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POSITIVE SOLUTIONS TO A FOUR-POINT BOUNDARY VALUE PROBLEM OF HIGHER-ORDER DIFFERENTIAL EQUATION WITH A P-LAPLACIAN†

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ABSTRACT. In this paper, we obtain the existence of positive solutions for a quasi-linear four-point boundary value problem of higher-order differential equation. By using the fixed point index theorem and imposing some conditions on f, the existence of positive solutions to a higher-order four-point boundary value problem with a p-Laplacian is obtained.

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1. Introduction

In this paper, we study the existence of positive solutions to the following boundary value problem with a p-Laplacian

$$(\phi_p(u^{(n-1)}(t)))' + f(t, u(t), u'(t), \dots, u^{(n-2)}(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$\begin{cases} u^{(i)}(0) = 0, & 0 \leqslant i \leqslant n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(\xi) = 0, & n \geqslant 3, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(\eta) = 0, & n \geqslant 3, \end{cases} \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = \phi_p^{-1}$, $1/p + 1/q = 1$, $0 < \xi < \eta < 1$, $\alpha > 0$, $\beta \geqslant 0$, $\gamma > 0$, $\delta \geqslant 0$.

To the best of our knowledge, no one has obtained results with the boundary condition (1.2). We notice that when $\xi \to 0, \eta \to 1$, (1.2) approaches to the Sturm-Liouville boundary condition. Therefore, we call BVP (1.1)-(1.2) a generalized Sturm-Liouville boundary value problem. The study of multi-point

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boundary value problems for linear second-order differential equations was initiated by Il'in and Moiseev [1]. Since then there has been much current attention focused on the study of nonlinear multi-point boundary value problems see [1-5]. Moreover, in recent years, some literatures begin to investigate multi-point BVPs for higher-order differential equations. Such as the type given by (1.1) and (1.2). But such BVPs are different from the two-point ones, there are difficulties to be overcome.

For example, in [8], the authors considered the following n-order boundary value problem with a p-Laplacian

$$(\phi_p(u^{(n-1)}(t)))' + g(t)f(u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0, \quad 0 < t < 1, \tag{1.3}$$

$$\begin{cases} u^{(i)}(0) = 0, & 0 \leqslant i \leqslant n-3, \\ \alpha\phi_p(u^{(n-2)}(0)) - \beta\phi_p(u^{(n-1)}(\xi)) = 0, & n \geqslant 3, \\ \gamma\phi_p(u^{(n-2)}(1)) + \delta\phi_p(u^{(n-1)}(\eta)) = 0, & n \geqslant 3, \end{cases} \tag{1.4}$$

They claimed in Lemma 2.3 that "By the equations of boundary condition, we have $u^{(n-1)}(\xi) \geqslant 0$, $u^{(n-1)}(\eta) \leqslant 0$. Then there exists a constant $\delta \in [\xi, \eta] \subset (0, 1)$ such that $u^{(n-1)}(\delta) = 0$". Unfortunately, this statement is not true. So their results for the existence of positive solutions are not acceptable. We give a counter example to their claim as follow.

Consider the following BVP for $n = 3$, $p = 2$,

$$u''' + a = 0, \quad 0 < t < 1, \tag{1.5}$$

$$\begin{cases} u(0) = 0, \\ u'(0) - \dfrac{1}{3}u''\left(\dfrac{5}{6}\right) = 0, \\ u'(1) + \dfrac{1}{2}u''\left(\dfrac{8}{9}\right) = 0, \end{cases} \tag{1.6}$$

where a is a positive number. After some calculation, the solution of BVP (1.5)-(1.6) can be expressed as follow:

$$u(t) = -\frac{a}{6}t^3 + \frac{a}{3}t^2 - \frac{a}{18}t, \ for\ t \in [0, 1].$$

Moreover, we have $u''(t) = -at + \frac{2}{3}a$, then it is easy to see that $u''\left(\frac{2}{3}\right) = 0$. Thus $\sigma = \frac{2}{3} \notin \left[\frac{5}{6}, \frac{8}{9}\right] \subset [0, 1]$. The key point to obtain a positive solution for BVP (1.1)-(1.2) is to find suitable conditions imposed on f to ensure that the maximum point σ of $u^{(n-2)}(t)$ locates between ξ and η. Only in this way, one can obtain $u^{(n-1)}(0)$, $u^{(n-1)}(1) \geqslant 0$ and then $u^{(n-1)}(t) \geqslant 0$ by the concavity of $u^{(n-1)}(t)$.

2. Preliminaries

Theorem 2.1 ([6]). *Suppose E is a real Banach space, $K \subset E$ is a cone, let $\Omega_r = \{u \in K : \|u\| \leqslant r\}$. Let operator $T : \Omega_r \to K$ be completely continuous and satisfy $Tx \neq x$, $\forall x \in \partial\Omega_r$. Then*

(S1) $i(T, \Omega_r, K) = 1$, *if* $\|Tx\| \leqslant \|x\|$, $\forall x \in \partial\Omega_r$;
(S2) $i(T, \Omega_r, K) = 0$, *if* $\|Tx\| \geqslant \|x\|$, $\forall x \in \partial\Omega_r$.

Firstly, for $y \in C[0,1]$, we give some conclusions with respect to the following boundary value problem

$$(\phi_p(u^{(n-1)}(t)))' + y(t) = 0, \qquad 0 < t < 1, \tag{2.1}$$

$$\begin{cases} u^{(i)}(0) = 0, & 0 \leqslant i \leqslant n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(\xi) = 0, & n \geqslant 3, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(\eta) = 0, & n \geqslant 3. \end{cases} \tag{2.2}$$

Lemma 2.2. *Let $y \in C[0,1]$, $y(t) \geqslant 0$, $y(t) \not\equiv 0$. Then BVP (2.1)-(2.2) has a unique solution*

$$u(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2}) ds_{n-2} ds_{n-3} \ldots ds_1, \tag{2.3}$$

where

$$w(t) = \begin{cases} \dfrac{\beta}{\alpha}\phi_q\left(\displaystyle\int_\xi^\sigma y(r)dr\right) + \displaystyle\int_0^t \phi_q\left(\int_s^\sigma y(r)dr\right) ds, & 0 \leqslant t \leqslant \sigma, \\ \dfrac{\delta}{\gamma}\phi_q\left(\displaystyle\int_\sigma^\eta y(r)dr\right) + \displaystyle\int_t^1 \phi_q\left(\int_\sigma^s y(r)dr\right) ds, & \sigma \leqslant t \leqslant 1, \end{cases} \tag{2.4}$$

where σ is a solution of the following equation

$$V_1(t) - V_2(t) = 0, \quad t \in [0,1], \tag{2.5}$$

where

$$V_1(t) = \frac{\beta}{\alpha}\phi_q\left(\int_\xi^t y(r)dr\right) + \int_0^t \phi_q\left(\int_s^t y(r)dr\right) ds,$$
$$V_2(t) = \frac{\delta}{\gamma}\phi_q\left(\int_t^\eta y(r)dr\right) + \int_t^1 \phi_q\left(\int_t^s y(r)dr\right) ds.$$

Proof. We notice that $V_1(t)$ is an increasing function about $t \in [0,1]$. Since $\beta \geqslant 0$, we have $V_1(0) \leqslant 0$, $V_1(1) > 0$. At the same time, $V_2(t)$ is a decreasing function about $t \in [0,1]$. Since $\delta \geqslant 0$, we have $V_2(0) > 0$, $V_2(1) \leqslant 0$. So, there must be an intersection point between 0 and 1 for $V_1(t)$ and $V_2(t)$, which is a solution of (2.5). Secondly, it is easy to verify that (2.3) is a solution of BVP (2.1)-(2.2). Next we prove that the solution of BVP (2.1)-(2.2) can be expressed as equation (2.3). Set u is a solution of BVP (2.1)-(2.2). Then $(\phi_p(u^{(n-1)}(t)))' = -y(t) \leqslant 0$ implies $u^{(n)}(t) \leqslant 0$. So there exists a constant $\sigma \in (0,1)$ (if $\sigma = 0, 1$, we can deduce a contradiction easily) such that $u^{(n-1)}(\sigma) = 0$. If it doesn't hold, without loss of generality, we suppose that $u^{(n-1)}(t) > 0$, for $t \in (0,1)$, then $u^{(n-2)}(t)$ is an increasing function about $t \in (0,1)$. From (2.2), we have

$$u^{(n-2)}(0) = \frac{\beta}{\alpha} u^{(n-1)}(\xi) > 0, \quad u^{(n-2)}(1) = -\frac{\delta}{\gamma} u^{(n-1)}(\eta) < 0,$$

which is a contradiction.

Now by integrating the equation (2.1) on $(\sigma, 1)$, we have

$$\phi_p(u^{(n-1)}(t)) = \phi_p(u^{(n-1)}(\sigma)) - \int_\sigma^t y(s)ds,$$

then

$$u^{(n-1)}(t) = -\phi_q\left(\int_\sigma^t y(s)ds\right), \tag{2.6}$$

next

$$u^{(n-2)}(t) = u^{(n-2)}(\sigma) - \int_\sigma^t \phi_q\left(\int_\sigma^s y(r)dr\right)ds. \tag{2.7}$$

Let $t = \eta$ on (2.6), we have

$$u^{(n-1)}(\eta) = -\phi_q\left(\int_\sigma^\eta y(s)ds\right).$$

By boundary condition (2.2), we get

$$u^{(n-2)}(1) = \frac{\delta}{\gamma}\phi_q\left(\int_\sigma^\eta y(s)ds\right).$$

By (2.7), we have

$$u^{(n-2)}(\sigma) = \frac{\delta}{\gamma}\phi_q\left(\int_\sigma^\eta y(s)ds\right) + \int_\sigma^1 \phi_q\left(\int_\sigma^s y(r)dr\right)ds. \tag{2.8}$$

Then

$$u^{(n-2)}(t) = \frac{\delta}{\gamma}\phi_q\left(\int_\sigma^\eta y(s)ds\right) + \int_t^1 \phi_q\left(\int_\sigma^s y(r)dr\right)ds. \tag{2.9}$$

Then by integrating equation (2.9) for n-2 times on (0,1), we have

$$\begin{aligned} u(t) = &\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}} \frac{\delta}{\gamma}\phi_q\left(\int_\sigma^\eta y(s)ds\right)ds_{n-2}ds_{n-3}\ldots ds_1 \\ &+\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\int_{s_{n-2}}^1 \phi_q\left(\int_\sigma^s y(r)dr\right)dsds_{n-2}ds_{n-3}\ldots ds_1. \end{aligned}$$

Similarly, for $t \in (0, \sigma)$, by integrating the equation (2.1) on $(0, \sigma)$, we have

$$\begin{aligned} u(t) = &\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}} \frac{\beta}{\alpha}\phi_q\left(\int_\xi^\sigma y(s)ds\right)ds_{n-2}ds_{n-3}\ldots ds_1 \\ &+\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\int_0^{s_{n-2}} \phi_q\left(\int_s^\sigma y(r)dr\right)dsds_{n-2}ds_{n-3}\ldots ds_1. \end{aligned}$$

Hence, for $t \in [0, 1]$, $u(t)$ can be expressed as equation

$$u(t) = \int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}} w(s_{n-2})ds_{n-2}ds_{n-3}\ldots ds_1,$$

where $w(t)$ is given by (2.4). The proof is completed.

Let E be a real Banach space defined by

$$E = \left\{ u \in C^{n-2}[0,1] : u^{(i)}(0) = 0,\ 0 \leqslant i \leqslant n-3 \right\}$$

with the norm $\|u\| = \max_{t\in[0,1]} \left|u^{(n-2)}(t)\right|$.

Lemma 2.3. *Let $y(t) \in C[0,1]$ satisfy conditions in Lemma 2.2. Suppose*

$$\max_{t\in[0,\xi]\cup[\eta,1]} y(t) \leqslant \Gamma \min_{t\in[\xi,\eta]} y(t),$$

where

$$\Gamma = \left(\min \left\{ \frac{\delta q(\eta-\xi)^{q-1} + \gamma(\eta-\xi)^q}{\gamma \xi^q},\ \frac{\beta q(\eta-\xi)^{q-1} + \alpha(\eta-\xi)^q}{\alpha(1-\eta)^q} \right\} \right)^{p-1}.$$

Then solution $u(t)$ of problem (2.1) satisfies that $u^{(n-2)}(t)$ is concave on $[0,1]$, $u^{(n-2)}(t) \geqslant 0$, $u^{(n-2)}(t)$ is increasing on $[0,\xi]$ and decreasing on $[\eta,1]$. Moreover,

$$u^{(n-2)}(t) \geqslant \omega(t)\|u^{(n-2)}\|, \quad \text{for } t \in [0,1], \tag{2.10}$$

where $\omega(t) = \min\{\frac{1}{\eta}t, \frac{1}{1-\xi}(1-t)\}$.

Proof. Clearly , $u^{(n-2)}(t)$ is concave on [0,1]. Now we prove that $u^{(n-2)}(t)$ is increasing on $[0,\xi]$. If it doesn't hold, then there exists a constant $\sigma \in (0,\xi)$ such that $u^{(n-1)}(\sigma) = 0$.

$$\begin{aligned}
u^{(n-2)}(\sigma) &= \frac{\beta}{\alpha}\phi_q\left(\int_\xi^\sigma y(r)dr\right) + \int_0^\sigma \phi_q\left(\int_s^\sigma y(r)dr\right)ds \\
&< \int_0^\xi \phi_q\left(\int_s^\xi y(r)dr\right)ds \\
&\leqslant \int_0^\xi \phi_q\left(\int_s^\xi dr\right)ds \cdot \phi_q(\max_{r\in[0,\xi]} y(r)) \\
&\leqslant \int_0^\xi \phi_q\left(\int_s^\xi dr\right)ds \cdot \phi_q(\Gamma)\phi_q(\min_{r\in[\xi,\eta]} y(r)) \\
&\leqslant \frac{1}{q}\xi^q\left(\frac{\delta q(\eta-\xi)^{q-1}}{\gamma\xi^q} + \frac{(\eta-\xi)^q}{\xi^q}\right)\phi_q(\min_{r\in[\xi,\eta]} y(r)) \\
&\leqslant \frac{\delta}{\gamma}\phi_q\left(\int_\xi^\eta y(r)dr\right) + \int_\xi^\eta\int_\xi^s y(r)drds \\
&\leqslant \frac{\delta}{\gamma}\phi_q\left(\int_\xi^\eta y(r)dr\right) + \int_\sigma^\eta\int_\sigma^s y(r)drds \\
&\leqslant \frac{\delta}{\gamma}\phi_q\left(\int_\xi^\eta y(r)dr\right) + \int_\sigma^1\int_\sigma^s y(r)drds \\
&= u^{(n-2)}(\sigma),
\end{aligned}$$

which is a contradiction. Therefore, $u^{(n-2)}(t)$ is increasing on $[0,\xi]$. By similar argument, we obtain that $u^{(n-2)}(t)$ is decreasing on $[\eta,1]$. Thus, with the boundary condition (2.2), we have

$$u^{(n-2)}(0)=\frac{\beta}{\alpha}u^{(n-1)}(\xi)\geqslant 0,\quad u^{(n-2)}(1)=-\frac{\delta}{\gamma}u^{(n-1)}(\eta)\geqslant 0.$$

So, $u^{(n-2)}(t)\geqslant 0$ on [0,1]. Furthermore, from the process of the proof , we can deduce that the constant $\sigma\in(0,1)$ mentioned in Lemma 2.2 satisfies $\sigma\in[\xi,\eta]\subset(0,1)$. And $u^{(n-2)}(t)\geqslant 0$ implied $u(t)\geqslant 0$ since equation (2.3). At last by the concavity of $u^{(n-2)}(t)$, for $t\in(0,\sigma]$, we have

$$\frac{u^{(n-2)}(t)}{t}\geqslant\frac{u^{(n-2)}(\sigma)}{\sigma}\geqslant\frac{1}{\eta}\|u\|,$$

i.e.,

$$u^{(n-2)}(t)\geqslant\frac{t}{\eta}\|u\|.$$

At the same time, for $t\in[\sigma,1)$, we have

$$\frac{u^{(n-2)}(t)}{1-t}\geqslant\frac{u^{(n-2)}(\sigma)}{1-\sigma}\geqslant\frac{1}{1-\xi}\|u\|,$$

i.e.,

$$u^{(n-2)}(t)\geqslant\frac{1-t}{1-\xi}\|u\|.$$

Set $\omega(t)=\min\{\frac{t}{\eta},\frac{1-t}{1-\xi}\}$. Thus, (2.10) holds. The proof is completed.

3. Existence of single positive solution for BVP (1.1)-(1.2)

In this section, we present our main results with respect to BVP (1.1)-(1.2). Set $k>2$ such that $\xi,\eta\in\left[\frac{1}{k},1-\frac{1}{k}\right]$.

Lemma 3.1. *Set* $\tilde{\omega}=\min\left\{\omega(\frac{1}{k}),\omega(1-\frac{1}{k})\right\}$. *Then the solution* $u(t)$ *of problem (2.1) satisfies:*

$$u(t)\leqslant u'(t)\leqslant\ldots\leqslant u^{(n-3)}(t),\quad t\in[0,1]$$

and for $k>2$, *we have*

$$u^{(n-3)}(t)\leqslant\frac{1}{\tilde{\omega}}u^{(n-2)}(t),\quad t\in\left[\frac{1}{k},1-\frac{1}{k}\right].$$

Proof. If $u(t)$ is a solution of problem (2.1), then $u^{(n-2)}(t)$ is concave, and $u^{(i)}(t)\geqslant 0$, $i=0,1,\ldots,n-2$, $t\in[0,1]$. Thus we have

$$u^{(i)}(t)=\int_0^t u^{(i+1)}(s)ds\leqslant tu^{(i+1)}(t)\leqslant u^{(i+1)}(t),\quad i=0,1,\ldots,n-4,$$

i.e., $u(t)\leqslant u'(t)\leqslant\ldots\leqslant u^{(n-3)}(t)$, $t\in[0,1]$.

Secondly, for $t \in [\frac{1}{k}, 1-\frac{1}{k}]$, we have $u^{(n-2)}(t) \geqslant \omega(t)\|u^{(n-2)}\| \geqslant \tilde{\omega}\|u^{(n-2)}\|$. By $u^{(n-3)}(t) = \int_0^t u^{(n-2)}(s)ds \leqslant \|u^{(n-2)}\|$, we have

$$u^{(n-3)}(t) \leqslant \|u^{(n-2)}\| \leqslant \frac{1}{\tilde{\omega}} u^{(n-2)}(t), \quad t \in \left[\frac{1}{k}, 1-\frac{1}{k}\right].$$

The proof is completed.

Denote

$$\frac{2}{M} \leqslant \min_{\sigma \in [\xi,\eta]} \left(\frac{\beta}{\alpha}(\sigma-\xi)^{q-1} + \frac{1}{q}\left(\sigma - \frac{1}{k}\right)^q + \frac{\delta}{\gamma}(\eta-\sigma)^{q-1} + \frac{1}{q}\left(1-\frac{1}{k}-\sigma\right)^q \right),$$

$$\frac{2}{m} \geqslant \max_{\sigma \in [\xi,\eta]} \left(\frac{\beta}{\alpha}(\sigma-\xi)^{q-1} + \frac{1}{q}\sigma^q + \frac{\delta}{\gamma}(\eta-\sigma)^{q-1} + \frac{1}{q}(1-\sigma)^q \right),$$

$$\begin{aligned} M_R = &\max\{f(t,u_0,u_1,\dots,u_{n-2}), \quad t \in [0,\xi] \cup [\eta,1], \\ &0 \leqslant u_0 \leqslant u_1 \leqslant \dots \leqslant u_{n-3} \leqslant \frac{R}{\tilde{\omega}}, \quad \omega(t)R \leqslant u_{n-2} \leqslant R\}, \end{aligned}$$

$$\begin{aligned} N_R = &\min\{f(t,u_0,u_1,\dots,u_{n-2}), \ t \in [\xi,\eta], \\ &0 \leqslant u_0 \leqslant u_1 \leqslant \dots \leqslant u_{n-3} \leqslant \frac{R}{\tilde{\omega}}, \ \omega(t)R \leqslant u_{n-2} \leqslant R\}, \end{aligned}$$

$$\begin{aligned} f_{\theta\rho,\rho} = &\min\left\{ \frac{f(t,u_0,u_1,\dots,u_{n-2})}{\phi_p(\rho)}, \ t \in [\theta, 1-\theta], \right. \\ &\left. 0 \leqslant u_0 \leqslant u_1 \leqslant \dots \leqslant u_{n-3} \leqslant \frac{\rho}{\tilde{\omega}}, \omega(t)\rho \leqslant u_{n-2} \leqslant \rho \right\}, \end{aligned}$$

$$\begin{aligned} f^{0,\rho} = &\max\left\{ \frac{f(t,u_0,u_1,\dots,u_{n-2})}{\phi_p(\rho)}, \ t \in [0,1], \right. \\ &\left. 0 \leqslant u_0 \leqslant u_1 \leqslant \dots \leqslant u_{n-3} \leqslant \frac{\rho}{\tilde{\omega}}, \omega(t)\rho \leqslant u_{n-2} \leqslant \rho \right\}, \end{aligned}$$

$$f^a = \lim_{u_{n-2}\to a} \sup \max_{0\leqslant t\leqslant 1,\ 0\leqslant u_i \leqslant \frac{u_{n-2}}{\tilde{\omega}}} \left\{ \frac{f(t,u_0,u_1,\dots,u_{n-2})}{\phi_p(u_{n-2})} \right\},$$

$$f_a = \lim_{u_{n-2}\to a} \inf \min_{0\leqslant t\leqslant 1,\ 0\leqslant u_i \leqslant \frac{u_{n-2}}{\tilde{\omega}}} \left\{ \frac{f(t,u_0,u_1,\dots,u_{n-2})}{\phi_p(u_{n-2})} \right\},$$

where $a = 0, \infty$, $0 \leqslant i \leqslant n-3$ and $\tilde{\omega}$ is mentioned by Lemma 3.1. In the follows, we give the assumptions in this paper:

(H1) $f : C([0,1] \times [0,+\infty)^{n-1}, [0,+\infty))$, $f(t,0,\dots,0) \not\equiv 0$;
(H2) There exists a constant $R_0 > 0$ such that $M_R \leqslant \Gamma N_R$, for all $R \in [0, R_0]$;
(H3) $M_R \leqslant \Gamma N_R$, for all $R > 0$.

Define a cone $K \subset E$ by

$$\begin{aligned} K = &\{u \in E : u^{(n-2)}(t) \geqslant 0, u^{(n-2)}(t) \text{ is concave}, \\ &u^{(n-2)}(t) \text{ is increasing on } [0,\xi] \text{ and decreasing on } [\eta,1]\}. \end{aligned}$$

Define an operator $T : K \to C^{n-1}[0,1]$ by

$$(Tu)(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2})ds_{n-2}ds_{n-3}\dots ds_1,$$

where

$$w(t) = \begin{cases} \frac{\beta}{\alpha}\phi_q\left(\int_\xi^\sigma f(r,u(r),u'(r),\dots,u^{(n-2)}(r))dr\right) \\ \qquad + \int_0^t \phi_q\left(\int_s^\sigma f(r,u(r),u'(r),\dots,u^{(n-2)}(r))dr\right)ds, & 0 \leqslant t \leqslant \sigma, \\ \frac{\delta}{\gamma}\phi_q\left(\int_\sigma^\eta f(r,u(r),u'(r),\dots,u^{(n-2)}(r))dr\right) \\ \qquad + \int_t^1 \phi_q\left(\int_\sigma^s f(r,u(r),u'(r),\dots,u^{(n-2)}(r))dr\right)ds, & \sigma \leqslant t \leqslant 1. \end{cases}$$

Theorem 3.2. *Suppose that conditions (H1), (H2) hold. There exist positive constants r, ρ such that f also satisfies:*

(H4) $f_{\frac{1}{k}r,r} \geqslant \phi_p(M)$;

(H5) $f^{0,\rho} \leqslant \phi_p(m)$.

Then BVP (1.1)-(1.2) has a solution u such that $\|u\|$ lies between r and ρ.

Proof. Set $\Omega_R = \{x \in K \mid \|x\| < R\}$. Then $T : \overline{\Omega}_R \to K$ is completely continuous. Because

$$(Tu)^{(n-1)}(t) = \begin{cases} \phi_q\left(\int_t^\sigma f(s,u(s),\dots,u^{(n-2)}(s))ds\right) \geqslant 0, & 0 \leqslant t \leqslant \sigma, \\ -\phi_q\left(\int_\sigma^t f(s,u(s),\dots,u^{(n-2)}(s))ds\right) \leqslant 0, & \sigma \leqslant t \leqslant 1 \end{cases}$$

is continuous, decreasing on $[0,1]$ and satisfies

$$(Tu)^{(n-1)}(\sigma) = 0.$$

Then, $(Tu)^{(n-2)}(t)$ is concave and $(Tu)^{(n-2)}(\sigma) = \max_{t\in[0,1]}(Tu)^{(n-2)}(t)$. Since condition (H1), we obtain that $(Tu)^{(n-2)}(t)$ is increasing on $[0,\xi]$ and decreasing on $[\eta,1]$, also, we have $(Tu)^{(n-2)}(t) \geqslant 0$. This shows that $T\overline{\Omega}_R \subset K$. Furthermore, it is easy to check by Arzela-Ascoli Theorem that $T : \overline{\Omega}_R \to K$ is completely continuous.

It is clear that $u \in E$ is a positive of BVP (1.1)-(1.2) if and only if u is a fixed point of operator T. Without loss of generality, we suppose $r < \rho$. Let $\Omega_r = \{u \in K : \|u\| < r\}$ and $\Omega_\rho = \{u \in K : \|u\| < \rho\}$. For any $u \in \partial\Omega_r$, if $Tu = u$, then T has a fixed point satisfying $\|u\| = r$. If not, by (2.10), we have

$$r = \|u\| \geqslant u^{(n-2)}(t) \geqslant \omega(t)\|u\| \geqslant \omega(t)r, \quad t \in \left[\frac{1}{k}, 1 - \frac{1}{k}\right].$$

One side, we have

$$
\begin{aligned}
\|Tu\| =&(Tu)^{(n-2)}(\sigma)\\
\geqslant&\frac{\beta}{2\alpha}\phi_q\left(\int_\xi^\sigma f(r,u(r),u'(r),\ldots,u^{(n-2)}(r))dr\right)\\
&\qquad+\frac{1}{2}\int_{\frac{1}{k}}^{\sigma}\phi_q\left(\int_s^\sigma f(r,u(r),u'(r),\ldots,u^{(n-2)}(r))dr\right)ds\\
&+\frac{\delta}{2\gamma}\phi_q\left(\int_\sigma^\eta f(r,u(r),u'(r),\ldots,u^{(n-2)}(r))dr\right)\\
&\qquad+\frac{1}{2}\int_\sigma^{1-\frac{1}{k}}\phi_q\left(\int_\sigma^s f(r,u(r),u'(r),\ldots,u^{(n-2)}(r))dr\right)ds\\
\geqslant&\frac{1}{2}Mr\left[\frac{\beta}{\alpha}\phi_q\left(\int_\xi^\sigma dr\right)+\int_{\frac{1}{k}}^\sigma\phi_q\left(\int_s^\sigma dr\right)ds\right.\\
&\qquad\left.+\frac{\delta}{\gamma}\phi_q\left(\int_\sigma^\eta dr\right)+\int_\sigma^{1-\frac{1}{k}}\phi_q\left(\int_\sigma^s dr\right)ds\right]\\
\geqslant&\frac{1}{2}Mr\cdot\frac{2}{M}=r=\|u\|.
\end{aligned}
$$

Therefore, we have

$$\|Tu\|\geqslant\|u\|,\quad for\ u\in\partial\Omega_r.$$

Then by Theorem 2.1, we have

$$i(T,\Omega_r,K)=0. \tag{3.1}$$

On the other hand, for $u\in\partial\Omega_\rho$, if $Tu=u$, then T has a fixed point satisfying $\|u\|=\rho$. If not, $\rho=\|u\|\geqslant\omega(t)\rho$, we have

$$
\begin{aligned}
\|Tu\| =&(Tu)(\sigma)\\
\leqslant&\frac{1}{2}\rho m\left[\frac{\beta}{\alpha}\phi_q\left(\int_\xi^\sigma dr\right)+\int_0^\sigma\phi_q\left(\int_s^\sigma dr\right)ds\right.\\
&\qquad\left.+\frac{\delta}{\gamma}\phi_q\left(\int_\sigma^\eta dr\right)+\int_\sigma^1\phi_q\left(\int_\sigma^s dr\right)ds\right]\\
\leqslant&\frac{1}{2}m\rho\cdot\frac{2}{m}=\rho=\|u\|.
\end{aligned}
$$

Thus, by Theorem 2.1, we have

$$i(T,\Omega_\rho,K)=1. \tag{3.2}$$

Hence, by (3.1) and (3.2) with $r<\rho$, we have

$$i(T,\Omega_\rho\setminus\bar{\Omega}_r,K)=1.$$

Then operator T has a fixed point $u\in(\bar{\Omega}_\rho\setminus\Omega_r)$ and $r\leqslant\|u\|\leqslant\rho$. The proof is completed.

Theorem 3.3. *Suppose that (H1), (H3) and the following conditions hold:*
(H6) $f^0 < \phi_p(m)$;
(H7) $f_\infty > \phi_p(\frac{M}{\tilde{\omega}})$.
Then BVP (1.1)-(1.2) has a positive solution.

Proof. By condition (H6),

$$f^0 = \lim_{u_{n-2}\to 0} \sup \max_{0\leqslant t\leqslant 1,\ 0\leqslant u_i\leqslant u_{n-2}} \left\{ \frac{f(t,u_0,u_1,\ldots,u_{n-2})}{\phi_p(u_{n-2})} \right\},$$

then for any $\varepsilon > 0$, there exists a small enough positive number $r > 0$, for $0 \leqslant u_{n-2} \leqslant r$, and $u_{n-2} \neq 0$, we have

$$f(t,u_0,u_1,\ldots,u_{n-2}) \leqslant (\phi_p(m)+\varepsilon)\phi_p(u_{n-2}) \leqslant (\phi_p(m)+\varepsilon)\phi_p(r). \tag{3.3}$$

Thus, by (3.3), we get that condition (H5) holds.

Next, by condition (H7),

$$f_\infty = \lim_{u_{n-2}\to \infty} \inf \min_{0\leqslant t\leqslant 1,\ 0\leqslant u_i\leqslant u_{n-2}} \left\{ \frac{f(t,u_0,u_1,\ldots,u_{n-2})}{\phi_p(u_{n-2})} \right\},$$

then for any $\varepsilon > 0$, there exists a large enough positive number $\rho \neq r$, for $u_{n-2} \geqslant \omega(t)\rho \geqslant \tilde{\omega}\rho$, for $t \in [\frac{1}{k}, 1-\frac{1}{k}]$, we have

$$\begin{aligned} f(t,u_0,u_1,\ldots,u_{n-2}) &\geqslant \left(\phi_p\left(\frac{M}{\tilde{\omega}}\right) - \varepsilon\right) \cdot \phi_p(u_{n-2}) \\ &\geqslant \left(\phi_p\left(\frac{M}{\tilde{\omega}}\right) - \varepsilon\right) \cdot \phi_p(\tilde{\omega}\rho) \\ &\geqslant (\phi_p(M) - \varepsilon\phi_p(\tilde{\omega})) \cdot \phi_p(\rho). \end{aligned} \tag{3.4}$$

Thus, by (3.4), condition (H4) holds. Hence, by Theorem 3.2, we obtain that BVP (1.1)-(1.2) has a positive solution. The proof is completed.

Theorem 3.4. *Suppose that (H1), (H3) and the following conditions hold:*
(H8) $f_0 > \phi_p(\frac{M}{\tilde{\omega}})$;
(H9) $f^\infty < \phi_p(m)$.
Then BVP (1.1)-(1.2) has a positive solution.

Proof. By condition (H8),

$$f_0 = \lim_{u_{n-2}\to 0} \inf \min_{0\leqslant t\leqslant 1,\ 0\leqslant u_i\leqslant u_{n-2}} \left\{ \frac{f(t,u_0,u_1,\ldots,u_{n-2})}{\phi_p(u_{n-2})} \right\},$$

then for any $\varepsilon > 0$, there exists a small enough positive number $r > 0$ such that for $0 \leqslant u_{n-2} \leqslant r$, $u_{n-2} \neq 0$, we have

$$f(t,u_0,u_1,\ldots,u_{n-2}) \geqslant \left(\phi_p\left(\frac{M}{\tilde{\omega}}\right) - \varepsilon\right) \phi_p(u_{n-2}).$$

Thus, for $t \in \left[\frac{1}{k}, 1-\frac{1}{k}\right]$,

$$\begin{aligned} f(t, u_0, u_1, \ldots, u_{n-2}) &\geqslant \left(\phi_p\left(\frac{M}{\tilde{\omega}}\right) - \varepsilon\right)\phi_p(u_{n-2}) \\ &\geqslant \left(\phi_p\left(\frac{M}{\tilde{\omega}}\right) - \varepsilon\right)\phi_p(\omega(t)r) \\ &\geqslant \left(\phi_p\left(\frac{M}{\tilde{\omega}}\right) - \varepsilon\right)\phi_p(\tilde{\omega}r). \end{aligned}$$

Furthermore, we have

$$\frac{f(t, u_0, u_1, \ldots, u_{n-2})}{\phi_p(r)} \geqslant \phi_p(M) - \varepsilon\phi_p(\tilde{\omega}). \tag{3.5}$$

So, by (3.5), condition (H4) holds.

Next, by condition (H9),

$$f^\infty = \lim_{u_{n-2}\to\infty} \sup \max_{0\leqslant t\leqslant 1,\ 0\leqslant u_i\leqslant u_{n-2}} \left\{\frac{f(t, u_0, u_1, \ldots, u_{n-2})}{\phi_p(u_{n-2})}\right\},$$

then for any $\varepsilon > 0$, there exists a large enough positive number $r \neq \rho$, as $u_{n-2} \geqslant \rho$, we have

$$f(t, u_0, u_1, \ldots, u_{n-2})) \leqslant (\phi_p(m) + \varepsilon) \cdot \phi_p(u_{n-2}). \tag{3.6}$$

If f is unbounded, by the continuity of f on $[0,1] \times [0,+\infty)^{n-1}$, then there exists a constant $\rho \leqslant \rho_1 \leqslant R$ and a point $(\hat{t}, \hat{u}_0, \ldots, \hat{u}_{n-2}) \in [0,1] \times [0,+\infty)^{n-1}$ such that $\rho \leqslant \hat{u}_{n-2} \leqslant \rho_1$ and

$$f(t, u_0, u_1, \ldots, u_{n-2}) \leqslant f(\hat{t}, \hat{u}_0, \ldots, \hat{u}_{n-2}), \quad t \in [0,1],\ u_{n-2} \in [0, \rho^*].$$

So, by $\rho \leqslant \hat{u}_{n-2} \leqslant \rho_1$, for $0 \leqslant t \leqslant 1$, $0 \leqslant u_{n-2} \leqslant \rho_1$, we get

$$\begin{aligned} f(t, u_0, u_1, \ldots, u_{n-2}) &\leqslant f(\hat{t}, \hat{u}_0, \ldots, \hat{u}_{n-2}) \\ &\leqslant (\phi_p(m) + \varepsilon)\phi_p(\hat{u}_{n-2}) \\ &\leqslant (\phi_p(m) + \varepsilon)\phi_p(\rho_1). \end{aligned} \tag{3.7}$$

So, by (3.7), condition (H5) holds.

If f is bounded, then there exists a constant D such that $f(t, u_0, \ldots, u_{n-2}) \leqslant D$, for $t \in [0,1]$, $u_{n-2} \in [0,+\infty)$. Set $\rho_2 > \rho$ and satisfies $\phi_p(m\rho_2) \geqslant D$. Thus, for $t \in [0,1]$, $u_{n-2} \in [0, \rho_2]$,

$$f(t, u_0, \ldots, u_{n-2}) \leqslant D \leqslant \phi_p(m\rho_2) \leqslant (\phi_p(m) + \varepsilon)\phi_p(\rho_2). \tag{3.8}$$

So, by (3.8), condition (H5) holds. Hence, by Theorem 3.2, we obtain that BVP (1.1)-(1.2) has a positive solution. The proof is completed.

4. Existence of twin positive solutions for BVP (1.1)-(1.2)

Theorem 4.1. *Suppose that conditions (H1),(H3),(H5) hold. Assume that f also satisfies*

(H10) $f_0 = +\infty$;
(H11) $f_\infty = +\infty$.

Then BVP (1.1)-(1.2) has at least two positive solutions u_1, u_2 such that

$$0 < \|u_1\| < \rho < \|u_2\|.$$

Proof. By condition (H10), for any $L > M$, there exists a constant $\rho_1 \in (0, \rho)$ such that

$$f(t, u_0, u_1, \ldots, u_{n-2}) \geqslant \phi_p(Lu_{n-2}), \quad t \in [0,1],\ u_{n-2} \in (0, \rho_1),\ u_{n-2} \neq 0. \quad (4.1)$$

Set $\Omega_{\rho_1} = \{u \in K : \|u\| < \rho_1\}$, for any $u \in \partial\Omega_{\rho_1}$, by (4.1) and Lemma 2.3, similarly to the proof in Theorem 3.2, we have

$$\|Tu\| \geqslant \|u\|, \quad for\ u \in \partial\Omega_{\rho_1}.$$

Then by Theorem 2.1, we have

$$i(T, \Omega_{\rho_1}, K) = 0. \quad (4.2)$$

Next, by condition (H11), for any $L' > M$, there exists a constant $0 < \rho_2$ such that

$$f(t, u_0, u_1, \ldots, u_{n-2}) \geqslant \phi_p(L'u_{n-2}), \quad t \in [0,1],\ u_{n-2} > \rho_2. \quad (4.3)$$

We choose a constant $\rho_3 > \max\{\rho, \frac{\rho_2}{\tilde{\omega}}\}$, obviously , $\rho_1 < \rho < \rho_3$. Set $\Omega_{\rho_3} = \{u \in K : \|u\| < \rho_3\}$. For any $u \in \partial\Omega_{\rho_3}$, by Lemma 2.3, we have

$$u^{(n-2)}(t) \geqslant \omega(t)\|u\| \geqslant \tilde{\omega}\rho_3 > \rho_2, \ for\ t \in \left[\frac{1}{k}, 1 - \frac{1}{k}\right].$$

Then by (4.3) and similar to the proof in Theorem 3.2, we obtain

$$\|Tu\| \geqslant \|u\|, \quad for\ u \in \partial\Omega_{\rho_3}.$$

Then by Theorem 2.1, we have

$$i(T, \Omega_{\rho_3}, K) = 0. \quad (4.4)$$

Finally, set $\Omega_\rho = \{u \in K : \|u\| < \rho\}$. For any $u \in \partial\Omega_\rho$, by (H5), and Lemma 2.3 and also similar to the proof in Theorem 3.2, we have

$$\|Tu\| \leqslant \|u\|, \quad for\ u \in \partial\Omega_\rho.$$

Then by Theorem 2.1, we have

$$i(T, \Omega_\rho, K) = 1. \quad (4.5)$$

So, by (4.2), (4.4) and (4.5) and $\rho_1 < \rho < \rho_3$, we have

$$i(T, \Omega_\rho \setminus \overline{\Omega}_{\rho_1}, K) = 1, \quad i(T, \Omega_{\rho_3} \setminus \overline{\Omega}_\rho, K) = -1.$$

Then T have fixed point $u_1 \in \Omega_\rho \setminus \overline{\Omega}_{\rho_1}$ and $u_2 \in \Omega_{\rho_3} \setminus \overline{\Omega}_\rho$. It is easy to see that u_1, u_2 are both positive solutions of BVP (1.1)-(1.2) with $0 < \|u_1\| < \rho < \|u_2\|$. The proof is completed.

Theorem 4.2. *Suppose that conditions (H1),(H3),(H4) hold. Assume that f also satisfies*

(H12) $f^0 = 0$;
(H13) $f^\infty = 0$.

Then BVP (1.1)-(1.2) has at least two positive solutions u_1, u_2 such that

$$0 < \|u_1\| < r < \|u_2\|.$$

Proof. By condition (H12), for any small enough $\varepsilon \in (0, m)$, there exists a constant $\rho_1 \in (0, r)$ such that

$$f(t, u_0, u_1, \ldots, u_{n-2}) \leqslant \phi_p(\varepsilon u_{n-2}), \quad t \in [0,1], \quad u_{n-2} \in (0, \rho_1]. \tag{4.6}$$

Set $\Omega_{\rho_1} = \{u \in E : \|u\| < \rho_1\}$, for any $u \in \partial\Omega_{\rho_1}$, by (4.6), we have

$$\|Tu\| \leqslant \|u\|, \quad for\ u \in \partial\Omega_{\rho_1}.$$

Then by Theorem 2.1, we have

$$i(T, \Omega_{\rho_1}, K) = 1. \tag{4.7}$$

Next, let $f^*(t,x) = \max_{0 \leqslant u_{n-2} \leqslant x} f(t, u_0, u_1, \ldots, u_{n-2})$, we notice that $f^*(t,x)$ is monotone increasing with respect to $x \geqslant 0$. Then from $f^\infty = 0$, it is easy to see that

$$\lim_{x\to\infty} \sup \max_{t\in[0,1]} \frac{f^*(t,x)}{\phi_p(x)} = 0.$$

Therefore, for any $\varepsilon \in (0, m)$, there exists a constant $\rho_2 > r$ such that

$$f^*(t,x) \leqslant \phi_p(\varepsilon x), \quad for\ t \in [0,1], \ x \geqslant \rho_2. \tag{4.8}$$

Set $\Omega_{\rho_2} = \{u \in E : \|u\| < \rho_2\}$, for any $u \in \partial\Omega_{\rho_2}$, by (4.8), we have

$$\begin{aligned}
\|Tu\| =& (Tu)^{(n-2)}(\sigma) \\
=& \frac{\beta}{2\alpha}\phi_q\left(\int_\xi^\sigma f(r, u(r), u'(r), \ldots, u^{(n-2)}(r))dr\right) \\
&\quad + \frac{1}{2}\int_0^\sigma \phi_q\left(\int_s^\sigma f(r, u(r), u'(r), \ldots, u^{(n-2)}(r))dr\right) ds \\
&+ \frac{\delta}{2\gamma}\phi_q\left(\int_\sigma^\eta f(r, u(r), u'(r), \ldots, u^{(n-2)}(r))dr\right) \\
&\quad + \frac{1}{2}\int_\sigma^1 \phi_q\left(\int_\sigma^s f(r, u(r), u'(r), \ldots, u^{(n-2)}(r))dr\right) ds
\end{aligned}$$

$$\leqslant \frac{\beta}{2\alpha}\phi_q\left(\int_\xi^\sigma f^*(r,\rho_2)dr\right) + \frac{1}{2}\int_0^\sigma \phi_q\left(\int_s^\sigma f^*(r,\rho_2)dr\right)ds$$
$$+ \frac{\delta}{2\gamma}\phi_q\left(\int_\sigma^\eta f^*(r,\rho_2)dr\right) + \frac{1}{2}\int_\sigma^1 \phi_q\left(\int_\sigma^s f^*(r,\rho_2)dr\right)ds$$
$$\leqslant \frac{\varepsilon\rho_2}{2}\left[\frac{\beta}{\alpha}\phi_q\left(\int_\xi^\sigma dr\right) + \int_0^\sigma \phi_q\left(\int_s^\sigma dr\right)ds \right.$$
$$\left. + \frac{\delta}{\gamma}\phi_q\left(\int_\sigma^\eta dr\right) + \int_\sigma^1 \phi_q\left(\int_\sigma^s dr\right)ds\right]$$
$$\leqslant \frac{\varepsilon\rho_2}{2}\cdot\frac{2}{m} \leqslant \rho_2 = \|u\|,$$

i.e.,

$$\|Tu\| \leqslant \|u\|, \quad for \quad u \in \partial\Omega_{\rho_2}.$$

Then by Theorem 2.1, we have

$$i(T,\Omega_{\rho_2},K) = 1. \tag{4.9}$$

Finally, set $\Omega_r = \{u \in K : \|u\| < r\}$, for any $u \in \partial\Omega_r$, by (H4) and Lemma 2.3, we have

$$\|Tu\| \geqslant \|u\|, \quad for \quad u \in \partial\Omega_r.$$

Then by Theorem 2.1, we have

$$i(T,\Omega_r,K) = 0. \tag{4.10}$$

So, by (4.7), (4.9) and (4.10) and $\rho_1 < r < \rho_2$, we have

$$i(T,\Omega_r \setminus \overline{\Omega}_{\rho_1},K) = -1, \quad i(T,\Omega_{\rho_2} \setminus \overline{\Omega}_r,K) = 1.$$

Then T has fixed point $u_1 \in \Omega_r \setminus \overline{\Omega}_{\rho_1}$ and $u_2 \in \Omega_{\rho_2} \setminus \overline{\Omega}_r$. It is easy to see that u_1, u_2 are both positive solutions of BVP (1.1)-(1.2) with $0 < \|u_1\| < r < \|u_2\|$. The proof is completed. □

5. Some examples

Example 1. Consider the following 3-order boundary value problem with a p-Laplacian

$$(\phi_p(u''(t)))' + \left[8 + \frac{u'^{\frac{1}{2}}}{100+u^2}\right] = 0, \quad 0 < t < 1, \tag{5.1}$$

$$\begin{cases} u(0) = 0, \\ u'(0) - u''\left(\frac{1}{4}\right) = 0, \\ u'(1) + 2u''\left(\frac{1}{2}\right) = 0, \end{cases} \tag{5.2}$$

where

$$\alpha = \beta = \gamma = 1,\ \delta = 2,\ p = \frac{3}{2},\ q = 3,\ \xi = \frac{1}{4},\ \eta = \frac{1}{2},\ k = 4.$$

After some simple calculation, we have

$$M \geqslant \frac{1344}{37},\ m \leqslant \frac{96}{13},\ \Gamma = \sqrt{\frac{13}{8}},\quad \min_{t\in[\frac{1}{4},\frac{3}{4}]} \omega(t) = \frac{1}{3},\quad \min_{t\in[\frac{1}{4},\frac{1}{2}]} \omega(t) = \frac{1}{2}.$$

We set $R = 100, r = 1, \rho = 64$, and without loss of generality, let $M = 37, m = 7$, then we have

$$M_R = 8.1,\quad N_R \approx 8.07.$$

We can verify that (H1) holds. At the same time, after some calculation, we also have (H2), (H3) hold. Then, by Theorem 3.2, BVP (5.1)-(5.2) has a positive solution.

Example 2. Consider the following 3-order boundary value problem with a p-Laplacian

$$(\phi_p(u''(t)))' + \left[e^{t+1}u^{\frac{1}{8}} + (u')^{\frac{1}{4}} + 2(u')^{\frac{1}{2}}\right] = 0,\quad 0 < t < 1, \tag{5.3}$$

$$\begin{cases} u(0) = 0, \\ u'(0) - 2u''\left(\dfrac{1}{4}\right) = 0, \\ u'(1) + 2u''\left(\dfrac{1}{2}\right) = 0, \end{cases} \tag{5.4}$$

where

$$\alpha = \gamma = 1,\ \beta = \delta = 2,\ p = \frac{3}{2},\ q = 3,\ \xi = \frac{1}{4},\quad \eta = \frac{1}{2},\ k = 4.$$

After some simple calculation, we have

$$M \geqslant \frac{576}{23},\ m \leqslant \frac{96}{13},\ \Gamma = \sqrt{\frac{25}{8}},\quad \min_{t\in[\frac{1}{4},\frac{1}{2}]} \omega(t) = \frac{1}{2}.$$

Without loss of generality, we set $M = 25, m = 7$. It is clear that $f^0 = +\infty,\ f_\infty = +\infty$. After some calculation, we have

$$\begin{aligned} \lim\sup_{R\to+\infty} (M_R - \Gamma N_R) &= e^2(4R)^{\frac{1}{8}} + (R)^{\frac{1}{4}} + 2(R)^{\frac{1}{2}} \\ &\quad - \sqrt{\frac{25}{8}}\left[\left(\frac{R}{2}\right)^{\frac{1}{4}} + 2\left(\frac{R}{2}\right)^{\frac{1}{2}}\right] \leqslant 0. \end{aligned}$$

Therefore, condition (H1) holds. At the same time, for the large enough ρ, we also have,

$$f^{0,\rho} = e^3(4\rho)^{\frac{1}{8}} + \rho^{\frac{1}{4}} + 2\rho^{\frac{1}{2}} \leqslant (7\rho)^{\frac{1}{2}}.$$

So, condition (H3) holds. Hence, by Theorem 4.1, BVP (5.3)-(5.4) has at least twin positive solutions.

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