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APPROXIMATION METHODS FOR FINITE FAMILY OF NONSPREADING MAPPINGS AND NONEXPANSIVE MAPPINGS IN HILERT SPACESE

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ABSTRACT. The purpose of this paper is to prove a weak convergence theorem for a common fixed points of finite family of nonspreading mappings and nonexpansive mappings in Hilbert spaces. The results presented in this paper extend and improve the results of Mondafi [A. Moudafi, Krasnoselski-Mann iteration for hierarchical fixed-point problems, Inverse Problems 23 (2007) 1635-1640], and Iemoto and Takahashi [S. Iemoto, W.Takahashi, Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space, Nonlinear Analysis (2009), doi:10.1016/j.na.2009.03.064].

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. C is a nonempty closed convex subset of H, A mapping $T : C \to C$ is said to be monexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$; If the set of fixed points of $T(T)$ $(i.e. F(T) = \{x \in C : Tx = x\})$ is nonempty, then T is said to be quasi-nonexpansive if

$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in C$ and $p \in F(T)$. A mapping F is said to be firmly nonexpansive if

$$\|Fx - Fy\| \leq \langle x - y, Fx - Fy \rangle \tag{1.1}$$

for all $x, y \in C$; see, for instance, Browder [1], Goebel and Kirk [2], Reich and Shoikhet [3], Iemoto and Takahashi [4]. It is known that any firmly mapping F

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is nonexpansive and of form $F = \frac{1}{2}(I + T)$ with some nonexpansive mapping T; see [2, 3] for instance.

The following nonlinear mapping is introduced by Kohsaka and Takahashi [5] recently. Let E be a smooth, strictly convex and reflexive Banach space, Let J be the duality mapping of E and Let C be a nonempty closed convex subset of E. Then, a mapping $S : C \to C$ is said to be nonspreading if

$$Q(Sx, Sy) + Q(Sy, Sx) \leq Q(Sx, y) + Q(Sy, x)$$

for all $x, y \in C$, where $Q(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. they considered such a mapping to study the resolvents of a maximal monotone operator in Banach space. In the case when E is a Hilbert space, we know that $Q(x, y) = \|x - y\|^2$ for all $x, y \in E$, then a nonspreading mapping $S : C \to C$ in a Hilbert space H is defined as follows:

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2 \tag{1.2}$$

for all $x, y \in C$. It is know that if the set of fixed points of a nonspreading mapping is nonempty, the nonspreading mapping is quasi-nonexpansive.

On the other hand, weak convergence theorems for two nonexpansive mappings T_1, T_2 of C into itself were introduced by Takahashi and Tamura[6]. They considered the following iterative procedure:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1\{\beta_n T_2 x_n + (1 - \beta_n)x_n\} \end{cases}$$

for $n = 1, 2, \ldots$, where $F(T_1)$ and $F(T_2)$ are nonempty. Mondafi[7] also considered another iterative scheme for two nonexpansive mappings T_1, T_2 of C into itself:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\{\beta_n T_1 x_n + (1 - \beta_n)T_2 x_n\} \end{cases}$$

for $n = 1, 2, \ldots$, where $F(T_1)$ and $F(T_2)$ are nonempty. Recently, Iemoto and Takahashi [4] proved weak convergence theorems for a spreading mapping and a nonexpansive mapping

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\{\beta_n S x_n + (1 - \beta_n)T x_n\} \end{cases}$$

for $n = 1, 2, \ldots$, where $F(S)$ and $F(T)$ are nonempty. In this paper, motivated by Mondafi[7], Iemoto and Takahashi[4], we introduce the following iterative sequence: Let S_i, where $i = 1, 2, \ldots N$, be a finite family of nonspreading mappings of H and F_i, where $i = 1, 2, \ldots N$, be a finite family of nonexpansive mappings of H, respectively. Let $F(S_i)$ denote the fixed point set of S_i and $F(T_i)$ denote the fixed point set of T_i, i.e., $F(S_i) = \{x \in H : S_i x = x\}$ and $F(T_i) = \{x \in H : T_i x = x\}$, Stating with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\{\beta_n S_n x_n + (1-\beta_n)T_n x_n\} \tag{1.3}$$

where $S_n = S_{n(mod)N}$ and $T_n = T_{n(mod)N}$ and the mod function takes values in $1, 2 \ldots, N$, then we prove that $\{x_n\}$ defined by (1.3) converges weakly to a common fixed point for finite family of two kinds of mappings in Hilbert spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. For the sequence $\{x_n\}$ in H, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$. Let C be a nonempty closed convex subset of H. Then for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \le \|x - y\|$ *for all* $y \in C$. Such a P_C is called the metric projection of H onto C. We know that P_C is firmly nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \Leftrightarrow \quad \langle x - z, z - y\rangle \ge 0 \ \ for\ all\ y \in C$$

We also know that for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$, (we usually call it Opial's condition); see [8,9] for more details. Further, in a Hilbert space, we have

$$2\langle x - y, z - w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.2}$$

Using (2.2), we can show that the following lemma [4].

Lemma 2.1. *Let C be a nonempty closed convex subset of H and let F be a firmly nonexpansive of C into itself. Then F is nonspreading.*

Lemma 2.2. *Let C be a nonempty closed convex subset of H, a mapping $F : C \to C$ is firmly nonexpansive if and only if*

$$\|Fx - Fy\|^2 \le \|x-y\|^2 - \|(x - Fx) - (y - Fy)\|^2 \tag{2.3}$$

for all $x, y \in C$.

Proof. for all$x, y \in C$,

$$\begin{aligned}
\|Fx - Fy\|^2 &\le \|x-y\|^2 - \|(x-Fx)-(y-Fy)\|^2 \\
&\Longleftrightarrow \|Fx - Fy\|^2 \le \|x-y\|^2 \\
&\quad - \langle (x-y)-(Fx-Fy), (x-y)-(Fx-Fy)\rangle \\
&\Longleftrightarrow \|Fx - Fy\|^2 \le \|x-y\|^2 - \langle (x-y)-(x-y)\rangle \\
&\quad + 2\langle x-y, Fx-Fy\rangle - \langle Fx-Fy, Fx-Fy\rangle \\
&\Longleftrightarrow \|Fx - Fy\|^2 \le \langle x-y, Fx-Fy\rangle - \|Fx-Fy\|^2 \\
&\Longleftrightarrow \|Fx - Fy\|^2 \le \langle x-y, Fx-Fy\rangle
\end{aligned}$$

Iemoto and Takahashi [4] proved the following:

Lemma 2.3. *Let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and let $A = I - S$. Then*

$$\|Ax - Ay\|^2 \le \langle x-y, Ax-Ay\rangle + \frac{1}{2}(\|Ax\|^2 + \|Ay\|^2)$$

for all $x, y \in C$.

Lemma 2.4. *Let C be a nonempty closed convex subset of H, Then a mapping $S : C \to C$ is nonspreading if and only if*

$$\|Sx - Sy\|^2 \le \|x-y\|^2 + 2\langle x - Sx, y - Sy\rangle \tag{2.4}$$

for all $x, y \in C$.

Nonspreading mappings have been investigated no long time, we give two examples of this mappings:

Example 1. Let $E = (-\infty, +\infty)$ be endowed with the Euclidean norm $\|x-y\| = |x-y|$. Assumed that $C = [0, +\infty)$ and $S : C \to C$ is defined by

$$Sx = \begin{cases} x, & x \in [0,1) \\ 1, & x \in [1, +\infty), \end{cases}$$

We can prove S is nonpreading mappings.

Example 2. Let $E = (-\infty, +\infty)$ be endowed with the Euclidean norm $\|x-y\| = |x-y|$. Assumed that $K = [-3, 3]$ and $S : K \to K$ is defined by

$$Sx = \begin{cases} 0, & x \in [-2,2], \\ |\, x+1 \,|, & x \in [-3,-2), \\ |\, x-1 \,|, & x \in (-2,3]. \end{cases}$$

Then, S is not nonexpansive but nonspreading.

We know form Kohsaka and Takahashi [5] the following theorems.

Theorem 2.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let S be a nonspreading mapping of C into itself. Then the $F(S)$ is closed and convex. Furthermore, the following are equivalent:*
(i) There exists $x \in C$ such that $\{S^n x\}$ is bounded;
(ii) $F(S)$ is nonempty.

The following lemma is in [10,11].

Lemma 2.5. *Let $\{\alpha_n\}, \{\beta_n\}$ be sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$, then $\liminf_{n\to\infty} \beta_n = 0$.*

Tan and Xu [12] proved the following: see also [13,14].

Lemma 2.6. *Suppose that $\{s_n\}$ and $\{e_n\}$ are sequences of nonnegative real numbers such that $s_{n+1} \leq s_n + e_n$ for all $n \in N$. If $\sum_{n=1}^{\infty} e_n < \infty$, then $\liminf_{n\to\infty} s_n$ exists.*

Using Lemma 2.3, Iemoto and Takahashi [4] proved the following result which be said to the demiclosedness of a nonspreading mapping and essentially used in the proof of our main theorem.

lemma 2.7. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself such that $F(S) \neq \emptyset$. Then S is demiclosed, i.e., $x_n \rightharpoonup p$ and $\|x_n - Sx_n\| \to 0$ imply $p \in F(S)$.*

3. Main results

Now we prove our main results.

Theorem 3.1. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $S_i : C \to C$ for $i = 1, 2, \ldots, N$ be a family of spreading mappings of C into itself and $T_i : C \to C$ for $i = 1, 2, \ldots, N$ be a family of nonexpansive mappings of C into itself, such that $F_1 \bigcap T_2 \neq \emptyset$, where $F_1 := \bigcap_{i=1}^{N} F(S_i)$ and $F_2 := \bigcap_{i=1}^{N} F(T_i)$. A sequence of $\{x_n\}$ defined by (1.3), where $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequence in [0,1]. Then, the following hold:*

(i) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $\{x_n\}$ converges weakly to $p \in F_1$;

(ii) If $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\{x_n\}$ converges weakly to $p \in F_2$;

(iii) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges weakly to a common fixed point of $F_1 \bigcap F_2$.

Proof. Following lines of the proof of Theorem 3.1, we show that (i)-(iii) in turn:
(i) The proof is divided into three steps.

1. First, we show that $\{x_n\}$ is bounded. Indeed, let $U_n = \beta_n S_n + (1-\beta_n)T_n$, for $p \in F_1 \bigcap F_2$, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n U_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|U_n x_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\ &= \|x_n - p\|\end{aligned}$$

for all $n \in N$. Hence $\{x_n\}$ is bounded and there exists $\lim_{n\to\infty}\|x_n - p\|$.

2. In this part, we show that $\|x_n - S_i x_n\| \to 0$, $i = 1, 2, \ldots, N$. Put $z_{n+1} = (1-\alpha_n)x_n + \alpha_n S_n x_n$ and $A_n = I - S_n$. Then we have form Lemma 2.2 and $A_n p = 0$ that

$$\begin{aligned}\|z_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n S_n x_n - p\|^2 \\ &= \|x_n - p - \alpha_n x_n + \alpha_n S_n x_n\| = \|x_n - p - \alpha_n A_n x_n\|^2 \\ &= \|x_n - p\|^2 - 2\alpha_n\langle x_n - p, A_n x_n\rangle + \alpha_n^2\|A_n x_n\|^2 \\ &\leq \|x_n - p\|^2 - 2\alpha_n\{\|A_n x_n - A_n p\|^2 \\ &\quad - \frac{1}{2}(\|A_n x_n\|^2 + \|A_n p\|^2)\} + \alpha_n^2\|A_n x_n\|^2 \\ &\leq \|x_n - p\|^2 - 2\alpha_n(\|A_n x_n\|^2 - \frac{1}{2}\|A_n x_n\|^2) + \alpha_n^2\|A_n x_n\|^2 \\ &= \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|A_n x_n\|^2\end{aligned}$$

and hence

$$\alpha_n(1-\alpha_n)\|A_n x_n\|^2 \leq \|x_n - p\|^2 - \|z_{n+1} - p\|^2 \tag{3.1}$$

We also have

$$\begin{aligned}\|x_{n+1} - z_{n+1}\| &= \|(1-\alpha_n)x_n + \alpha_n U_n x_n - (1-\alpha_n)x_n - S_n x_n\| \\ &= \alpha_n\|\beta_n S_n x_n + (1-\beta_n)T_n x_n - S_n x_n\| \\ &= \alpha_n(1-\beta_n)\|T_n x_n - S_n x_n\|.\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$,we have $\|x_n - z_n\| \to 0$ and hence $\lim_{n\to\infty}\|z_n - p\| = \lim_{n\to\infty}\|x_n - p\|$, thus form (3.1) and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ have

$$\lim_{n\to\infty}\|A_n x_n\| = \lim_{n\to\infty}\|x_n - S_n x_n\| = 0.$$

Now for all $i = 1, 2, \ldots, N$, from Lemma 2.4, we have

$$\begin{aligned}\|x_n - S_{n+i}x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\| + \|S_{n+i}x_{n+i} - S_{n+i}x_n\| \\ &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\| \\ &\quad + \{\|x_{n+i} - x_n\|^2 + 2\langle x_{n+i} - S_{n+i}x_{n+i}, x_n - S_{n+i}x_n\rangle\}^{\frac{1}{2}}\end{aligned}$$

which on taking the limit $n \to \infty$ yields

$$\lim_{n\to\infty}\|x_n - S_{n+i}x_n\| = 0$$

for all $i = 1, 2, \dots, N$. Consequently, we have

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0$$

3. Finally, we show that $x_n \rightharpoonup p$.

Since $\{x_n\}$ is bounded sequence, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to p. From Lemma 2.7, we obtain $p \in F_1$. To show our conclusion, it is sufficient to show that for another subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $w \in F_1$, $p = w$. Before proving this, we show that for any $p \in F_1$, $\lim_{n\to\infty} \|x_n - p\|$ exists. We have that for all $p \in F_1$,

$$\begin{aligned}\|z_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n S_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n \|S_n x_n - p\| \\ &\le \|x_n - p\| \\ &\le \|z_n - p\| + \|x_n - z_n\|.\end{aligned}$$

From Lemma 2.6, $\lim_{n\to\infty} \|z_n - p\|$ exists. So, there exists $\lim_{n\to\infty} \|x_n - p\|$ for all $p \in F_1$ because $x_n - z_n \to 0$. Suppose that $p \neq w$. We have form Opial's theorem [9] that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\| &= \lim_{i\to\infty} \|x_{n_i} - p\| < \lim_{i\to\infty} \|x_{n_i} - w\| \\ &= \lim_{n\to\infty} \|x_n - w\| = \lim_{j\to\infty} \|x_{n_j} - w\| \\ &< \lim_{j\to\infty} \|x_{n_j} - p\| = \lim_{n\to\infty} \|x_n - p\|\end{aligned}$$

This is a contradiction. So, $\{x_n\}$ converges weakly to $p \in F_1$.

(ii) It follows from the proof of (i) that we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Put $z_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n x_n$ and $B_n = I - S_n$. we have

$$\begin{aligned}\|x_{n+1} - z_{n+1}\| &= \|(1-\alpha_n)x_n + \alpha_n U_n x_n - (1-\alpha_n)x_n - T_n x_n\| \\ &= \alpha_n \|\beta_n S_n x_n + (1-\beta_n)T_n x_n - T_n x_n\| \\ &= \alpha_n \beta_n \|S_n x_n - T_n x_n\|.\end{aligned}$$

So, $\{z_n\}$ is bounded since $\{x_n\}$ is bounded. For B is 1/2-inverse strongly monotone (i.e. $\frac{1}{2}\|B_n x - B_n y\|^2 \le \langle x - y, B_n x - B_n y\rangle$ and $B_n p = 0$, we have

$$\begin{aligned}\|z_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n T_n - p\|^2 \\ &= \|x_n - p - \alpha_n(I - T_n)x_n\|^2 \\ &= \|x_n - p\|^2 - 2\alpha_n \langle x_n - p, B_n x\rangle + \alpha_n^2 \|B_n x\|^2 \\ &= \|x_n - p\|^2 - 2\alpha_n \langle x_n - p, B_n x - B_n p\rangle + \alpha_n^2 \|B_n x\|^2 \\ &\le \|x_n - p\|^2 - \alpha_n \|B_n x_n - B_n p\|^2 + \alpha_n^2 \|B_n x\|^2 \\ &= \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|B_n x\|^2\end{aligned}$$

and hence

$$\alpha_n(1-\alpha_n)\|B_n x_n\|^2 \le \|x_n - p\|^2 - \|z_{n+1} - p\|^2 \tag{3.2}$$

Summing from n=1 to N, from (4) we have

$$
\begin{aligned}
&\sum_{n=1}^{N} \alpha_n(1-\alpha_n)\|B_n x_n\|^2 \\
&\le \|x_1 - p\|^2 + \sum_{n=1}^{N-1}(\|x_{n+1}-p\|^2 - \|z_{n+1}-p\|^2) - \|z_{N+1}-p\|^2 \\
&\le \|x_1 - p\|^2 + \sum_{n=1}^{N-1}(\|x_{n+1}-p\| + \|z_{n+1}-p\|)\|x_{n+1}-z_{n+1}\| \\
&\le \|x_1 - p\|^2 + \sum_{n=1}^{N-1}\alpha_n\beta_n(\|x_{n+1}-p\| + \|z_{n+1}-p\|)\|S_n x_n - T_n x_n\| \\
&\le \|x_1 - p\|^2 + M\sum_{n=1}^{N-1}\beta_n,
\end{aligned}
$$

where $M = \sup_{n\in N}\{(\|x_{n+1}-p\| + \|z_{n+1}-p\|)\|S_n x_n - T_n x_n\|\}$. Letting $N \to \infty$, form $\sum_{n=1}^{\infty}\beta_n < \infty$, we have

$$
\sum_{n=1}^{N}\alpha_n(1-\alpha_n)\|B_n x_n\|^2 \le \|x_1 - p\|^2 + M\sum_{n=1}^{N-1}\beta_n \le \infty.
$$

Since $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$, from Lemma 2.1 we get

$$
\liminf_{n\to\infty}\|B_n x_n\| = \liminf_{n\to\infty}\|x_n - T_n x_n\| = 0. \tag{3.3}
$$

We note that

$$
\begin{aligned}
&\|x_{n+1} - T_n x_{n+1}\| \\
&\le \|T_n x_{n+1} - T_n x_n\| + \|T_n x_n - U_n x_n\| + (1-\alpha_n)\|U_n x_n - x_n\| \\
&\le \|x_{n+1} - x_n\| + \|T_n x_n - U_n x_n\| + (1-\alpha_n)\|U_n x_n - x_n\| \\
&= \|(1-\alpha_n)x_n + \alpha_n U_n x_n - x_n\| + \|T_n x_n - U_n x_n\| \\
&\qquad + (1-\alpha_n)\|U_n x_n - x_n\| \\
&= \alpha_n\|U_n x_n - x_n\| + \|T_n x_n - U_n x_n\| + (1-\alpha_n)\|U_n x_n - x_n\| \\
&= \|U_n x_n - x_n\| + \|T_n x_n - \{\beta_n S_n x_n + (1-\beta_n)T_n x_n\}\| \\
&= \|\beta_n(S_n x_n - x_n) + (1-\beta_n)(T_n x_n - x_n)\| + \beta_n\|T_n x_n - S_n x_n\| \\
&\le (1-\beta_n)\|x_n - T_n x_n\| + \beta_n(\|S_n x_n - x_n\| + \|T_n x_n - S_n x_n\|) \\
&\le (1-\beta_n)\|x_n - T_n x_n\| + \beta_n(\|x_n - T_n x_n\| + 2\|T_n x_n - S_n x_n\|) \\
&\le \|x_n - T_n x_n\| + 2\beta_n\|T_n x_n - S_n x_n\|.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}\beta_n < \infty$, from Lemma 2.6 there exists the limit of $\{\|x_n - T_n x_n\|\}$. Therefore, form (3.3) we get

$$
\lim_{n\to\infty}\|x_n - T_n x_n\| = 0.
$$

Now for all $i = 1, 2, \dots, N$, from Lemma 2.4, we have

$$\begin{aligned} |x_n - T_{n+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|x_{n+i} - x_n\| \end{aligned}$$

which on taking the limit $n \to \infty$ yields

$$\lim_{n\to\infty} \|x_n - T_{n+i}x_n\| = 0$$

for all $i = 1, 2, \dots, N$. Consequently, we have

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$$

Since $\{x_n\}$ is bounded sequence, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to p. Since a nonexpansive mapping T is demiclosed, we have $p \in F_2$. As in the proof of (i), $\{x_n\}$ converges weakly to $p \in F_2$.

(iii) We obtain from (1.3) that

$$x_{n+1} = \beta_n\{(1-\alpha_n)x_n + \alpha_n S_n x_n\} + (1-\beta_n)\{(1-\alpha_n)x_n + \alpha_n T_n x_n\}$$

for all $n \in N$. Further, putting $W_n = \beta_n\{(1-\alpha_n)I + \alpha_n S_n\} + (1-\beta_n)\{(1-\alpha_n)I + \alpha_n T_n\}$, we can rewrite (1.3) by $x_{n+1} = W_n x_n$. We first show that $\{x_n\}$ converges weakly to some point on F_1. For any $p \in F_1 \bigcap F_2$, we have

$$\begin{aligned} \|(1-\alpha_n)x_n + \alpha_n S_n - p\|^2 = (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|S_n x_n - p\|^2 \\ - \alpha_n(1-\alpha_n)\|x_n - S_n x_n\|^2 \end{aligned}$$

and hence

$$\begin{aligned} \alpha_n(1-\alpha_n)\|x_n - S_n x_n\|^2 &= (\|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n S_n x_n - p\|^2) \\ &\quad - \alpha_n\|x_n - p\|^2 + \alpha_n\|S_n x_n - p\|^2 \\ &\le (\|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n S_n x_n - p\|^2) \quad (3.4) \\ &\quad - \alpha_n\|x_n - p\|^2 + \alpha_n\|x_n - p\|^2 \\ &= \|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n S_n x_n - p\|^2 \end{aligned}$$

by the proof (i),(ii), we have

$$\begin{aligned} \|W_n x_n - p\|^2 &\le \beta_n\|(1-\alpha_n)x_n + \alpha_n S_n x_n - p\|^2 \\ &\quad + (1-\beta_n)\|(1-\alpha_n)x_n + \alpha_n T_n x_n - p\|^2 \\ &\le \beta_n\|(1-\alpha_n)x_n + \alpha_n S_n x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 \\ &= \|x_n - p\|^2 \end{aligned}$$

and hence

$$\begin{aligned}&\|x_n-p\|^2-\beta_n\|(1-\alpha_n)x_n+\alpha_nS_nx_n-p\|^2-(1-\beta_n)\|x_n-p\|^2\\&=\beta_n(\|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_nS_nx_n-p\|^2)\\&\leq\|x_n-p\|^2-\|W_nx_n-p\|^2\end{aligned}$$

As $n\to\infty$, We have

$$\begin{aligned}0&\leq(1-\beta_n)\beta_n(\|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_nS_nx_n-p\|^2)\\&\leq(1-\beta_n)(\|x_n-p\|^2-\|W_nx_n-p\|^2)\\&=(1-\beta_n)(\|x_n-p\|^2-\|x_{n+1}-p\|^2)\to0\end{aligned}$$

Since $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$, we have

$$\lim_{n\to}(\|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_nS_nx_n-p\|^2)=0$$

since $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, from (3.4) we get

$$\lim_{n\to\infty}\|x_n-S_nx_n\|=0.$$

since $S_{n(mod)N}$,for all $i=1,2,\ldots,N$, we have

$$\lim_{n\to\infty}\|x_n-S_ix_n\|=0$$

As in the proof of (i), we obtain from Lemma (2.7) that if $\{x_n\}$ converges weakly to p, then $p\in F_1$. We also show that such p is in F_2. In fact, we have that for any $p\in F_1\bigcap F_2$ we have

$$\begin{aligned}\|(1-\alpha_n)x_n+\alpha_nT_n-p\|^2&=(1-\alpha_n)\|x_n-p\|^2+\alpha_n\|T_nx_n-p\|^2\\&\quad-\alpha_n(1-\alpha_n)\|x_n-T_nx_n\|^2\end{aligned}$$

and hence

$$\begin{aligned}\alpha_n(1-\alpha_n)\|x_n-T_nx_n\|^2&=(\|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_nT_nx_n-p\|^2)\\&\quad-\alpha_n\|x_n-p\|^2+\alpha_n\|T_nx_n-p\|^2\\&\leq(\|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_nT_nx_n-p\|^2)\quad(3.5)\\&\quad-\alpha_n\|x_n-p\|^2+\alpha_n\|x_n-p\|^2\\&=\|x_n-p\|^2-\|(1-\alpha_n)x_n+\alpha_nT_nx_n-p\|^2\end{aligned}$$

by the proof (i)-(ii), we have

$$\begin{aligned}\|W_nx_n-p\|^2&\leq\beta_n\|(1-\alpha_n)x_n+\alpha_nT_nx_n-p\|^2\\&\quad+(1-\beta_n)\|(1-\alpha_n)x_n+\alpha_nT_nx_n-p\|^2\\&\leq\beta_n\|x_n-p\|^2+(1-\beta_n)\|(1-\alpha_n)x_n+\alpha_nT_nx_n-p\|^2\\&\leq\beta_n\|x_n-p\|^2+(1-\beta_n)\|x_n-p\|^2\\&=\|x_n-p\|^2\end{aligned}$$

and hence

$$\begin{aligned}
&\|x_n - p\|^2 - \beta_n\|(1-\alpha_n)x_n + \alpha_n T_n x_n - p\|^2 - (1-\beta_n)\|x_n - p\|^2 \\
&= \beta_n(\|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n T_n x_n - p\|^2) \\
&\leq \|x_n - p\|^2 - \|W_n x_n - p\|^2
\end{aligned}$$

As $n \to \infty$, We have

$$\begin{aligned}
0 &\leq (1-\beta_n)\beta_n(\|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n T_n x_n - p\|^2) \\
&\leq (1-\beta_n)(\|x_n - p\|^2 - \|W_n x_n - p\|^2) \\
&= (1-\beta_n)(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \to 0
\end{aligned}$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to}(\|x_n - p\|^2 - \|(1-\alpha_n)x_n + \alpha_n T_n x_n - p\|^2) = 0$$

since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, from (3.5) we get

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

Further since $T_{n(mod)N}$,for all $i = 1, 2, \dots, N$, we have

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$$

Since $\{x_n\}$ converges weakly to p, then $p \in F_2$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_j}\} \rightharpoonup v$. Then we have $p = v$. In fact, if$p \neq v$,then we have that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p\| &= \lim_{i\to\infty} \|x_{n_i} - p\| < \lim_{i\to\infty} \|x_{n_i} - v\| \\
&= \lim_{n\to\infty} \|x_n - v\| = \lim_{j\to\infty} \|x_{n_j} - v\| \\
&< \lim_{j\to\infty} \|x_{n_j} - p\| = \lim_{n\to\infty} \|x_n - p\|
\end{aligned}$$

This is a contradiction. So, we have $p = v$. Therefore we conclude that $\{x_n\}$ converges weakly to $p \in F_1 \bigcap F_2$.

Put $\beta = 1$ and $\beta = 0$, we get the following Corollary form Theorem 3.1.

Corollary 3.1. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $S_i : C \to C$ for $i = 1, 2, \dots, N$ be a family of spreading mappings of C into itself such that $F_1 \neq \emptyset$. For arbitrary $x_1 \in C$, defined a sequence $\{x_n\}$ as following*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S_n x_n$$

where $S_n = S_{n(modN)}$ and the mod function takes values in $1, 2 \dots, N$, $\alpha_n \subset [0,1]$. If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, then $\{x_n\}$ converges weakly to $p \in F_1$.

Proof. Putting $\beta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1 ,we get the conclusion.

Corollary 3.2. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $F_i : C \to C$ for $i = 1, 2, \dots, N$ be a family of nonexpansive* *mappings of C into itself such that $F_2 \neq \emptyset$. For arbitrary $x_1 \in C$, defined a sequence $\{x_n\}$ as following*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n$$

where $T_n = T_{n(modN)}$ and the mod function takes values in $1, 2 \ldots, N$, $\alpha_n \subset [0, 1]$. If $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then $\{x_n\}$ converges weakly to $p \in F_2$.

Proof. Putting $\beta_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.1 ,we get the conclusion.

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