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PARAMETER-UNIFORM NUMERICAL METHOD FOR A SYSTEM OF COUPLED SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATIONS WITH MIXED TYPE BOUNDARY CONDITIONS

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ABSTRACT. In this paper, a numerical method for a weakly coupled system of two singularly perturbed convection-diffusion second order ordinary differential equations with the mixed type boundary conditions is presented. Parameter-uniform error bounds for the numerical solution and also to numerical derivative are established. Numerical results are provided to illustrate the theoretical results.

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1. Introduction

Many problems arising from various fields of physical interest such as biochemical kinetics, fluid dynamics, plasma physics, mechanical and electrical systems, the observed phenomenon, for examples, linearized Navier-Stokes equation at high Reynolds number, drift diffusion equation of semiconductor device modeling, mechanical oscillator, chemical reactor, etc, described by differential equation involving a small parameter, affecting the highest derivative term. Singularly perturbed Initial Value Problems (IVPs)/ Boundary Value Problems (BVPs) in Ordinary Differential Equations (ODEs) are characterized by the presence of a small parameter ($0 < \varepsilon \ll 1$) that multiplies the highest derivative term. Solution of such problems exhibit sharp boundary and/or interior layers when the small parameter ε is much smaller than 1. The numerical solution

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of such problems exhibit significant difficulties, particularly when the diffusion coefficient is small. Therefore, the interest in developing and analyzing efficient numerical methods for singularly perturbed problems has increased enormously (see [1, 2, 3, 4] and the reference are therein). Parameter-uniform numerical methods have been developed for a single singularly perturbed differential equation, but for system of equations, only few results are reported in the literature, even though, the coupled system of differential equations appear in many applications, notably optimal control problems and in certain resistance capacitor electrical circuits.

Robust parameter-uniform numerical methods for a system of singularly perturbed ordinary differential equations have been examined by a few authors [6] - [15]. Matthews [6], examined a Dirichlet problem for a system of two coupled singularly perturbed reaction-diffusion ODEs. It is shown that a parameter robust numerical method can be constructed which gives first order convergence. Valanarasu et al. [8] considered the same problem and they suggested an asymptotic initial value method. This method involves solving a set of IVPs and terminal value problems by fitted operator method. In [10], a parameter-uniform finite difference method for a system of coupled singularly perturbed convection-diffusion equations is presented. It is proved that the scheme converges almost first-order uniformly with respect to the small parameter. In [14], the author considered a system of reaction - convection - diffusion type equations. The author briefly summarized the stability and convergence results for (upwind) finite difference discretization. In [15], a system of coupled convection-diffusion equations having diffusion parameters of different magnitudes is discussed. A robust convergence with respect to the perturbation parameters is obtained. In [12], the author presented a computational method for a weakly coupled system of singularly perturbed reaction-diffusion equations with discontinuous source term.

While many finite difference methods have been proposed to approximate such solutions, there has been much less research into the finite-difference approximation of their derivatives, even though such approximations are desirable in certain applications (flux or drag). It should be noted that for convection-diffusion problems, the attainment of high accuracy in a computed solution does not automatically lead to good approximation of derivatives of the true solution. In [4], for singularly perturbed convection-diffusion problems with continuous convection coefficient and source term for single differential equation estimates for numerical derivatives have been derived. Here the scaled derivative is taken on whole domain whereas N. Kopteva et al. [16] have obtained approximation of derivatives with scaling in the boundary layer region and without scaling in the outer region. R. Mythili Priyadharshini et al. [13], have determined estimate for the scaled derivative in the boundary layer region and non-scaled derivative in the outer region for the system of singularly perturbed convection-diffusion equations with Dirichlet boundary conditions.

Motivated by the works of [5, 9, 10, 11, 13], in this article, we consider the following class of problems:

$$\begin{cases} L_1\bar{y} \equiv -\varepsilon y_1'' - a_1(x)y_1' + b_{11}(x)y_1 + b_{12}(x)y_2 = f_1(x), \\ L_2\bar{y} \equiv -\varepsilon y_2'' - a_2(x)y_2' + b_{21}(x)y_1 + b_{22}(x)y_2 = f_2(x), \quad x \in \Omega, \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} B_{10}y_1(0) \equiv \beta_{10}y_1(0) - \varepsilon\beta_{11}y_1'(0) = A_1, \quad B_{11}y_1(1) \equiv \gamma_{11}y_1(1) + \gamma_{12}y_1'(1) = B_1, \\ B_{20}y_2(0) \equiv \beta_{20}y_2(0) - \varepsilon\beta_{21}y_2'(0) = A_2, \quad B_{21}y_2(1) \equiv \gamma_{21}y_2(1) + \gamma_{22}y_2'(1) = B_2. \end{cases} \tag{2}$$

Assume that

$$a_1(x) \geq \alpha_1 > 0, \quad a_2(x) \geq \alpha_2 > 0,$$
$$b_{12}(x) \leq 0, \quad b_{21}(x) \leq 0,$$
$$\{b_{11}(x) + b_{12}(x)\} \geq 0, \quad \{b_{22}(x) + b_{21}(x)\} \geq 0,$$

where $\bar{y} = (y_1, y_2)^T$, $y_1, y_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ and the functions $a_i(x)$, $f_i(x)$, $b_{ij}(x)$ are sufficiently smooth on $\bar{\Omega}$, $\Omega = (0,1)$, $0 < \varepsilon \leq 1$, $\beta_{j0}, \beta_{j1} \geq 0$, $\beta_{j0} + \varepsilon\beta_{j1} \geq 1$, $\gamma_{j2} \geq 0$ and $\gamma_{j1} - \gamma_{j2} \geq 1$, for $i, j = 1, 2$. Let $\alpha = \min\{\alpha_1, \alpha_2\}$. The above system can be written in the matrix form as

$$\mathbf{L}\bar{y} \equiv \begin{pmatrix} L_1\bar{y} \\ L_2\bar{y} \end{pmatrix} \equiv \begin{pmatrix} -\varepsilon\frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon\frac{d^2}{dx^2} \end{pmatrix} \bar{y} - \mathbf{A}(x)\bar{y}' + \mathbf{B}(x)\bar{y} = \bar{f}(x), \quad x \in \Omega,$$

with the boundary conditions

$$\begin{pmatrix} B_{10}y_1(0) \\ B_{20}y_2(0) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad \begin{pmatrix} B_{11}y_1(1) \\ B_{21}y_2(1) \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where

$$\mathbf{A}(x) = \begin{pmatrix} a_1(x) & 0 \\ 0 & a_2(x) \end{pmatrix}, \quad \mathbf{B}(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}, \quad \bar{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}.$$

The present paper extends the results available in [5] for a single one - dimensional singularly perturbed convection-diffusion equation with mixed boundary conditions to a weakly coupled system of two singularly perturbed convection-diffusion equations with mixed boundary conditions. In this paper, we obtain parameter-uniform approximations not only to the solution but also to its derivatives. Thus in this paper, motivated by the works of [9] and [16], bounds on the errors in approximating the first derivative of the solution with a weight in the fine mesh and without a weight in the coarse mesh are obtained.

Through out this paper, C denotes a generic constant (sometimes subscripted) is independent of the singular perturbation parameter ε and the dimension of the discrete problem N. Let $y : D \to \mathbb{R}$, $D \subset \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the maximum norm $\|y\|_D = \sup_{x \in D} |y(x)|$. In case of vectors $\bar{y}$, we define $|\bar{y}(x)| = (|y_1(x)|, |y_2(x)|)^T$ and $\|\bar{y}\|_D = \max\{\|y_1\|_D, \|y_2\|_D\}$.

2. Preliminaries

In the following, the maximum principle, stability result and derivative estimates are established for BVP (1)-(2).

Theorem 1 (Maximum Principle). *Suppose that a function*

$$\bar{y}(x) = (y_1(x), y_2(x))^T, \quad y_1, y_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$$

satisfies $B_{j0}y_j(0) \geq 0$, $B_{j1}y_j(1) \geq 0$, *for* $j = 1, 2$ *and* $\boldsymbol{L}\bar{y}(x) \geq \bar{0}$, $\forall x \in \Omega$. *Then* $\bar{y}(x) \geq \bar{0}$, $\forall x \in \bar{\Omega}$.

Proof. Define $\bar{s}(x) = (s_1(x), s_2(x))^T$ as

$$s_1(x) = s_2(x) = 2 - x.$$

Then s_1, $s_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$, $\bar{s}(x) > \bar{0}$, for all $x \in \bar{\Omega}$ and $\mathbf{L}\bar{s}(x) > \bar{0}$, $x \in \Omega$. So, we further define

$$\mu = \max\left\{\max_{x\in\bar{\Omega}}\left(-\frac{y_1}{s_1}\right), \max_{x\in\bar{\Omega}}\left(-\frac{y_2}{s_2}\right)\right\}$$

Assume that the theorem is not true. Then $\mu > 0$ and there exists a point $x_0 \in \bar{\Omega}$, such that either $(\frac{-y_1}{s_1})(x_0) = \mu$ or $(\frac{-y_2}{s_2})(x_0) = \mu$ or both. Also $(\bar{y} + \mu\bar{s})(x) \geq \bar{0}$, $\forall x \in \bar{\Omega}$.

Case (i): $(y_1 + \mu s_1)(x_0) = 0$, for $x_0 = 0$. It implies that $(y_1 + \mu s_1)$ attains its minimum at x_0. Therefore,

$$\begin{aligned} 0 &< B_{10}(y_1 + \mu s_1)(x_0) \\ &= \beta_{10}(y_1 + \mu s_1)(x_0) - \varepsilon\beta_{11}(y_1 + \mu s_1)'(x_0) \\ &\leq 0, \end{aligned}$$

which is a contradiction.

Case (ii): $(y_1 + \mu s_1)(x_0) = 0$, for $x_0 \in \Omega$. It implies that $(y_1 + \mu s_1)$ attains its minimum at x_0. Therefore,

$$\begin{aligned} 0 &< L_1\bar{y}(x) \\ &\equiv -\varepsilon(y_1 + \mu s_1)''(x) - a_1(x)(y_1 \\ &\quad + \mu s_1)'(x) + b_{11}(x)(y_1 + \mu s_1)(x) + b_{12}(x)(y_2 + \mu s_2)(x) \\ &\leq 0, \end{aligned}$$

which is a contradiction.

Case (iii): $(y_1 + \mu s_1)(x_0) = 0$, for $x_0 = 1$. It implies that $(y_1 + \mu s_1)$ attains its minimum at x_0. Therefore,

$$\begin{aligned} 0 &< B_{11}(y_1 + \mu s_1)(x_0) \\ &= \gamma_{11}(y_1 + \mu s_1)(x_0) + \gamma_{12}(y_1 + \mu s_1)'(x_0) \\ &\leq 0, \end{aligned}$$

which is a contradiction.

Case (iv): $(y_2 + \mu s_2)(x_0) = 0$, for $x_0 = 0$. Similar to Case(i), it leads to a contradiction.

Case (v): $(y_2 + \mu s_2)(x_0) = 0$, for $x_0 \in \Omega$. Similar to Case(ii), it leads to a contradiction.

Case (vi): $(y_2 + \mu s_2)(x_0) = 0$, for $x_0 = 1$. Similar to Case(iii), it leads to a contradiction.

Hence $\bar{y}(x) \geq \bar{0}$, $\forall x \in \bar{\Omega}$.

In the rest of the problem for continuous case the norm $\|\cdot\|$ means $\|\cdot\|_{\bar{\Omega}}$.

Theorem 2 (Stability Result). *If $y_1, y_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ then for $j = 1, 2$*

$$|y_j(x)| \leq C \max\{|B_{10}y_1(0)|, |B_{11}y_1(1)|, |B_{20}y_2(0)|, |B_{21}y_2(1)|, \|L_1\bar{y}\|_\Omega, \|L_2\bar{y}\|_\Omega\},$$

where $x \in \bar{\Omega}$.

Proof. Set

$$M = C \max\{|B_{10}y_1(0)|, |B_{11}y_1(1)|, |B_{20}y_2(0)|, |B_{21}y_2(1)|, \|L_1\bar{y}\|_\Omega, \|L_2\bar{y}\|_\Omega\}.$$

It is easy to see that

$$M(2\beta_{10} + \varepsilon\beta_{11}, 2\beta_{20} + \varepsilon\beta_{21})^T \pm (B_{10}y_1(0), B_{20}y_2(0))^T$$

and

$$M(\gamma_{11} - \gamma_{12}, \gamma_{21} - \gamma_{22})^T \pm (B_{11}y_1(1), B_{21}y_2(1))^T$$

are non-negative. Further

$$\begin{aligned}\mathbf{L}(M(2-x, 2-x)^T \pm \bar{y}(x)) &= M\mathbf{A}(x) + M(2-x)\begin{pmatrix} b_{11}(x) + b_{12}(x) \\ b_{21}(x) + b_{22}(x) \end{pmatrix} \pm \bar{f}(x) \\ &\geq \begin{pmatrix} M\alpha_1 \pm f_1(x) \\ M\alpha_2 \pm f_2(x) \end{pmatrix} \geq \bar{0},\end{aligned}$$

by a proper choice of C. Applying Theorem 1 implies that $M(2-x, 2-x) \pm \bar{y}(x) \geq \bar{0}, x \in \bar{\Omega}$, and the desired result follows.

Lemma 1. *Let $\bar{y} = (y_1, y_2)^T$ be the solution of (1)-(2). Then, for $j = 1, 2$*

$$\|y_j^{(k)}\| \leq C\varepsilon^{-k} \max\{\|f_j\|, \|\bar{y}\|\}, \quad \text{for } k = 1, 2$$

$$\|y_j^{(3)}\| \leq C\varepsilon^{-3} \max\{\|f_j\|, \|f_j'\|, \|\bar{y}\|\},$$

where C depends on $\|a_1\|$, $\|a_2\|$, $\|a_1'\|$, $\|a_2'\|$, $\|b_{11}\|$, $\|b_{12}\|$, $\|b_{11}'\|$, $\|b_{12}'\|$, $\|b_{21}\|$, $\|b_{22}\|$, $\|b_{21}'\|$ and $\|b_{22}'\|$.

Proof. Using the technique adopted in [4, pp. 44,45], the present theorem can be proved.

The sharper bounds on the derivatives of the solution are obtained by decomposing the solution $\bar{y}$ into regular and singular components as, $\bar{y} = \bar{v} + \bar{w}$, where $\bar{v} = (v_1, v_2)^T$ and $\bar{w} = (w_1, w_2)^T$. The regular component $\bar{v}$ can be written in the form $\bar{v} = \bar{v}_0 + \varepsilon\bar{v}_1 + \varepsilon^2\bar{v}_2$, where $\bar{v}_0 = (v_{01}, v_{02})^T$, $\bar{v}_1 = (v_{11}, v_{12})^T$, $\bar{v}_2 = (v_{21}, v_{22})^T$ are defined respectively to be the solutions of the problems

$$\mathbf{A}(x)\bar{v}_0' + \mathbf{B}(x)\bar{v}_0 = \bar{f}(x),\ x \in \Omega,$$
$$\begin{pmatrix} B_{11}v_{01}(1) \\ B_{21}v_{02}(1) \end{pmatrix} = \begin{pmatrix} B_{11}y_1(1) \\ B_{21}y_2(1) \end{pmatrix},$$
$$\mathbf{A}(x)\bar{v}_1' + \mathbf{B}(x)\bar{v}_1 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix} \bar{v}_0,$$
$$\begin{pmatrix} B_{11}v_{11}(1) \\ B_{21}v_{12}(1) \end{pmatrix} = \bar{0},$$

and

$$\mathbf{L}\bar{v}_2 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix} \bar{v}_1,$$
$$\begin{pmatrix} B_{10}v_{21}(0) \\ B_{20}v_{22}(0) \end{pmatrix} = \bar{0},\ \begin{pmatrix} B_{11}v_{21}(1) \\ B_{21}v_{22}(1) \end{pmatrix} = \bar{0}.$$

Thus the regular component $\bar{v}$ is the solution of

$$\mathbf{L}\bar{v} = \bar{f}(x),\ x \in \Omega \tag{3}$$
$$\begin{pmatrix} B_{10}v_1(0) \\ B_{20}v_2(0) \end{pmatrix} = \begin{pmatrix} B_{10}v_{01}(0) + \varepsilon(B_{10}v_{11}(0)) \\ B_{20}v_{02}(0) + \varepsilon(B_{20}v_{12}(0)) \end{pmatrix},\ \begin{pmatrix} B_{11}v_1(1) \\ B_{21}v_2(1) \end{pmatrix} = \begin{pmatrix} B_{11}y_1(1) \\ B_{21}y_2(1) \end{pmatrix}. \tag{4}$$

Then the singular component $\bar{w}$ is the solution of

$$\mathbf{L}\bar{w} = \bar{0}, \tag{5}$$
$$\begin{pmatrix} B_{10}w_1(0) \\ B_{20}w_2(0) \end{pmatrix} = \begin{pmatrix} B_{10}y_1(0) - B_{10}v_1(0) \\ B_{20}y_2(0) - B_{20}v_2(0) \end{pmatrix},\quad \begin{pmatrix} B_{11}w_1(1) \\ B_{21}w_2(1) \end{pmatrix} = \bar{0}. \tag{6}$$

The following lemma provides the bound on the derivatives of the regular and singular components of the solution $\bar{y}$.

Lemma 2. *The solution $\bar{y}$ can be decomposed into the sum $\bar{y} = \bar{v} + \bar{w}$, where, $\bar{v}$ and $\bar{w}$ are regular and singular components respectively. Further, these components and their derivatives satisfy the bounds for $j = 1, 2$*

$$\|v_j^{(k)}\| \le C(1 + \varepsilon^{2-k}),\quad k = 0, 1, 2, 3,$$

and

$$|w_j^{(k)}(x)| \le C\varepsilon^{-k}e^{-\alpha x/\varepsilon},\quad k = 0, 1, 2, 3,\quad \forall x \in \bar{\Omega}.$$

Proof. Using appropriate barrier functions, applying Theorem 1 and adopting the method of proof used in [4, p. 46], the present lemma can be proved. □

3. Discrete problem

The system of BVP (1)-(2) is now discretised using a fitted mesh method composed of a standard finite difference operator on a fitted piecewise uniform mesh

$$\bar{\Omega}_\varepsilon^N = \{x_i \mid x_i = 2i\sigma/N,\ 0 \le i \le N/2;\ x_i = x_{i-1} + 2(1-\sigma)/N,\ N/2+1 \le i \le N\}$$

condensing at the boundary point $x_0 = 0$ with two different mesh widths $h = \frac{2\sigma}{N}$ and $H = \frac{2(1-\sigma)}{N}$. The transition parameter σ is chosen to satisfy $\sigma = \min\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\}$. The resulting fitted mesh finite difference method is to find $\bar{Y}(x_i) = (Y_1(x_i), Y_2(x_i))^T$ for $i = 0, 1, 2, \ldots, N$ such that for $x_i \in \Omega_\varepsilon^N$,

$$\begin{cases} L_1^N \bar{Y}(x_i) \equiv -\varepsilon\delta^2 Y_1(x_i) - a_1(x_i)D^+Y_1(x_i) \\ \qquad\qquad +b_{11}(x_i)Y_1(x_i) + b_{12}(x_i)Y_2(x_i) = f_1(x_i), \\ L_2^N \bar{Y}(x_i) \equiv -\varepsilon\delta^2 Y_2(x_i) - a_2(x_i)D^+Y_2(x_i) \\ \qquad\qquad +b_{21}(x_i)Y_1(x_i) + b_{22}(x_i)Y_2(x_i) = f_2(x_i), \end{cases} \tag{7}$$

$$\begin{cases} B_{10}^N Y_1(x_0) \equiv \beta_{10}Y_1(x_0) - \varepsilon\beta_{11}D^+Y_1(x_0) = A_1, \\ B_{11}^N Y_1(x_N) \equiv \gamma_{11}Y_1(x_N) + \gamma_{12}D^-Y_1(x_N) = B_1, \\ B_{20}^N Y_2(x_0) \equiv \beta_{20}Y_2(x_0) - \varepsilon\beta_{21}D^+Y_2(x_0) = A_2, \\ B_{21}^N Y_2(x_N) \equiv \gamma_{21}Y_2(x_N) + \gamma_{22}D^-Y_2(x_N) = B_2. \end{cases} \tag{8}$$

The finite difference operator δ^2 is the central difference operator defined as

$$\delta^2 U_j(x_i) = \frac{(D^+ - D^-)U_j(x_i)}{(x_{i+1} - x_{i-1})/2}, \quad \text{for } j = 1, 2,$$

where

$$D^+U_j(x_i) = \frac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i} \quad \text{and} \quad D^-U_j(x_i) = \frac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}}.$$

The difference operator $\mathbf{L}^N$ can be defined as for $x_i \in \Omega_\varepsilon^N$,

$$\begin{aligned} \mathbf{L}^N \bar{Y}(x_i) &\equiv \begin{pmatrix} L_1^N \bar{Y}(x_i) \\ L_2^N \bar{Y}(x_i) \end{pmatrix} \\ &\equiv \begin{pmatrix} -\varepsilon\delta^2 & 0 \\ 0 & -\varepsilon\delta^2 \end{pmatrix} \bar{Y}(x_i) - \mathbf{A}(x_i)D^+\bar{Y}(x_i) + \mathbf{B}(x_i)\bar{Y}(x_i) = \bar{f}(x_i), \end{aligned}$$

$$\begin{pmatrix} B_{10}^N Y_1(x_0) \\ B_{20}^N Y_2(x_0) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \qquad \begin{pmatrix} B_{11}^N Y_1(x_N) \\ B_{21}^N Y_2(x_N) \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

3.1. Numerical solution estimates. Analogous to the continuous results stated in Theorem 1 and Theorem 2 one can prove the following results.

Theorem 3. *For any mesh function* $\bar{\Psi}(x_i)$ *assume that*

$$B_{j0}^N\Psi_j(x_0) \geq 0, \quad B_{j1}^N\Psi_j(x_N) \geq 0$$

for $j = 1, 2$ *and* $\boldsymbol{L}^N\bar{\Psi}(x_i) \geq \bar{0}$, *for all* $i = 1, \ldots, N-1$. *Then* $\bar{\Psi}(x_i) \geq \bar{0}$, *for all* $i = 0, 1, \ldots, N$.

Proof. Define $\bar{S}(x_i) = (S_1(x_i), S_2(x_i))^T$ as $S_1(x_i) = S_2(x_i) = 2 - x_i$. Then $\bar{S}(x_i) > \bar{0}$ for all $x_i \in \bar{\Omega}_\varepsilon^N$. So, we further define

$$\xi = \max\left\{ \max_{x_i\in\bar{\Omega}_\varepsilon^N} (\frac{-\Psi_1}{S_1})(x_i), \ \max_{x_i\in\bar{\Omega}_\varepsilon^N} (\frac{-\Psi_2}{S_2})(x_i) \right\}.$$

Assume that the theorem is not true. Then $\xi > 0$ and $(\bar{\Psi}+\xi\bar{S})(x_i) = \bar{0}$. Further there exists a $i^* \in \{0, 1, 2, \ldots, N\}$ such that $(\bar{\Psi}+\xi\bar{S})(x_{i^*}) \geq \bar{0}$ and we consider the following cases:

Case (i): $(\bar{\Psi}+\xi\bar{S})(x_{i^*}) = \bar{0}$, for $i^* = 0$. Therefore,

$$\begin{aligned} 0 &\leq B_{10}^N(\Psi_1+\xi S_1)(x_{i^*}) \\ &= \beta_{10}(\Psi_1+\xi S_1)(x_{i^*}) - \varepsilon\beta_{11}D^+(\Psi_1+\xi S_1)(x_{i^*}) \\ &= -\varepsilon\beta_{11}\frac{(\Psi_1+\xi S_1)(x_{i^*+1}) - (\Psi_1+\xi S_1)(x_{i^*})}{x_{i^*+1}-x_{i^*}} < 0, \end{aligned}$$

which is a contradiction. Similarly $B_{20}^N(\Psi_2+\xi S_2)(x_{i^*}) < 0$, which is again a contradiction.

Case (ii): $(\bar{\Psi}+\xi\bar{S})(x_{i^*}) = \bar{0}$, for $0 < i^* < N$. Therefore,

$$\begin{aligned} 0 &\leq L_1^N(\bar{\Psi}+\xi\bar{S})(x_{i^*}) \\ &= -\varepsilon\delta^2(\Psi_1+\xi S_1)(x_{i^*}) - a_1(x_{i^*})D^+(\Psi_1+\xi S_1)(x_{i^*}) \\ &\quad + b_{11}(\Psi_1+\xi S_1)(x_{i^*}) + b_{12}(\Psi_2+\xi S_2)(x_{i^*}) < 0, \end{aligned}$$

which is a contradiction. Similarly $L_2^N(\bar{\Psi}+\xi\bar{S})(x_{i^*}) < 0$, which is again a contradiction.

Case (iii): $(\bar{\Psi}+\xi\bar{S})(x_{i^*}) = \bar{0}$, for $i^* = N$. Therefore,

$$\begin{aligned} 0 &\leq B_{1N}^N(\Psi_1+\xi S_1)(x_{i^*}) \\ &= \gamma_{11}(\Psi_1+\xi S_1)(x_{i^*}) + \gamma_{12}D^-(\Psi_1+\xi S_1)(x_{i^*}) \\ &= \gamma_{12}\frac{(\Psi_1+\xi S_1)(x_{i^*}) - (\Psi_1+\xi S_1)(x_{i^*-1})}{x_{i^*}-x_{i^*-1}} < 0, \end{aligned}$$

which is a contradiction. Similarly, $B_{2N}^N(\Psi_2+\xi S_2)(x_{i^*}) < 0$, which is again a contradiction.

Hence $\bar{\Psi}(x_i) \geq \bar{0}$, $\forall x_i \in \bar{\Omega}_\varepsilon^N$.

Theorem 4. *Consider the scheme (7)-(8). If $\bar{Z}(x_i) = (Z_1(x_i), Z_2(x_i))^T$ is any mesh function then, for all $x_i \in \bar{\Omega}_\varepsilon^N$,*

$$|Z_j(x_i)| \leq C \max\{|B_{10}^N Z_1(x_0)|, |B_{11}^N Z_1(x_N)|, |B_{20}^N Z_2(x_0)|, |B_{21}^N Z_2(x_N)|, \max_{1\leq i \leq N-1} |L_1^N \bar{Z}(x_i)|, \max_{1\leq i \leq N-1} |L_2^N \bar{Z}(x_i)|\}, \; j = 1, 2.$$

Proof. Set

$$M = C \max\{|B_{10}^N Z_1(x_0)|, |B_{11}^N Z_1(x_N)|, |B_{20}^N Z_2(x_0)|, |B_{21}^N Z_2(x_N)|, \max_{1\leq i \leq N-1} |L_1^N \bar{Z}(x_i)|, \max_{1\leq i \leq N-1} |L_2^N \bar{Z}(x_i)|\}.$$

Define the mesh functions

$$\bar{W}^{\pm}(x_i) = \begin{pmatrix} M(2-x_i) \\ M(2-x_i) \end{pmatrix} \pm \bar{Z}(x_i).$$

Then we have

$$B_{j0}^N W_j^{\pm}(x_0) \geq 0, \quad B_{j1}^N W_j^{\pm}(x_N) \geq 0 \quad \text{for } j = 1, 2$$

and $\mathbf{L}^N \bar{W}^{\pm}(x_i) \geq \bar{0}$. By Theorem 3 we get the required result.

The discrete solution $\bar{Y}(x_i)$ can be decomposed into the sum

$$\bar{Y}(x_i) = \bar{V}(x_i) + \bar{W}(x_i)$$

where $\bar{V}(x_i)$ and $\bar{W}(x_i)$ are regular and singular components respectively defined as

$$\mathbf{L}^N \bar{V}(x_i) = \bar{f}(x_i), \; i = 1, \ldots, N-1, \tag{9}$$

$$\begin{pmatrix} B_{10}^N V_1(x_0) \\ B_{20}^N V_2(x_0) \end{pmatrix} = \begin{pmatrix} B_{10} v_1(0), \\ B_{20} v_2(0) \end{pmatrix}, \quad \begin{pmatrix} B_{11}^N V_1(x_N) \\ B_{21}^N V_2(x_N) \end{pmatrix} = \begin{pmatrix} B_{11} v_1(1) \\ B_{21} v_2(1) \end{pmatrix} \tag{10}$$

and

$$\mathbf{L}^N \bar{W}(x_i) = \bar{0}, \; i = 1, \ldots, N-1, \tag{11}$$

$$\begin{pmatrix} B_{10}^N W_1(x_0) \\ B_{20}^N W_2(x_0) \end{pmatrix} = \begin{pmatrix} B_{10} w_1(0), \\ B_{20} w_2(0) \end{pmatrix}, \quad \begin{pmatrix} B_{11}^N W_1(x_N) \\ B_{21}^N W_2(x_N) \end{pmatrix} = \bar{0}. \tag{12}$$

The error in the numerical solution can be written in the form

$$(\bar{Y} - \bar{y})(x_i) = (\bar{V} - \bar{v})(x_i) + (\bar{W} - \bar{w})(x_i).$$

Lemma 3. *At each mesh point $x_i \in \bar{\Omega}_\varepsilon^N$ the regular component of the error satisfies the estimate*

$$|(\bar{V} - \bar{v})(x_i)| \leq \begin{pmatrix} C(2-x_i)N^{-1} \\ C(2-x_i)N^{-1} \end{pmatrix},$$

where $\bar{v}$ and $\bar{V}$ are the solutions of (3)-(4) and (9)-(10) respectively.

Proof. We have

$$\begin{aligned}| B_{10}^N(V_1 - v_1)(x_0) | &= |\beta_{10}(V_1 - v_1)(x_0) - \beta_{11}\varepsilon D^+(V_1 - v_1)(x_0)| \\ &\le C\beta_{11}\varepsilon(x_{i+1} - x_i)\|v_1^{(2)}\| \le CN^{-1}\end{aligned}$$

and

$$\begin{aligned}| B_{1N}^N(V_1 - v_1)(x_N) | &= |\gamma_{11}(V_1 - v_1)(x_N) + \gamma_{12}D^-(V_1 - v_1)(x_N)| \\ &\le C\gamma_{12}(x_i - x_{i-1})\|v_1^{(2)}\| \le CN^{-1}.\end{aligned}$$

Similarly, $|B_{20}^N(V_2 - v_2)(x_0)| \le CN^{-1}$ and $|B_{2N}^N(V_2 - v_2)(x_N)| \le CN^{-1}$. By standard local truncation error estimate and Lemma 2, we have

$$|L_j^N(V_j - v_j)(x_i)| \le CN^{-1}, \; j = 1, 2.$$

Using the barrier functions

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} C(2 - x_i)N^{-1} \\ C(2 - x_i)N^{-1} \end{pmatrix} \pm (\bar{V} - \bar{v})(x_i)$$

and applying Theorem 3, we get $\bar{\Psi}^{\pm}(x_i) \ge \bar{0}$, for all $x_i \in \bar{\Omega}_\varepsilon^N$, which completes the proof.

Lemma 4. *At each mesh point $x_i \in \bar{\Omega}_\varepsilon^N$, the singular component of the error satisfies the estimate*

$$|(\bar{W} - \bar{w})(x_i)| \le \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix},$$

where $\bar{w}$ and $\bar{W}$ are the solutions of (5)-(6) and (11)-(12) respectively.

Proof. We have

$$\begin{aligned}|B_{10}^N(W_1 - w_1)(x_0)| &= |\beta_{10}(W_1 - w_1)(x_0) - \beta_{11}\varepsilon D^+(W_1 - w_1)(x_0)| \\ &\le \beta_{11}\varepsilon(x_{i+1} - x_i)|w_1''(x_i)| \\ &\le CN^{-1}\ln N\end{aligned}$$

and

$$\begin{aligned}|B_{1N}^N(W_1 - w_1)(x_N)| &= |\gamma_{11}(W_1 - w_1)(x_N) + \gamma_{12}D^-(W_1 - w_1)(x_N)| \\ &\le \gamma_{12}(x_i - x_{i-1})|w_1''(x_i)| \\ &\le CN^{-1}.\end{aligned}$$

Similarly, we have

$$|B_{20}^N(W_2 - w_2)(x_0)| \le CN^{-1}\ln N \quad \text{and} \quad |B_{2N}^N(W_2 - w_2)(x_N)| \le CN^{-1}.$$

We consider first the case $\sigma = \frac{1}{2}$ and so $\varepsilon^{-1} \le C\ln N$ and $h = H = N^{-1}$. By classical argument and using Lemma 2, we obtain $|L_1^N(\bar{W} - \bar{w})(x_i)| \le$

$C\varepsilon^{-2}N^{-1}e^{-\alpha x_{i-1}/\varepsilon}$. We introduce the mesh functions

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha-\gamma)}\varepsilon^{-1}N^{-1}(Y_1(x_i)+\frac{1}{\gamma_1}) \\ \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha-\gamma)}\varepsilon^{-1}N^{-1}(Y_2(x_i)+\frac{1}{\gamma_1}) \end{pmatrix} \pm (\bar{W}-\bar{w})(x_i)$$

where γ is any constant satisfying $0<\gamma<\alpha$ and

$$Y_1(x_i) = \frac{\lambda^{N-i}+(\gamma\gamma_2/\gamma_1\varepsilon)-1}{\beta_1(\lambda^N+(\gamma\gamma_2/\gamma_1\varepsilon)-1)+\beta_2\gamma\lambda^{N-1}},$$

where $\lambda = 1+\frac{\gamma h}{\varepsilon}$, $\beta_1 = \min\{\beta_{10},\beta_{20}\}$, $\beta_2 = \min\{\beta_{11},\beta_{21}\}$, $\gamma_1 = \min\{\gamma_{11},\gamma_{21}\}$, $\gamma_2 = \min\{\gamma_{12},\gamma_{22}\}$ and let $Y_2(x_i)\equiv Y_1(x_i)$. It is easy to see that for $1\le i\le N-1$,

$$(\varepsilon\delta^2+\gamma D^+)Y_1(x_i) = 0,$$
$$\beta_1 Y_1(x_0)-\varepsilon\beta_2 D^+Y_1(x_0) = 1,\ \gamma_1 Y_1(x_N)+\gamma_2 D^-Y_1(x_N) = 0$$

and $D^+Y_1(x_i)\le 0$. Then we have

$$\begin{aligned} B_{10}^N\Psi_1^{\pm}(x_0) &\equiv \beta_{10}\Psi_1^{\pm}(x_0)-\varepsilon\beta_{11}D^+\Psi_1^{\pm}(x_0) \\ &\ge \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha-\gamma)}\varepsilon^{-1}N^{-1}(\beta_1 Y_1(x_0)-\varepsilon\beta_2 D^+Y_1(x_0)+\frac{\beta_1}{\gamma_1})\pm CN^{-1}\ln N \\ &\ge 0, \end{aligned}$$

$$\begin{aligned} B_{1N}^N\Psi_1^{\pm}(x_N) &\equiv \gamma_{11}\Psi_1^{\pm}(x_N)+\gamma_{12}D^-\Psi_1^{\pm}(x_N) \\ &\ge \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha-\gamma)}\varepsilon^{-1}N^{-1}(\gamma_1 Y_1(x_N)+\gamma_2 D^-Y_1(x_N)+\frac{\gamma_1}{\gamma_1})\pm CN^{-1} \\ &\ge 0 \end{aligned}$$

and

$$\begin{aligned} L_1^N\bar{\Psi}^{\pm}(x_i) &\equiv -\varepsilon\delta^2\Psi_1^{\pm}(x_i)-a_1(x_i)D^+\Psi_1^{\pm}(x_i) \\ &\quad + b_{11}(x_i)\Psi_1^{\pm}(x_i)+b_{12}(x_i)\Psi_2^{\pm}(x_i),\quad x_i\in\Omega_\varepsilon^N \\ &\ge \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha-\gamma)}\varepsilon^{-1}N^{-1}(-\varepsilon\delta^2 Y_1(x_i)-a_1(x_i)D^+Y_1(x_i)+b_{11}(x_i)(Y_1(x_i) \\ &\quad +\frac{1}{\gamma_1})+b_{12}(x_i)(Y_2(x_i)+\frac{1}{\gamma_1}))\pm C\varepsilon^{-2}N^{-1}e^{-\alpha x_{i-1}/\varepsilon} \\ &\ge \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha-\gamma)}\varepsilon^{-1}N^{-1}(-(\alpha-\gamma)D^+Y_1(x_i) \\ &\quad +(b_{11}(x_i)+b_{12}(x_i))(Y_1(x_i)+\frac{1}{\gamma_1}))\pm C\varepsilon^{-2}N^{-1}e^{-\alpha x_{i-1}/\varepsilon} \\ &\ge 0. \end{aligned}$$

Similarly,

$$B_{20}^N\Psi_2^{\pm}(x_0)\ge 0,\quad B_{2N}^N\Psi_2^{\pm}(x_N)\ge 0,\quad L_2^N\bar{\Psi}^{\pm}(x_i)\ge 0,\ x_i\in\Omega_\varepsilon^N.$$

Hence by Theorem 3, we get $\bar{\Psi}^{\pm}(x_i) \geq \bar{0}$, which leads to the required result.

Now consider the case $\sigma = \dfrac{2\varepsilon}{\alpha} \ln N$. Suppose that $x_i \in [\sigma, 1]$. Using the triangle inequality we have $|(\bar{W} - \bar{w})(x_i)| \leq |\bar{W}(x_i)| + |\bar{w}(x_i)|$. Using Lemma 2, we have $|\bar{w}(x_i)| \leq \begin{pmatrix} CN^{-2} \\ CN^{-2} \end{pmatrix}$. To obtain a similar bound for $|\bar{W}(x_i)|$, consider the mesh function $Y_1(x_i)$, which is the solution of the constant coefficient discretised problem

$$\varepsilon \delta^2 Y_1(x_i) + \alpha D^+ Y_1(x_i) = 0, \;\; i = 1, \dots, N-1,$$
$$Y_1(x_0) = 1, \quad \gamma_1 Y_1(x_N) + \gamma_2 D^- Y_1(x_N) = 0,$$

where γ_1, γ_2 are defined as before and further let $Y_2(x_i) \equiv Y_1(x_i)$. Also, $D^+ Y_1(x_i) \leq 0$, for $0 \leq i \leq N-1$. Let

$$\beta = \max\{|\beta_{10} W_1(x_0) - \varepsilon \beta_{11} D^+ W_1(x_0)|, |\beta_{20} W_2(x_0) - \varepsilon \beta_{21} D^+ W_2(x_0)|\}$$

and introduce the mesh functions

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} \beta Y_1(x_i) \\ \beta Y_2(x_i) \end{pmatrix} \pm \bar{W}(x_i).$$

Then we have

$$\begin{aligned} B_{10}^N \Psi_1^{\pm}(x_0) &\equiv \beta_{10} \Psi_1^{\pm}(x_0) - \varepsilon \beta_{11} D^+ \Psi_1^{\pm}(x_0) \\ &\geq \beta(\beta_{10} Y_1(x_0) - \varepsilon \beta_{11} D^+ Y_1(x_0)) \pm B_{10}^N W_1(x_0) \\ &\geq 0, \end{aligned}$$

$$\begin{aligned} B_{1N}^N \Psi_1^{\pm}(x_N) &\equiv \gamma_{11} \Psi_1^{\pm}(x_N) + \gamma_{12} D^- \Psi_1^{\pm}(x_N) \\ &\geq \beta(\gamma_{11} Y_1(x_N) + \gamma_{12} D^- Y_1(x_N) \pm B_{1N}^N W_1(x_N) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} L_1^N \bar{\Psi}^{\pm}(x_i) &\equiv -\varepsilon \delta^2 \Psi_1^{\pm}(x_i) - a_1(x_i) D^+ \Psi_1^{\pm}(x_i) \\ &\quad + b_{11}(x_i) \Psi_1^{\pm}(x_i) + b_{12}(x_i) \Psi_2^{\pm}(x_i), \; x_i \in \Omega_\varepsilon^N \\ &\geq \beta(-\varepsilon \delta^2 Y_1(x_i) - a_1(x_i) D^+ Y_1(x_i) \\ &\quad + b_{11}(x_i) Y_1(x_i) + b_{12}(x_i) Y_2(x_i)) \pm L_1^N W_1(x_i) \\ &\geq \beta(-(a_1(x_i) - \alpha) D^+ Y_1(x_i) + (b_{11}(x_i) + b_{12}(x_i)) Y_1(x_i)) \\ &\geq 0. \end{aligned}$$

Similarly,

$$B_{20}^N \Psi_2^{\pm}(x_0) \geq 0, \quad B_{2N}^N \Psi_2^{\pm}(x_N) \geq 0, \quad L_2^N \bar{\Psi}^{\pm}(x_i) \geq 0, \;\; x_i \in \Omega_\varepsilon^N.$$

Using the procedure adopted in [5, Lemma 4.2] and hence by Theorem 3, we get

$$|\bar{W}(x_i)| \leq \begin{pmatrix}\beta Y_1(x_i)\\ \beta Y_2(x_i)\end{pmatrix} \leq \begin{pmatrix}CN^{-2}\\ CN^{-2}\end{pmatrix}, \text{ for } x_i \in [\sigma, 1].$$

Now for $x_i \in [0, \sigma)$, the proof follows the same lines as for the case $\sigma = 1/2$, except that we use the discrete maximum principle on $[0, \sigma]$ and the already established bound $|\bar{W}(x_{N/2})| \leq \begin{pmatrix}CN^{-2}\\ CN^{-2}\end{pmatrix}$. We now present a detailed proof. For all i, $0 \leq i \leq N/2$, we introduce the mesh functions

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix}\frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha-\gamma)}\sigma\varepsilon^{-1}N^{-1}Y_1(x_i) + C_2N^{-2}\\ \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha-\gamma)}\sigma\varepsilon^{-1}N^{-1}Y_2(x_i) + C_2N^{-2}\end{pmatrix} \pm (\bar{W} - \bar{w})(x_i)$$

where $Y_1(x_i)$ is the solution of the problem

$$\varepsilon\delta^2 Y_1(x_i) + \gamma D^+ Y_1(x_i) = 0, \;\; i = 1, \ldots, N-1,$$

$$\beta_1 Y_1(x_0) - \varepsilon\beta_2 D^+ Y_1(x_0) = 1, \quad \gamma_1 Y_1(x_{N/2}) + \gamma_2 D^- Y_1(x_{N/2}) = 0$$

and γ, β_1, β_2, γ_1, γ_2 are defined as before.

Thus, for all i, $0 \leq i \leq N/2$,

$$Y_1(x_i) = \frac{\lambda^{N/2-i} + (\gamma\gamma_2/\gamma_1\varepsilon) - 1}{\varepsilon[\lambda^{N/2} + (\gamma\gamma_2/\gamma_1\varepsilon) - 1]},$$

where, $\lambda = 1 + \frac{\gamma h}{\varepsilon}$ and $D^+Y_1(x_i) \leq 0$. Let $Y_2(x_i) \equiv Y_1(x_i)$. It is easy to show that $B_{j0}^N \Psi_j^{\pm}(x_0) \geq 0$, $B_{j1}^N \Psi_i^{\pm}(x_{N/2}) \geq 0$, for $j = 1, 2$ and $\mathbf{L}^N \bar{\Psi}^{\pm}(x_i) \geq \bar{0}$, for $0 < i \leq N/2 - 1$. Then by Theorem 3 we get $\bar{\Psi}^{\pm}(x_i) \geq \bar{0}$. Thus we get the required result.

Theorem 5. *Let $\bar{y}(x) = (y_1(x), y_2(x))^T$ be the solution of (1)-(2) and let $\bar{Y}(x_i) = (Y_1(x_i), Y_2(x_i))^T$ be the corresponding numerical solution of (7)-(8). Then we have*

$$\sup_{0<\varepsilon\leq 1} \|Y_1 - y_1\|_{\bar{\Omega}_\varepsilon^N} \leq CN^{-1}\ln N \quad \textit{and} \quad \sup_{0<\varepsilon\leq 1} \|Y_2 - y_2\|_{\bar{\Omega}_\varepsilon^N} \leq CN^{-1}\ln N.$$

Proof. Proof follows immediately, if one applies the above Lemmas 3 and 4 to $\bar{Y} - \bar{y} = \bar{V} - \bar{v} + \bar{W} - \bar{w}$.

4. Analysis on derivative approximation

In this section, we give an $\varepsilon-$uniform error estimate between the scaled derivative of the continuous solution and the corresponding numerical solution in the fine mesh region. Further, in the coarse mesh, an estimate is obtained without scaling the derivative.

We note that the errors

$$e_j(x_i) \equiv Y_j(x_i) - y_j(x_i),$$

satisfy the equations

$[\varepsilon\delta^2+a_j(x_i)D^+]e_j(x_i) = [b_{j1}(x_i)e_1(x_i)+b_{j2}(x_i)e_2(x_i)]-\text{truncation error}, j = 1,2$

where, by Theorem 5, $[b_{j1}(x_i)e_1(x_i) + b_{j2}(x_i)e_2(x_i)] = O(N^{-1}\ln N)$. In the proofs of the following lemmas and theorems, we use the above equations, they are necessary single equation. Hence the analysis carried out in [4, §3.5] for single equation and [13, §4] for system of equations can be applied immediately with a slight modifications where ever necessary. Therefore, proofs for some lemmas are omitted; for some of the theorems short proves are given.

4.1. Numerical derivative estimates.

Lemma 5. *For each mesh point $x_i \in \Omega_\varepsilon^N \cup \{0\}$ and all $x \in \bar{\Omega}_i = [x_i, x_{i+1}]$, we have*

$$|\varepsilon(D^+y_j(x_i) - y_j'(x))| \le CN^{-1}\ln N, \quad \text{for} \quad x_i < \sigma, \; j = 1,2,$$

and

$$|D^+y_j(x_i) - y_j'(x)| \le CN^{-1}, \quad \text{for} \quad x_i \ge \sigma, \; j = 1,2,$$

where $(y_1(x), y_2(x))^T$ is the solution of (1)-(2).

Lemma 6. *At each mesh point $x_i \in \Omega_\varepsilon^N \cup \{0\}$, we have for $j = 1,2$*

$$\max_{0\le i<N/2} |\varepsilon D^+(V_j-v_j)(x_i)| \le CN^{-1} \quad \text{and} \quad \max_{N/2\le i<N} |D^+(V_j-v_j)(x_i)| \le CN^{-1},$$

where v_j and V_j are the solutions of (3)-(4) and (9)-(10) respectively.

Proof. We denote the error and the local truncation error, respectively at each mesh point by $e_j(x_i) = V_j(x_i) - v_j(x_i)$ and $\tau_j(x_i) = L_j^N e_j(x_i)$, for $j = 1,2$. Since $|e_j(x_i)| \le CN^{-1}$, we have

$$|\varepsilon D^+e_j(x_{N-1})| = |\frac{\varepsilon(e_j(x_N) - e_j(x_{N-1}))}{x_N - x_{N-1}}| \le C\varepsilon N^{-1}. \tag{13}$$

Now we write $\tau_j(x_i) = L_j^N e_j(x_i)$ in the form,

$$\begin{aligned} &\varepsilon D^+e_j(x_k) - \varepsilon D^+e_j(x_{k-1}) + \frac{1}{2}(x_{k+1} - x_{k-1})a_j(x_k)D^+e_j(x_k) \\ &= \frac{1}{2}(x_{k+1} - x_{k-1})([b_{j1}(x_k)e_1(x_k) + b_{j2}(x_k)e_2(x_k)] - \tau_j(x_k)). \end{aligned} \tag{14}$$

Summing and rearranging for each i, $0 \le i < N-1$, we get

$$\begin{aligned} |\varepsilon D^+e_j(x_i)| \le |\varepsilon D^+e_j(x_{N-1})| &+ \frac{1}{2}\sum_{k=i+1}^{N-1}(x_{k+1} - x_{k-1})(|\tau_j(x_k)| \\ &+ |[b_{j1}(x_k)e_1(x_k) + b_{j2}(x_k)e_2(x_k)]|) \\ &+ |\frac{1}{2}\sum_{k=i+1}^{N-1}(x_{k+1} - x_{k-1})a_j(x_k)D^+e_j(x_k)|. \end{aligned}$$

Using the telescoping effect for the last term, $|e_j(x_i)| \leq CN^{-1}$ and $\|a_j'\| \leq C$, we get for all i, $0 \leq i < N$,

$$|\varepsilon D^+ e_j(x_i)| \leq CN^{-1}. \tag{15}$$

Over the region $[0, \sigma)$, the result follows immediately from (15).

For $i \geq N/2$, we rewrite (14) in the form

$$\begin{aligned}(1+\rho_k)D^+e_j(x_k) &= D^+e_j(x_{k-1}) \\ &+ \frac{\rho_k}{a_j(x_k)}(b_{j1}(x_k)e_1(x_k) + b_{j2}(x_k)e_2(x_k) - \tau_j(x_k))\end{aligned} \tag{16}$$

where $\rho_k = \dfrac{a_j(x_k)(x_{k+1} - x_{k-1})}{\varepsilon}$. Summing these equations from $k = 1$ to $k = N/2$, we get

$$|D^+e_j(\sigma)| \leq |D^+e_j(x_0)| \frac{(1+\rho)^{-(N/2-1)}}{1+\rho_{N/2}} + CN^{-1} \leq CN^{-1}, \ \rho = \frac{\alpha h}{\varepsilon}.$$

Summing the equations in (16) from $k = N/2$ to $k = i < N$, we have

$$|D^+e_j(x_i)| \leq |D^+e_j(\sigma)| \frac{(1+\bar{\rho})^{-(i-\frac{N}{2}-1)}}{1+\rho_i} + CN^{-1} \leq CN^{-1}, \ \bar{\rho} = \frac{\alpha H}{\varepsilon},$$

which completes the proof.

Lemma 7. *For $\sigma = 1/2$, we have for all $x_i \in \bar{\Omega}_\varepsilon^N$,*

$$|\varepsilon(W_j - w_j)(x_i)| \leq C(2 - x_i)N^{-1}\ln N, \quad \textit{for} \quad j = 1, 2,$$

where w_j and W_j are the solutions of (5)-(6) and (11)-(12) respectively.

Proof. Use the barrier functions

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} \frac{C}{\alpha}\varepsilon^{-2}(2 - x_i)N^{-1} \\ \frac{C}{\alpha}\varepsilon^{-2}(2 - x_i)N^{-1} \end{pmatrix} \pm (\bar{W} - \bar{w})(x_i)$$

to establish the required bound.

Lemma 8. *At each mesh point $x_i \in \Omega_\varepsilon^N \cup \{0\}$, we have for $j = 1, 2$*

$$\max_{0 \leq i < N/2} |\varepsilon D^+(W_j - w_j)(x_i)| \leq CN^{-1}\ln N$$

and

$$\max_{N/2 \leq i < N} |D^+(W_j - w_j)(x_i)| \leq CN^{-1},$$

where w_j and W_j are the solution of (5)-(6) and (11)-(12) respectively.

Proof. Consider the case $x_i \geq \sigma$. From the particular choice of transition parameter, we have for $j = 1, 2$, $|W_j(x_i)| \leq CN^{-2}$ and $|w_j(x_i)| \leq CN^{-2}$, for all $x_i \geq \sigma$. Hence for $j = 1, 2$

$$|D^+(W_j - w_j)(x_i)| \leq CN^{-1}, \ x_i \in [\sigma, 1].$$

For $x_i = \sigma$, we write $L_1^N \bar{W}(\sigma) = 0$ in the form

$$\varepsilon D^+ W_1(x_{N/2-1}) = \varepsilon D^+ W_1(\sigma) + \frac{h+H}{2} a_1(\sigma) D^+ W_1(\sigma) + \frac{1}{2} b_{11}(\sigma)(h+H) W_1(\sigma) + \frac{1}{2} b_{12}(\sigma)(h+H) W_2(\sigma).$$

Using the estimate obtained at the point σ and from the proof of Lemma 4, we obtain $|\varepsilon D^+ W_1(x_{N/2-1})| \le CN^{-1}$. Now consider $x_i \in [0, \sigma)$. For convenience we introduce the notation $\hat{e}_j(x_i) = (\hat{W}_j - \hat{w}_j)(x_i)$ and $\hat{\tau}_j(x_i) = L_j^N \hat{e}(x_i)$, $j = 1, 2$. We have already established that

$$|\hat{e}_j(x_i)| \le CN^{-1} \ln N \quad \text{and} \quad |\hat{\tau}_j(x_i)| \le C\sigma\varepsilon^{-2} N^{-1} e^{-\alpha x_{i-1}/\varepsilon}, \quad j = 1, 2. \tag{17}$$

We write the equation $\hat{\tau}_j(x_i) = L_j^N \hat{e}(x_i)$ in the form

$$\varepsilon D^+(\hat{e}_j(x_k) - \hat{e}_j(x_{k-1})) + \frac{1}{2} a_j(x_k)(x_{k+1} - x_{k-1}) D^+ \hat{e}_j(x_k) = -\frac{1}{2}(x_{k+1} - x_{k-1}) \hat{\tau}_j(x_k) + [b_{j1}(x_k)\hat{e}_1(x_k) + b_{j2}(x_k)\hat{e}_2(x_k)].$$

Summing these equations from $x_k = x_i > 0$ to $x_k = \sigma - h$, we get

$$\varepsilon D^+ \hat{e}_j(x_i) = \varepsilon D^+ \hat{e}_j(x_{N/2-1}) + \sum_{k=i+1}^{\frac{N}{2}-1} a_j(x_k)(\hat{e}_j(x_{k+1}) - \hat{e}_j(x_k)) - \sum_{k=i+1}^{\frac{N}{2}-1} h\hat{\tau}_j(x_k) + \sum_{k=i}^{\frac{N}{2}-1} [b_{j1}(x_k)\hat{e}_1(x_k) + b_{j2}(x_k)\hat{e}_2(x_k)], \quad j = 1, 2.$$

Hence using the result at the point $x_{N/2-1}$ and (17), we have

$$|\varepsilon D^+ \hat{e}_j(x_i)| \le CN^{-1}(\ln N + \frac{\sigma\alpha h/\varepsilon}{\varepsilon(1 - e^{-\alpha h/\varepsilon})}).$$

But $y = \alpha h/\varepsilon = 2N^{-1} \ln N$ and $B(y) = \frac{y}{1-e^{-y}}$ are bounded and it follows that $|\varepsilon D^+ \hat{e}_j(x_i)| \le CN^{-1} \ln N$, $j = 1, 2$ as required.

When $\sigma = 1/2$, using the above arguments and Lemma 7, we get

$$|\varepsilon D^+ \hat{e}_j(x_i)| \le CN^{-1} \ln N, \quad j = 1, 2.$$

which is the required result.

Theorem 6. *Let $\bar{y}$ be the solution of (1) - (2) and $\bar{Y}$ be the corresponding numerical solution of (7) - (8). Then, we have for $j = 1, 2$,*

$$\sup_{0<\varepsilon\le 1} \|\varepsilon(D^+ Y_j(x_i) - y_j')\|_{\bar{\Omega}_i} \le CN^{-1} \ln N, \quad \textit{for} \quad 0 \le i < N/2,$$

and

$$\sup_{0<\varepsilon\le 1} \|D^+ Y_j(x_i) - y_j'\|_{\bar{\Omega}_i} \le CN^{-1}, \quad \textit{for} \quad N/2 \le i \le N - 1.$$

Proof. Following the method of proof adopted in [4, Theorem 3.17], using the Lemmas 6 and 8 we get the required result.

Remark 1. Let $\widetilde{Y}_j$, $j = 1, 2$, denote the piecewise linear interpolant of the finite difference solution $\{Y_j(x_i)\}_{i=0}^{N}$. As done in [4, p.66], we get for $j = 1, 2$

$$\sup_{0<\varepsilon\leq 1} \|\varepsilon(\widetilde{D}^+ Y_j - y_j')\|_{\bar{\Omega}_i} \leq CN^{-1}\ln N, enskipi = 1, \ldots, N/2 - 1,$$

and

$$\sup_{0<\varepsilon\leq 1} \|\widetilde{D}^+ Y_j - y_j'\|_{\bar{\Omega}_i} \leq CN^{-1}, \ \ i = N/2, \ldots, N,$$

where, $\widetilde{D}^+ Y_j(x) = D^+ Y_j(x_i)$, for $x \in [x_i, x_{i+1})$, $i = 0, \ldots, N$.

5. Numerical experiments

In this section, we verify experimentally the theoretical results obtained in this paper by two test problems.

Example 1. Consider the singularly perturbed boundary value problem :

$$\begin{aligned}
&-\varepsilon y_1''(x) - \frac{1}{1+x} y_1'(x) + 3y_1(x) - y_2(x) = 1 + x, \ x \in \Omega,\\
&-\varepsilon y_2''(x) - \frac{1}{1+x} y_2'(x) - y_1(x) + 3y_2(x) = \frac{1+x}{2}, \ x \in \Omega,\\
&y_1(0) - \varepsilon y_1'(0) = 1, \ 2y_1(1) + y_1'(1) = 1, \ y_2(0) - \varepsilon y_2'(0) = 2, \ 2y_2(1) + y_2'(1) = 1.5.
\end{aligned}$$

Example 2. Consider the singularly perturbed boundary value problem :

$$\begin{aligned}
&-\varepsilon y_1''(x) - 3y_1'(x) + 3y_1(x) - y_2(x) = 1 + e^{-x}, \ x \in \Omega,\\
&-\varepsilon y_2''(x) - y_2'(x) - y_1(x) + 3y_2(x) = 1 - e^{-x}, \ x \in \Omega,\\
&3y_1(0) - \varepsilon y_1'(0) = 0, 2y_1(1) + y_1'(1) = 1, 3y_2(0) - \varepsilon y_2'(0) = 2, 2y_2(1) + y_2'(1) = 2.
\end{aligned}$$

Let $(Y_1^N, Y_2^N)^T$ be a numerical approximation for the exact solution $(y_1, y_2)^T$ on the mesh Ω_ε^N and N is the number of mesh points. Since the exact solutions are not available for the above test problems, for a finite set of values $\varepsilon \in R_\varepsilon = \{2^0, 2^{-1}, \ldots, 2^{-25}\}$, we compute the maximum pointwise error for $j = 1, 2$,

$$S_{\varepsilon,j}^N = \|Y_j^N - \widetilde{Y}_j^{2048}\|_{\bar{\Omega}_\varepsilon^N}$$

$$D_{\varepsilon,j}^N = \begin{cases} \max|\varepsilon(D^+ Y_j^N - \widetilde{D}^+ Y_j^{2048})(x_i)|, & \text{for} \quad 0 \leq i < N/2\\ \max|D^+ Y_j^N - \widetilde{D}^+ Y_j^{2048})(x_i)|, & \text{for} \quad N/2 \leq i \leq N-1,\end{cases}$$

where $\widetilde{Y}_j^{2048}$ is the piecewise linear interpolant of the mesh function Y_j^{2048} onto $[0, 1]$. From these values the $\varepsilon-$uniform maximum pointwise difference

$$S_j^N = \max_{\varepsilon\in R_\varepsilon} D_{\varepsilon,j}^N, \quad D_j^N = \max_{\varepsilon\in R_\varepsilon} D_{\varepsilon,j}^N, \quad j = 1, 2$$

are formed for each available value of N satisfying N, $2N \in R_N$. Approximations to the $\varepsilon-$uniform order of local convergence are defined, for all $N, 4N \in R_N$, by

$$r_j^N = \log_2(\frac{S_j^N}{S_j^{2N}}), \quad p_j^N = \log_2(\frac{D_j^N}{D_j^{2N}}), \quad j = 1, 2.$$

Surface plots of the maximum error for the solution as well as scaled first derivative of the above test problems are presented. Figure 1 shows the numerical solution of the Examples 1 and 2 respectively. In Figures 2 and 4, we observe that as $\varepsilon \to 0$, the maximum error for the numerical solution $(Y_1, Y_2)^T$ to the exact solution $(y_1, y_2)^T$ of the Examples 1 and 2 respectively, decreases and gets stabilized at a constant value. Figures 3 - 5, we observe that as $\varepsilon \to 0$, the maximum error for the numerical scaled derivatives in the fine mesh region $(\varepsilon D^+Y_1, \varepsilon D^+Y_2)^T$ to the exact scaled derivatives $(\varepsilon y_1', \varepsilon y_2')^T$ of the Examples 1 and 2 respectively, decreases and gets stabilized at a constant value. Tables 1 and 4 present $\varepsilon-$uniform maximum pointwise error and $\varepsilon-$uniform order of local convergence to the numerical solution of the Examples 1 and 2 respectively. Tables 2 and 3 present $\varepsilon-$uniform maximum pointwise error and $\varepsilon-$uniform order of local convergence to the scaled derivatives in the fine mesh region and the non scaled derivative in the coarse mesh region for the Example 1. Tables 5 and 6 present $\varepsilon-$uniform maximum pointwise error and $\varepsilon-$uniform order of local convergence to the scaled derivatives in the fine mesh region and the non scaled derivative in the coarse mesh region for the Example 2.

Table 1. *Values of S_1^N, r_1^N and S_2^N, r_2^N for the solution components y_1 and y_2 (Example 1) respectively*

	Number of mesh points N				
	64	128	256	512	1024
S_1^N	2.2195e-2	1.3034e-2	7.0987e-3	3.4194e-3	1.2446e-3
r_1^N	7.6795e-1	8.7665e-1	1.0538	1.4581	-
S_2^N	2.9883e-2	1.7628e-2	9.6085e-3	4.6303e-3	1.6857e-3
r_2^N	7.6146e-1	8.7549e-1	1.0532	1.4578	-

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Fig. 1. *Approximate solutions of the Examples 1 and 2 for* $\varepsilon = 2^{-7}$ *with* $N = 128$.

Fig. 2. *Surface plots of the maximum pointwise errors as a function of* N *and* ε *for the solutions* $(Y_1, Y_2)^T$ *of the Example 1.*

Fig. 3. *Surface plots of the maximum pointwise errors as a function of* N *and* ε *for the scaled derivative* $(\varepsilon D^+Y_1, \varepsilon D^+Y_2)^T$ *of the Example 1.*

Fig. 4. *Surface plots of the maximum pointwise errors as a function of* N *and* ε *for the solution* $(Y_1, Y_2)^T$ *of the Example 2.*

Table 2. *Values of D_1^N, p_1^N and D_2^N, p_2^N for the scaled derivative components $\varepsilon y_1'$ and $\varepsilon y_2'$ in the fine mesh region (Example 1) respectively*

	Number of mesh points N				
	64	128	256	512	1024
D_1^N	1.3462e-2	9.6538e-3	5.9760e-3	3.1094e-3	1.1856e-3
p_1^N	4.7972e-1	6.9192e-1	9.4254e-1	1.3910	-
D_2^N	1.8066e-2	1.3091e-2	8.0877e-3	4.2105e-3	1.6061e-3
p_2^N	4.6470e-1	6.9477e-1	9.4174e-1	1.3904	-

Table 3. *Values of D_1^N, p_1^N and D_2^N, p_2^N for the derivative components y_1' and y_2' in the coarse mesh region (Example 1) respectively*

	Number of mesh points N				
	64	128	256	512	1024
D_1^N	2.4063e-2	1.2373e-2	5.9545e-3	2.5918e-3	8.7069e-4
p_1^N	9.5962e-1	1.0551	1.2000	1.5737	-
D_2^N	4.8640e-3	2.4367e-3	1.1569e-3	5.0026e-4	1.6750e-4
p_2^N	9.9721e-1	1.0747	1.2095	1.5785	-

Table 4. *Values of S_1^N, r_1^N and S_2^N, r_2^N for the solution y_1 and y_2 (Example 2) respectively*

	Number of mesh points N				
	64	128	256	512	1024
S_1^N	4.7772e-2	2.8723e-2	1.5992e-2	7.6877e-3	2.6599e-3
r_1^N	7.3396e-1	8.4486e-1	1.0567	1.5312	-
S_2^N	4.6237e-2	2.7172e-2	1.4720e-2	7.0696e-3	2.5692e-3
r_2^N	7.6693e-1	8.8434e-1	1.0581	1.4603	-

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Table 5. *Values of* D_1^N, p_1^N *and* D_2^N, p_2^N *for the scaled derivative components* $\varepsilon y_1'$ *and* $\varepsilon y_2'$ *in the fine mesh region (Example 2) respectively*

	Number of mesh points N				
	64	128	256	512	1024
D_1^N	2.2024e-2	1.7972e-2	1.1689e-2	6.2351e-3	2.4096e-3
p_1^N	2.2563e-1	5.6828e-1	8.9811e-1	1.4413	-
D_2^N	2.2024e-2	1.7972e-2	1.1689e-2	6.2351e-3	2.4096e-3
p_2^N	2.9333e-1	6.2060e-1	9.0667e-1	1.3716	-

Table 6. *Values of* D_1^N, p_1^N *and* D_2^N, p_2^N *for the derivative components* y_1' *and* y_2' *in the coarse mesh region (Example 2) respectively*

	Number of mesh points N				
	64	128	256	512	1024
D_1^N	1.7408e-3	9.1346e-4	4.7509e-4	2.2752e-4	7.3799e-5
p_1^N	9.3034e-1	9.4314e-1	1.0622	1.6243	-
D_2^N	1.1142e-3	5.4989e-4	2.5913e-4	1.1159e-4	3.7269e-5
p_2^N	1.0188	1.0855	1.2155	1.5822	-

Fig. 5. *Surface plots of the maximum pointwise errors as a function of* N *and* ε *for the scaled derivative* $(\varepsilon D^+Y_1, \varepsilon D^+Y_2)^T$ *of the Example 2.*

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