\Delta}{36} &\approx \left((z_1 - u)\cos(\eta) - v\sin(\eta)\right)^2 - (z_1 - u)\cdot\left(z_1 - u\cos(2\eta) - v\sin(2\eta)\right) \\ &= (u - z_1)^2\cos^2(\eta) + v^2\sin^2(\eta) - (u - z_1)u\cos(2\eta) + z_1(u - z_1) \\ &= (z_1^2 - u^2)\cos^2(\eta) + v^2\sin^2(\eta) + u^2 - z_1^2 \\ &= (u^2 + v^2 - z_1^2)\sin^2(\eta). \end{aligned}$$

Note that $u^2 + v^2 - z_1^2 = \left\|\frac{\mathbf{s}_1}{\mathbf{a}_0}\right\|^2 - z_1^2 = \|\mathbf{s}_1\|^2 - z_1^2 = \|\mathbf{P}_1\|_*^2$. Consequently, we obtain $\Delta \approx 36\sin^2(\eta)\|\mathbf{P}_1\|_*^2$. Thus, since $\|\mathbf{P}_1 - \mathbf{P}_0\|_*^2 = \|\mathbf{P}_1\|_*^2 \geq 0$ by assumption, for sufficiently small $\|\mathbf{d}_0\|$ and $\|\mathbf{d}_1\|$, the C^1 Hermite data H_C^1 is generically admissible, i.e. by Theorem 2, there generically exist eight MPH quartic interpolants satisfying H_C^1.

Theorem 5. *Let $H_C^1 = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0^*, \mathbf{D}_1^*\right\}$ be a C^1 Hermite data in $\mathbb{R}^{2,1}$. Assume that $\|\mathbf{P}_1 - \mathbf{P}_0\|_*^2 \geq 0$ and $\|\mathbf{d}_0\|$ and $\|\mathbf{d}_1\|$ are sufficiently small. There generically exist infinitely many C^1 interpolants satisfying H_C^1, each of which is given by the speed reparametrization of a MPH quartic.*

Proof. First, we modify the given C^1 Hermite data H_C^1 to another C^1 Hermite data $\widetilde{H_C^1} = \left\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0, \mathbf{D}_1\right\}$, where $\mathbf{D}_0 = \epsilon_0\mathbf{D}_0^*$ and $\mathbf{D}_1 = \epsilon_1\mathbf{D}_1^*$ for some sufficiently small positive real numbers ϵ_0 and ϵ_1. Then, by Theorem 4, there generically exist eight MPH quartic interpolants satisfying $\widetilde{H_C^1}$. In addition, by

Theorem 3, there exists a speed reparametrization from $\widetilde{H_C^1}$ to H_C^1. Thus, we can obtain eight speed reparametrized interpolants satisfying H_C^1.

Example 3. Recall the non-admissible C^1 Hermite data H_C^1 used in Example 2. As shown in the proof of Theorem 4, by handling ϵ_0 and ϵ_1, we can obtain suitably an admissible C^1 Hermite data $\widetilde{H_C^1} = \{\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0, \mathbf{D}_1\}$ where $\mathbf{D}_0 = \epsilon_0 \mathbf{D}_0^*$ and $\mathbf{D}_1 = \epsilon_1 \mathbf{D}_1^*$ for $\epsilon_0, \epsilon_1 \in \mathbb{R}^+$.

For this Hermite data H_C^1, it is sufficient to assign $\epsilon_0 = \dfrac{1}{2}$ and $\epsilon_1 = \dfrac{1}{2}$, for which $\widetilde{H_C^1}$ is admissible. Then, by Theorem 2, we can obtain eight MPH quartic interpolants $\alpha_j(t) = \Big(x_j(t), y_j(t), z_j(t)\Big)$ for $j = 1, \cdots, 8$, as follows:

$$\begin{aligned}
x_1 &= -17.92139524t^4 + 36.34279050t^3 - 19.42139526t^2 + 2.000000000t + 1,\\
y_1 &= 15.38644617t^4 - 29.27289235t^3 + 12.88644618t^2 + 2.000000000t + 1,\\
z_1 &= -14.79453532t^4 + 32.58907056t^3 - 19.79453521t^2 + 2.000000000t + 0.5,
\end{aligned}$$

$$\begin{aligned}
x_2 &= -0.9348766928t^4 + 2.369753385t^3 - 2.434876692t^2 + 2.000000000t + 1,\\
y_2 &= 18.76809818t^4 - 36.03619635t^3 + 16.26809816t^2 + 2.000000000t + 1,\\
z_2 &= 2.248080164t^4 - 1.496160367t^3 - 2.751919778t^2 + 2.000000000t + 0.5,
\end{aligned}$$

$$\begin{aligned}
x_3 &= -9.500747940t^4 + 19.50149588t^3 - 11.00074794t^2 + 2.000000000t + 1,\\
y_3 &= -2.596409998t^4 + 6.692819997t^3 - 5.096410000t^2 + 2.000000001t + 1,\\
z_3 &= 0.7914037095t^4 + 1.417192576t^3 - 4.208596282t^2 + 2.000000000t + 0.5,
\end{aligned}$$

$$\begin{aligned}
x_4 &= -12.08160094t^4 + 24.66320189t^3 - 13.58160094t^2 + 2.000000000t + 1,\\
y_4 &= 8.894921560t^4 - 16.28984312t^3 + 6.394921566t^2 + 2.000000000t + 1,\\
z_4 &= 11.98777284t^4 - 20.97554568t^3 + 6.987772831t^2 + 2.000000000t + 0.5,
\end{aligned}$$

$$\begin{aligned}
x_5 &= -2.497386161t^4 + 5.494772321t^3 - 3.997386161t^2 + 2.000000000t + 1,\\
y_5 &= -3.308128658t^4 + 8.116257315t^3 - 5.808128657t^2 + 1.999999999t + 1,\\
z_5 &= -3.336346155t^4 + 9.672692311t^3 - 8.336346157t + 2.000000000t + 0.5,
\end{aligned}$$

$$\begin{aligned}
x_6 &= -0.8107982478t^4 + 2.121596495t^3 - 2.310798247t^2 + 2.000000000t + 1,\\
y_6 &= 2.156235740t^4 - 2.812471480t^3 - 0.3437642609t^2 + 2.000000000t + 1,\\
z_6 &= 1.270068909t^4 + 0.459862182t^3 - 3.729931090t^2 + 2.000000000t + 0.5,
\end{aligned}$$

$$x_7 = -0.1014349368t^4 + 0.7028698740t^3 - 1.601434938t^2 + 2.000000000t + 1,$$

$$
\begin{aligned}
y_7 &= -6.157326852t^4 + 13.81465371t^3 - 8.657326853t^2 + 2.000000000t + 1,\\
z_7 &= -4.968413565t^4 + 12.93682713t^3 - 9.968413573t^2 + 2.000000000t + 0.5,
\end{aligned}
$$

$$
\begin{aligned}
x_8 &= 3.520774458t^4 - 6.541548915t^3 + 2.020774458t^2 + 1.999999999t + 1,\\
y_8 &= -1.081623640t^4 + 3.663247280t^3 - 3.581623639t^2 + 2.000000000t + 1,\\
z_8 &= 0.6625911992t^4 + 1.674817602t^3 - 4.337408802t^2 + 2.000000000t + 0.5.
\end{aligned}
$$

In addition, by Theorem 3, there exist a speed reparametrization form $\widetilde{H^1_C}$ to H^1_C, and moreover, since $(\lambda_0, \lambda_1) = \left(\frac{1}{\epsilon_0}, \frac{1}{\epsilon_1}\right) = (2, 2)$ lies in the solvable region for the speed reparametrization, as shown in the proof of Theorem 17 in [16], we obtain the speed reparametrization $\tilde{t} = \phi(t)$ given by $\phi(t) = 2\,t^3 - 3\,t^2 + 2\,t$. Consequently, we can construct eight MPH quartic C^1 Hermite interpolants $\alpha_j(\phi(t))$ satisfying H^1_C for $j = 1, \cdots, 8$, as shown in Figure 1.

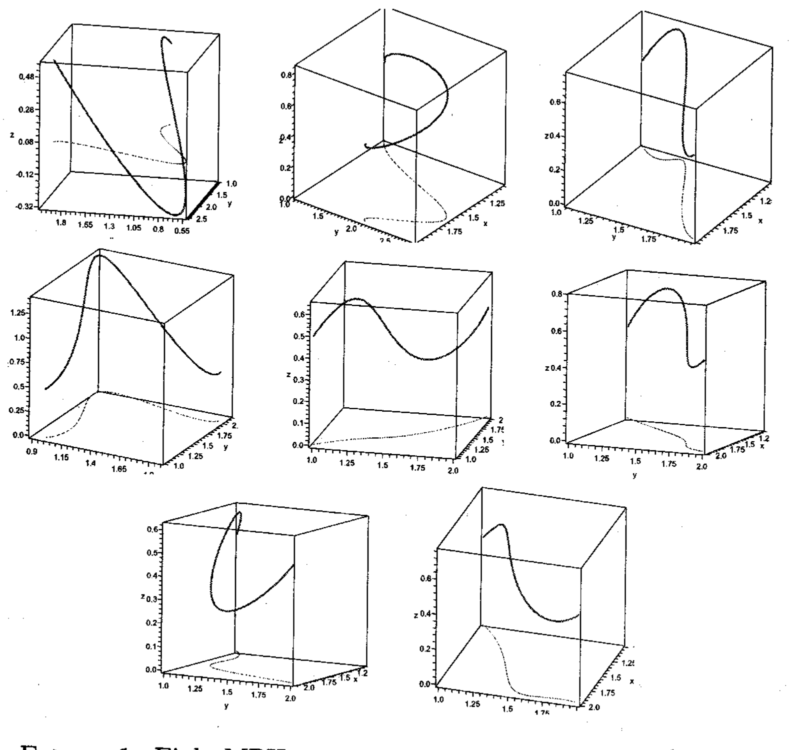

FIGURE 1. Eight MPH quartic interpolants satisfying H^1_C given in Remark 2. The MPH quartic interpolants are denoted by the thick solid curves and their spine curves are denoted by the thin dotted curves.

REFERENCES

1. G. Albrecht and R. T. Farouki, *Construction of C^2 Pythagorean-hodograph interpolating splines by the homotopy method*, Advances in Computational Mathematics **5(4)** (1996), 417–442.
2. H. I. Choi, C. Y. Han, H. P. Moon, K. H. Roh and N. S. Wee, *Medial Axis transform and offset curves by Minkowski Pythagorean hodograph curves*, Computer-Aided Design **31** (1999), 52–72.
3. H. I. Choi, D. S. Lee, and H. P. Moon, *Clifford algebra, spin representation, and rational parameterization of curves and surfaces*, Adv. Comput. Math. **17** (2002), 4–48.
4. R. T. Farouki, *Pythagorean-hodograph curves in practical uses.* In Barnhill,R.E.(Ed.), Geometric Processing for Design and Manufacturing. **SIAM**,(1992), 3–33.
5. R. T. Farouki, M. al-Kandari and T. Sakkalis, *Hermite interpolation by rotation-invariant spiral Pythagorean-hodograph curves*, Advances in computational Mathematics **17** (2002), 369–383.
6. R. T. Farouki and C. A. Neff, *Hermite interpolation by Pythagorean-hodograph quintics*, Mathematics of Computation **64** (1995), 1589-1609.
7. R. T. Farouki and T. Sakkalis, *Pythagorean hodographs*, IBM J. Res. Development **34** (1990), 736–752.
8. R. T. Farouki and T. Sakkalis, *Pythagorean-hodograph space curves*, Adv. Comput. Math. **2** (1994), 41–66.
9. R. T. Farouki and T. Sakkalis, *Pythagorean-hodograph space curves*, International J. Machine tools and Manufacture **39** (1999), 123–142.
10. R. T. Farouki and S. Shah, *Real-time CNC interpolators for Pythagorean-hodograph curves*, Comput. Aided Geom. Design **13** (1996), 583–600.
11. B. Jüttler, *Hermite interpolation by Pythagorean-hodograph curves of degree seven*, Mathematics of computation **70** (2001), 1089–1111.
12. G. I. Kim, *Higher dimensional PH curves*, Proc. Japan Acad., **78** (2002), Ser. A., 185–187.
13. G. I. Kim and M. H. Ahn, *C^1 Hermite interpolation using MPH quartic*, Comput. Aided Geom. Design **20** (2003), 469–492.
14. G. I. Kim, J. H. Kong and S. Lee, *First order Hermite interpolation with spherical Pythagorean-hodograph curves*, Journal of Applied Mathematics and computing **23** (2007), 73–86.
15. G.I. Kim and S. Lee, *Higher dimensional Minkowski Pythagorean hodograph curves*, J. Appl. Math. & Computing, **14** (2004), 405–413.
16. J. H. Kong, S. P. Jeong, S. Lee and G. I. Kim, *C^1 Hermite interpolation with simple planar PH curves by speed reparametrization*, Comput. Aided Geom. Design **25** (2008), 214–229.
17. H. P. Moon, *Minkowski Pythagorean hodographs*, Comput. Aided Geom. Design **16** (1999), 739–753.
18. Z. Šir and B. Jüttler, *Eucidean and Minkowski Pythagorean hodograph curves over planar cubics*, Comput. Aided Geom. Design **22** (2005), 753–770.

Gwang-Il Kim is a professor in Department of Mathematics, Gyeongsang National University, Jinju, Korea. He received his M. Sc from POSTECH in 1991 and Ph.D from the same University in 1996. His research interests focus on computer aided geometric design, dynamical systems and social networks.

Department of Mathematics and Research Institute of Natural Science, Gyeongsang National University, Jinju, 660-701 Korea

e-mail: gikim@gsnu.ac.kr (Gwang-Il Kim)