J. Appl. Math. & Informatics Vol. **28***(2010), No. 1 - 2, pp. 31 - 48*
Website: http://www.kcam.biz

AN SDFEM FOR A CONVECTION-DIFFUSION PROBLEM WITH NEUMANN BOUNDARY CONDITION AND DISCONTINUOUS SOURCE TERM

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ABSTRACT. In this article, we consider singularly perturbed Boundary Value Problems(BVPs) for second order Ordinary Differential Equations (ODEs) with Neumann boundary condition and discontinuous source term. A parameter-uniform error bound for the solution is established using the Streamline-Diffusion Finite Element Method (SDFEM) on a piecewise uniform meshes. We prove that the method is almost second order of convergence in the maximum norm, independently of the perturbation parameter. Further we derive superconvergence results for scaled derivatives of solution of the same problem. Numerical results are provided to substantiate the theoretical results.

AMS Mathematics Subject Classification : 6510, CR G1.7
Key words and phrases : Singularly perturbed problem, discontinuous source term, streamline-diffusion, finite element method, superconvergence, scaled derivative , boundary value problem.

1. Introduction

Singularly Perturbed Differential Equations appear in several branches of applied mathematics. Inparticular, convection-diffusion equations model many fluid flows such as water pollution problems, simulation of oil extraction from underground reservoirs, flows in chemical reactors, convective heat transport problems with large Peclet numbers and semiconductor device simulation. Analytical and numerical treatment of these equations have drawn much attention of many researchers [2, 4, 10, 13]. In general, classical numerical methods fail to produce good approximations for these equations. Hence one has to look for non-classical methods. A good number of articles have been appearing in the past three decades on non-classical methods which cover second order equations. Many authors have studied second order singularly perturbed probelms

Received February 27, 2009. October 16, 2009. *Corresponding author.

of both reaction-diffusion and convection-diffusion equations of discontinuous cases and references are therein [5, 6, 7, 8, 11]. Ingeneral the Galerkin FEM even on layer adapted mesh for convection-diffusion type equations does not yield satisfactory result because of the convergence of the stability problem of this method. Hence in [5], authors suggested a SDFEM to overcome this stability problem. This SDFEM was first introduced in [1] for a convection dominated convection-diffusion equation with smooth coefficients. The authors proposed a modification of the standard Galerkin finite element method that actually represents a Petrov-Galerkin FEM with the test functions adapted in such a way as to produce a small amount of artificial diffusion in the streamline direction, thereby enhancing stability. Therefore, this method is also known as the streamline-diffusion Petrov-Galerkin method. It can be also considered as the finite element method that adds weighted residuals to the standard Galerkin FEM. The SDFEM has been applied to numerical solving of single convection diffusion problem with neumann boundary condition and non-smooth source function and derive an error estimate of order $O(N^{-2}\ln^2 N)$ for Shishkin mesh, $O(N^{-2})$ for Bakhvalov-Shishkin mesh, in the maximum norm. $\|.\|_\infty$ denotes the L_∞ norm on $\bar{\Omega}$, $\|.\|_1$ denotes the L_1 norm on $\bar{\Omega}$ and maximum norm is defined as $\|u\| = \max_{x\in\bar{\Omega}} |u(x)|$.

Remark 1.0.1. Through out this paper, C denotes a generic constant that is independent of the parameter ε and N, the dimension of the discrete problem. We also assume $\varepsilon \le CN^{-1}$ as is generally the case in practice for convection-diffusion type equations [3].

In this paper, we consider the following BVP: Find $u \in C^0(\bar{\Omega}) \cap C^1(\Omega \cup \{1\}) \cap C^2(\Omega^- \cup \Omega^+)$ such that

$$Lu := -\varepsilon u''(x) + b(x)u'(x) + c(x)u(x) = f(x), \quad x \in (\Omega^- \cup \Omega^+), \tag{1.1}$$

$$u(0) = 0, \quad \varepsilon u'(1) = \gamma_0, \tag{1.2}$$

$$b(x) \ge \beta > 0, \quad c(x) \ge 0, \quad |[f](d)| \le C, \tag{1.3}$$

where $\Omega := (0,1)$, $\bar{\Omega} := [0,1]$, $\Omega^- = (0,d)$, $\Omega^+ = (d,1)$, $d \in \Omega$, ε is a small positive parameter. The function $f(x)$ is assumed to be sufficiently smooth function on $(\Omega^- \cup \Omega^+)$ and have a jump discontinuity at $x = d$. Further it is assumed that $f(x)$ and its derivatives have right and left limits at the point d. The function $b(x)$ and $c(x)$ are sufficiently smooth functions on $\bar{\Omega}$, and for any function w we denote its jump at $x = d$ as $[w](d) = w(d+) - w(d-)$.

Theorem 1.0.2. *For $k = 0, 1, 2, 3$ the solution u of the BVP (1.1)-(1.2) can be decomposed as $u = v_1 + w_1$ on $[0, d]$ and $u = v_2 + w_2$ on $[d, 1]$, where the regular components v_1 and v_2 satisfy*

$$Lv_1(x) = f(x), \qquad |v_1^{(k)}(x)| \le C(1 + \varepsilon^{(2-k)}), \quad x \in \Omega^-,$$

$$Lv_2(x) = f(x), \qquad |v_2^{(k)}(x)| \le C(1 + \varepsilon^{(2-k)}), \quad x \in \Omega^+,$$

$$|[v_1](d)| \le C, \quad |[v_1'](d)| \le C, \quad |[v_1''](d)| \le C,$$

while the layer solution components w_1 and w_2 satisfy

$$\begin{aligned} Lw_1(x) &= 0, \qquad |w_1^{(k)}(x)| \le C(\varepsilon|u'(1)| + \varepsilon)\varepsilon^{-k} e^{(-\frac{\beta}{\varepsilon}(d-x))}, \quad x \in \Omega^-, \\ Lw_2(x) &= 0, \qquad |w_2^{(k)}(x)| \le C(\varepsilon|u'(1)| + \varepsilon)\varepsilon^{-k} e^{(-\frac{\beta}{\varepsilon}(1-x))}, \quad x \in \Omega^+. \end{aligned}$$

Proof. Following the procedure adapted in [8], one can prove this theorem.

2. Finite element formulation

The standard weak formulation of (1.1) is: Find $u \in V$ such that

$$B(u,v) := (\varepsilon u', v') + (bu', v) + (cu, v) = f^*(v), \quad \forall v \in V, \tag{2.1}$$

where $V = \{v \in H^1(\Omega) : v(0) = 0\}$ denotes the special Sobolev space and $f^*(v) = (f,v) + \gamma_0 v(1)$ and $(.,.)$ is the inner product on $L_2(\Omega)$. Now we define a norm on V associated with the bilinear form $B(.,.)$, called energy norm as

$$|||u|||_V = [\varepsilon|u|_1^2 + \varrho\|u\|_0^2]^{\frac{1}{2}},$$

where $\|u\|_0 := (u,u)^{\frac{1}{2}}$ is the standard norm on $L_2(\Omega)$, $0 < \varrho \le c(x) - b'(x)/2, x \in \bar{\Omega}$, while $|u|_1 := \|u'\|_0$ is the usual semi-norm on V. It is obvious that B is a bilinear functional defined on $V \times V$. We now prove that it is coercive with respect to $|||.|||_V$, that is $|B(u,u)| \ge |||u|||_V^2$.

Lemma 2.0.3. *A bilinear functional B satisfies the coercive property with respect to $|||.|||_V$.*

Proof. Let $u \in V$. Then

$$\begin{aligned} B(u,u) &= \varepsilon|u|_1^2 + \int_0^1 bu'u\,dx + \int_0^1 cu^2 dx \\ &= \varepsilon|u|_1^2 + [\frac{b(1)}{2}(u(1))^2] - \int_0^1 \frac{b'}{2}u^2 dx + \int_0^1 cu^2 dx \\ &\ge \varepsilon|u|_1 + \varrho \int_0^1 u^2 dx \\ B(u,u) &\ge |||u|||_V^2. \end{aligned}$$

Hence B is coercive with respect to $|||.|||_V$.

Also we observe that B is continuous in the energy norm, that is, $|B(u,v)| \le \beta_0|||u|||_V \cdot |||v|||_V$ for some $\beta_0 > 0$. Further f^* is a bounded linear functional on V and also bounded by $|f^*(v)| \le C(\|v\|_0 + |\gamma_0 v(1)|)$. By Lax-Milgram Theorem, we conclude that the problem (2.1) has a unique stable solution.

2.1.Discretization of weak problem. Let $\bar{\Omega}_\varepsilon^N = \{x_0, x_1, ..., x_N\}$, $N \in \mathbb{N}$ be a given mesh and $h_i = x_i - x_{i-1}, i = 1(1)N$. Let $\bar{h}_i = \frac{h_i + h_{i+1}}{2}, i = 1(1)N-1$ and $\bar{h}_N = \frac{h_N}{2}$. We form the discrete problem as

$$B_h(u,v) := (\varepsilon u', v') + (bu', v) + \sum_{i=1}^{N} \bar{h}_i c_i u_i v_i + \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \delta_k(-\varepsilon u'' + bu' + cu)bv' dx \tag{2.2}$$

and

$$f_h^*(v) := \sum_{k=1, k\neq \frac{N}{2}}^{N} \bar{f}_k v_k + \frac{1}{2} h_{\frac{N}{2}} f_{\frac{N}{2}}^- v_{\frac{N}{2}}^- + \frac{1}{2} h_{\frac{N}{2}+1} f_{\frac{N}{2}}^+ v_{\frac{N}{2}}^+ + \gamma_0 v(1) + \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \delta_k f b v' dx.$$

The parameter δ_k is called the streamline-diffusion parameter and will be determined later. Now the discrete problem of (2.1) is: Find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = f_h^*(v_h), \quad \forall \quad v_h \in V_h, \tag{2.3}$$

where $V_h \subset V$ be the space of piecewise linear functions with the basis $\{\phi_i\}_{i=1}^N$ given by

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h_{i+1}}, & x \in [x_i, x_{i+1}] \\ 0, & x \notin [x_{i-1}, x_{i+1}], \end{cases}$$

for $i = 1(1)N-1$ and

$$\phi_N(x) = \begin{cases} \frac{x - x_{N-1}}{h_N}, & x \in [x_{N-1}, x_N] \\ 0, & x \notin [x_{N-1}, x_N]. \end{cases}$$

Here we define a discrete energy norm on V_h associated with the bilinear form $B_h(.,.)$, as

$$|||u_h|||_{V_h} = [\varepsilon |u_h|_1^2 + \varrho \|u_h\|_0^2 + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \delta_i b^2(x_i)(u_h'(x))^2 dx]^{\frac{1}{2}}.$$

It is obvious that B_h is a bilinear functional defined on $V_h \times V_h$. We can also prove that it is coercive with respect to $|||.|||_{V_h}$, that is $|B_h(u_h, u_h)| \geq \varsigma |||u_h|||_{V_h}^2$, for some $\varsigma > 0$. We see that B_h is continuous, that is, $|B_h(u_h, v_h)| \leq \beta_0' |||u_h|||_{V_h} \cdot |||v_h|||_{V_h}$ for some $\beta_0' > 0$. Further f_h^* is a bounded linear functional on V_h. By Lax-Milgram Theorem, the discrete problem (2.3) has a unique solution and it is also stable. The corresponding difference scheme is

$$L^N u_i := \begin{cases} -\varepsilon(D^+ u_i - D^- u_i) + \alpha_i D^+ u_i + \beta_i D^- u_i + \gamma_i u_i = f_h^*(\phi_i), \; i = 1(1)N-1, \\ \varepsilon(\frac{u_N - u_{N-1}}{h_N}) + \beta_N(\frac{u_N - u_{N-1}}{h_N}) + \gamma_N u_N = f_h^*(\phi_N), \quad i = N, \; u_0 = 0, \end{cases} \tag{2.4}$$

where $u_i = u_h(x_i)$, $D^+u_i = \frac{u_{i+1}-u_i}{h_{i+1}}$, $D^-u_i = \frac{u_i-u_{i-1}}{h_i}$ and for $i = 1(1)N-1$

$$\begin{aligned} \alpha_i &= h_{i+1}\int_{x_i}^{x_{i+1}} (b\phi'_{i+1}\phi_i + \delta_{i+1}b^2\phi'_{i+1}\phi'_i + \delta_{i+1}bc\phi_{i+1}\phi'_i)dx, \\ \beta_i &= -h_i\int_{x_{i-1}}^{x_i} (b\phi'_{i-1}\phi_i + \delta_i b^2\phi'_{i-1}\phi'_i + \delta_i bc\phi_{i-1}\phi'_i)dx, \\ \gamma_i &= \bar{h}_i\widehat{c_i} + \int_{x_{i-1}}^{x_i} \delta_i bc\phi'_i + \int_{x_i}^{x_{i+1}} \delta_{i+1}bc\phi'_i dx. \end{aligned}$$

and

$$\begin{aligned} \beta_N &= -h_N\int_{x_{N-1}}^{x_N} (b\phi'_{N-1}\phi_N + \delta_N b^2\phi'_{N-1}\phi'_N + \delta_N bc\phi_{N-1}\phi'_N)dx, \\ \gamma_N &= \bar{h}_N\widehat{c_N} + \int_{x_{N-1}}^{x_N} \delta_N bc\phi'_N dx. \end{aligned}$$

$$\widehat{c_i} = \frac{\bar{b}^2}{\beta^2}\|c\|_{\infty[x_{i-1},x_i]}.$$

We choose $\widehat{c_i}$ as above, to preserve an "$M-$ matrix" of the corresponing coefficient matrix and $\delta_i = 0$ if the local mesh step size is small enough, otherwise we derive δ_i from the condition $\alpha_{i-1} = 0$. Since δ_i is positive we get,

$$\delta_i = \begin{cases} 0, & h_i \le \frac{2\varepsilon}{\bar{b}}, \\ \left| \int_{x_{i-1}}^{x_i} b\phi'_i\phi_{i-1}\left(\int_{x_{i-1}}^{x_i}(b^2\phi'_i\phi'_{i-1} + bc\phi'_i\phi'_{i-1})\right)^{-1}\right|, & h_i > \frac{2\varepsilon}{\bar{b}}, \end{cases} \tag{2.5}$$

where $\bar{b} = \|b\|_\infty$. We have bounds on δ_i on $\bar{\Omega}$ as follows $\dfrac{h_i\beta}{\bar{b}(2\bar{b}+\bar{c}h_i)} \le \delta_i \le \dfrac{h_i\bar{b}}{2\beta^2}$.

3. General approach for proving error estimates

A general problem of the form $B(u,v) = (f,v) + \gamma_0 v(1), \quad \forall v \in V$ is discretized by: find $u_h \in V_h \subset V$ such that $B_h(u_h, v_h) = f_h^*(v_h), \quad \forall v_h \in V_h$. Since the above discrete problem has a unique solution and some interpolant $u^I \in V_h$ of u is well defined. We define a biorthogonal basis of V_h with respect to B_h to be the set of functions $\{\lambda^j\}_{j=1}^N$ which satisfy

$$B_h(\phi_i, \lambda^j) = \delta_{ij}, \qquad i = 1(1)N, \tag{3.1}$$

where δ_{ij} is the Kronecker symbol. Each function $v_h \in V_h$ can be uniquely represented as,

$$v_h = \sum_{i=1}^N B_h(v_h, \lambda^i)\phi_i.$$

Let us define a linear operator $P : V \longrightarrow V_h$ such that

$$Pv := \sum_{i=1}^N B_h(v, \lambda^i)\phi_i.$$

Obviously, P is a projection since $Pv_h = v_h$, for all $v_h \in V_h$. Further, for a consistent method we have $Pu = u_h$. The error $u - u_h$ can be represented as

$$u - u_h = u - u^I + P(u^I - u) + Pu - u_h \tag{3.2}$$

Let $K = Pu - u_h$, we shall call this as consistency error, since it vanishes in case of consistent FEM.

3.1. Shishkin and Bakhvalov-Shishkin meshes. Let $N > 4$ be a positive even integer and

$$\sigma_1 = \min\{\frac{d}{2}, \frac{\varepsilon}{\beta}\tau_0 \ln N\}, \quad \sigma_2 = \min\{\frac{1-d}{2}, \frac{\varepsilon}{\beta}\tau_0 \ln N\}, \qquad \tau_0 \geq 2.$$

Our mesh will be equidistant on $\bar{\Omega}_S$, where $\Omega_S = (0, d - \sigma_1) \cup (d, 1 - \sigma_2)$ and graded on $\bar{\Omega}_0$ where $\Omega_0 = (d - \sigma_1, d) \cup (1 - \sigma_2, 1)$. First we shall assume $\sigma_1 = \sigma_2 = \tau_0 \varepsilon / \beta \ln N$ as otherwise N^{-1} is exponentially small compared to ε. We choose the transition points to be

$$x_{N/4} = d - \sigma_1, \quad x_{N/2} = d, \quad x_{3N/4} = 1 - \sigma_2.$$

Because of the specific layers, here we have to use two mesh generating functions φ_1 and φ_2 which are both continuous and piecewise continuously differentiable and monotonically decreasing functions. The mesh points are

$$x_i = \begin{cases} \frac{4i}{N}(d - \sigma_1), & i = 0(1)N/4 \\ d - \frac{\tau_0}{\beta}\varepsilon\varphi_1(t_i), & i = N/4 + 1(1)N/2 \\ d + \frac{4}{N}(1 - d - \sigma_2)(i - N/2), & i = N/2 + 1(1)3N/4 \\ 1 - \frac{\tau_0}{\beta}\varepsilon\varphi_2(t_i), & i = 3N/4 + 1(1)N, \end{cases}$$

where $t_i = i/N$. We define mesh-characterizing functions ψ_1 and ψ_2 by

$$\varphi_i = -\ln \psi_i, \quad i = 1, 2,$$

with the following properties

$$\max |\psi'| = \begin{cases} C \ln N & \text{for Shishkin meshes} \\ C & \text{for Bakhvalov-Shishkin meshes.} \end{cases}$$

- Shishkin mesh

$$\psi_1(t) = e^{-2(1-2t)lnN}, \qquad \psi_2(t) = e^{-4(1-t)lnN},$$

- Bakhvalov-Shishkin mesh

$$\psi_1(t) = 1 - 2(1 - N^{-1})(1 - 2t), \quad \psi_2(t) = 1 - 4(1 - N^{-1})(1 - t).$$

The set of interior mesh points is denoted by $\Omega_\varepsilon^N = \bar{\Omega}_\varepsilon^N \setminus \{x_{N/2}\}$. Also, for the both meshes, on the coarse part Ω_S we have

$$h_i \leq CN^{-1}.$$

It is well known that on the layer part of the Shishkin mesh [11]

$$h_i \leq C\varepsilon N^{-1} \ln N$$

and of the Bakhvalov-Shishkin mesh we have

$$h_i \le \begin{cases} \frac{\tau_0}{\beta}\varepsilon N^{-1} \max|\psi_1'| \exp\left(\frac{\beta}{\tau_0\varepsilon}(d-x_{i-1})\right), & i = N/4+1(1)N/2, \\ \frac{\tau_0}{\beta}\varepsilon N^{-1} \max|\psi_2'| \exp\left(\frac{\beta}{\tau_0\varepsilon}(1-x_{i-1})\right), & i = 3N/4+1(1)N \end{cases}$$

and $\frac{h_i}{\varepsilon} \le CN^{-1} \max|\varphi'| \le C$.

3.2. Interpolation error.

Theorem 3.2.1. *On Bakhvalov-Shishkin mesh for $\tau_0 \ge 2$ and $\varepsilon \le CN^{-1}$ it holds*

$$|u(x) - u^I(x)| \le \begin{cases} C(N^{-1}\max|\psi'|)^2, & x \in \Omega_0, \\ CN^{-2}, & x \in \Omega_S, \end{cases}$$

where $\psi = \psi_i$ for $i = 1, 2$.

Proof. Let $x \in \Omega^-$, then $u(x) = v_1(x) + w_1(x)$ on Ω^-. Now,

$$u(x) - u^I(x) = v_1(x) - v_1^I(x) + w_1(x) - w_1^I(x).$$

For the regular part of the interpolation error $v_1(x) - v_1^I(x)$ we can use classical theorem which yields, that is, if $x \in \Omega_S \cap \Omega^-$,

$$|v_1(x) - v_1^I(x)| \le Ch_i^2 \max_{[x_{i-1},x_i]} |v_1''(x)| \le CN^{-2}.$$

For the singular part of the interpolation error $v_1(x) - v_1^I(x)$ that is, if $x \in [x_{i-1}, x_i] \subset \Omega_0 \cap \Omega^-$,

$$\begin{aligned} |v_1(x) - v_1^I(x)| &\le C(\frac{\tau_0}{\beta}\varepsilon)^2 (N^{-1}\max|\psi_1'|)^2 (e^{(\frac{\beta}{\tau_0\varepsilon}(d-x_{i-1}))})^2 \\ &\le C\varepsilon^2 [N^{-1}\max|\varphi_1'|]^2 \\ &\le CN^{-2}. \end{aligned}$$

For the layer part of the interpolation error $w_1(x) - w_1^I(x)$ we can also use the classical estimate on Ω_0. If $x \in [x_{i-1}, x_i] \subset \Omega_0 \cap \Omega^-$ we have

$$\begin{aligned} |w_1(x) - w_1^I(x)| &\le Ch_i^2 \max|w_1''(x)| \\ &\le C\varepsilon(N^{-1}\max|\psi_1'|)^2 e^{\frac{2\beta}{\tau_0\varepsilon}(d-x_{i-1})} \max_{x\in[x_{i-1},x_i]} |e^{(-\frac{\beta}{\varepsilon}(d-x))}| \\ &\le C\varepsilon N^{-2}(\max|\psi_1'|)^2. \end{aligned}$$

If $x \in [x_{i-1}, x_i] \subset \Omega_S \cap \Omega^-$, then

$$\begin{aligned} |w_1(x) - w_1^I(x)| &\le 2\|w_1(x)\|_\infty \\ &\le C\varepsilon \max_{x\in[x_{i-1},x_i]} e^{(-\frac{\beta}{\varepsilon}(d-x))} \\ &\le C\varepsilon N^{1-\tau_0}. \end{aligned}$$

Similarly we can prove the result on $x \in \Omega^+$.

Corollary 3.2.2. *On Shishkin meshes for $\tau_0 \geq 2$, the interpolation error is bounded as follows*

$$|u(x) - u^I(x)| \leq \begin{cases} CN^{-2}\ln^2 N, & x \in \Omega_0, \\ CN^{-2}, & x \in \Omega_S. \end{cases}$$

Proof. Using the previous theorem and the values for $\max|\psi'|$, one can prove the present corollary.

4. Pointwise error

The difference scheme (2.4) can be rewritten as

$$\begin{cases} -\frac{\varepsilon}{h_i}(p_{i+1}\frac{u_{i+1}-u_i}{h_{i+1}} - p_i\frac{u_i-u_{i-1}}{h_i}) + r_i\frac{u_i-u_{i-1}}{h_i} + q_i u_i = f_h^*(\phi_i), & i = 1(1)N-1, \\ \varepsilon(\frac{u_N-u_{N-1}}{h_N}) + \beta_N\frac{u_N-u_{N-1}}{h_N} + \gamma_N u_N = f_h^*(\phi_N), & i = N, \end{cases}$$

where $p_i = 1 - \frac{\alpha_{i-1}}{\varepsilon}$, $q_i = \frac{\gamma_i}{h_i}$, $r_i = \frac{\alpha_{i-1}+\beta_i}{h_i}, i = 1(1)N-1$. For further analysis of (3.2), we shall need additional assumptions for the mesh.

Lemma 4.0.3. *If $\varepsilon < d\|b\|_\infty N^{-1}$ and $\tau_0 N^{-1}\max|\varphi'| \leq 2(1-p)\beta/\|b\|_\infty$ for some $0 < p < 1$, then $p_i \geq p > 0$, $r_i \geq \frac{\beta}{2} > 0$, $q_i \geq \frac{\bar{b}^2}{2\beta^2}\|c\|_\infty > 0$, $i = 1(1)N$.*

Proof. Following the procedure adapted in [11], we can prove this Lemma.

From the above Lemma, One can prove that [12]

$$\|\lambda^j\|_\infty \leq C, \quad \|(\lambda^j)'\|_1 \leq C, \tag{4.1}$$

where λ^j is defined as the solution of the problem (3.1).

4.1. Projection error. From the error representation (3.2), the projection error at the mesh points is

$$P(u^I - u)(x_i) = B_h(u^I - u, \lambda^i), \quad x_i \in \bar{\Omega}_\varepsilon^N.$$

But the definition (2.2) of the bilinear form B_h and the properties of the interpolant u^I, we have

$$B_h(u^I - u, \lambda^i) = \begin{cases} \varepsilon\int_0^1 (u^I-u)'(\lambda^i)'dx + \int_0^1 b(u^I-u)'\lambda^i dx + \sum_{i=1}^N \bar{h}_i c_i(u^I-u)_i\lambda^i \\ +\sum_{k=1}^N \int_{x_{k-1}}^{x_k} \delta_k\varepsilon u'' b(\lambda^i)'dx + \sum_{k=1}^N \int_{x_{k-1}}^{x_k} \delta_k b^2(u^I-u)'(\lambda^i)'dx \\ +\sum_{k=1}^N \int_{x_{k-1}}^{x_k} \delta_k bc(u^I-u)(\lambda^i)'dx. \end{cases} \tag{4.2}$$

First, we have

$$|\varepsilon\int_0^1 (u^I-u)'(\lambda^i)'dx| = 0. \tag{4.3}$$

Integration by parts, smoothness of the function b and Corollary 3.2.2, (4.1) are used to obtain the second part

$$
\begin{aligned}
|\int_0^1 b(u^I-u)'\lambda^i dx| &= |-\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}(u^I-u)(b'\lambda^i+b(\lambda^i)')dx| \\
&\le C\|u-u^I\|_\infty(\|\lambda^i\|_\infty+\|(\lambda^i)'\|_1) \\
&\le C\|u-u^I\|_\infty
\end{aligned}
$$

and also the third part will be

$$
|\sum_{i=1}^{N}\bar{h}_i c_i (u^I-u)_i\lambda^i| \le CN^{-1}\|u^I-u\|_\infty\|\lambda^i\|_\infty. \tag{4.4}
$$

For the analysis of the fourth part of $B_h(u^I-u,\lambda^i)$ we shall use the decomposition from the Theorem 1.0.2 and $\delta_i \le CN^{-1}$. Then the smooth part of the solution, we have

$$
|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\delta_k\varepsilon v'' b(\lambda^i)'dx \,|\le C\varepsilon N^{-1}\|v''\|_\infty\|(\lambda^i)'\|_1 \le C\varepsilon N^{-1}.
$$

Again consider

$$
|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\delta_k\varepsilon w'' b(\lambda^i)'dx| \le \begin{cases} C\varepsilon N^{-1}(\sum_{k=1}^{N/4}\int_{x_{k-1}}^{x_k}|w_1''||(\lambda^i)'|dx \\ +\sum_{k=N/2+1}^{3N/4}\int_{x_{k-1}}^{x_k}|w_2''||(\lambda^i)'|dx). \end{cases} \tag{4.5}
$$

Since the mesh step h_i can be bounded below by $h_i \ge \max\{d, 1-d\}N^{-1}$ on Ω_S, then for the function $\lambda^i \in V_h$,

$$
\begin{aligned}
|(\lambda^i)'(x)| &= \frac{1}{h_k}|\lambda^i(x_k)-\lambda^i(x_{k-1})| \\
&\le CN\|\lambda^i\|_\infty \\
&\le CN, \forall x \in [x_{k-1},x_k] \subset \bar{\Omega}_S.
\end{aligned}
$$

From the equation (4.5), we have

$$
\begin{aligned}
&|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\delta_k\varepsilon w'' b(\lambda^i)'dx| \\
\le\ & C\varepsilon(\sum_{k=1}^{N/4}\int_{x_{k-1}}^{x_k}|w_1''|dx + \sum_{k=N/2+1}^{3N/4}\int_{x_{k-1}}^{x_k}|w_2''|dx) \\
\le\ & C\varepsilon(\sum_{k=1}^{N/4}\int_{x_{k-1}}^{x_k}\varepsilon^{-1}e^{(-\frac{\beta_1}{\varepsilon}(d-x))}dx + \sum_{k=N/2+1}^{3N/4}\int_{x_{k-1}}^{x_k}\varepsilon^{-1}e^{(-\frac{\beta}{\varepsilon}(1-x))}dx) \\
\le\ & C(\sum_{k=1}^{N/4}\int_{x_{k-1}}^{x_k}e^{(-\frac{\beta}{\varepsilon}(d-x))}dx + \sum_{k=N/2+1}^{3N/4}\int_{x_{k-1}}^{x_k}e^{(-\frac{\beta}{\varepsilon}(1-x))}dx).
\end{aligned}
$$

Since

$$e^{-\frac{\beta}{\varepsilon}(d-x)} \leq \begin{cases} C, & x \in \Omega^- \cap \Omega_0 \\ CN^{-\tau_0}, & x \in \Omega^- \cap \Omega_S, \end{cases} \qquad e^{-\frac{\beta}{\varepsilon}(1-x)} \leq \begin{cases} C, & x \in \Omega^+ \cap \Omega_0 \\ CN^{-\tau_0}, & x \in \Omega^+ \cap \Omega_S. \end{cases} \tag{4.6}$$

then $|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} \delta_k \varepsilon w'' b(\lambda^i)' dx| \leq C\varepsilon N^{1-\tau_0}$. Now

$$\begin{aligned} |\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} \delta_k b^2 (u^I - u)'(\lambda^i)' dx| &= |-\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} \delta_k (u^I - u)(b^2(\lambda^i)')' dx| \\ &\leq CN^{-1}\|u^I - u\|_\infty, \end{aligned}$$

and also

$$\begin{aligned} |\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} \delta_k bc(u^I - u)(\lambda^i)' dx| &\leq CN^{-1}(|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} (u^I - u)(\lambda^i)' dx|) \\ &\leq CN^{-1}\|u^I - u\|_\infty. \end{aligned}$$

Using the above results and $\varepsilon \leq CN^{-1}$, $\tau_0 \geq 2$ in (4.2) we have

$$\begin{aligned} |B_h(u^I - u, \lambda^i)| &\leq C(\|u^I - u\| + N^{-1}\|u^I - u\|_\infty + \varepsilon N^{-1} + \varepsilon N^{1-\tau_0}) \\ &\leq CN^{-2}\max|\psi'|^2. \end{aligned}$$

Therefore the projection error is

$$|B_h(u^I - u, \lambda^i)| \leq CN^{-2}\max|\psi'|^2. \tag{4.7}$$

4.2. Consistency error. Now it remains to estimate the consistency error in the representation (3.2). Since $K = Pu - u_h$ can be written as

$$K = \sum_{i=1}^{N}(B_h - B)(u, \lambda^i)\phi_i + \sum_{i=1}^{N}(f^* - f_h^*)(\lambda^i)\phi_i.$$

For a fixed point $x_i \in \bar{\Omega}_\varepsilon^N$, we have

$$\begin{aligned} K(x_i) &= (B_h - B)(u, \lambda^i) + (f^* - f_h^*)(\lambda^i) \\ &= \sum_{k=1}^{N}\bar{h}_k c_k u_k \lambda_k^i - \int_0^1 cu\lambda^i dx + \int_0^1 f\lambda^i dx - \sum_{k=1}^{N} f_{h,k}\lambda_k^i. \end{aligned}$$

Let $K^*(x_i) = \sum_{k=1}^{N}\bar{h}_k c_k u_k \lambda_k^i - \int_0^1 (cu)^I \lambda^i dx + \int_0^1 f^I \lambda^i dx - \sum_{k=1}^{N} f_{h,k}\lambda_k^i$. Then we can write

$$K(x_i) = K^*(x_i) + (\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} ((cu)^I - (cu))\lambda^i dx) - (\int_0^1 (f^I - f)\lambda^i dx). \tag{4.8}$$

Then

$$|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}((cu)^I-(cu))\lambda^i dx| \leq C(\|u-u^I\|_\infty+N^{-2}\|u\|_\infty)\|\lambda^i\|_1$$
$$\leq CN^{-2}\max|\psi'|^2,$$

and

$$|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}(f^I-f)\lambda^i dx| \leq C(\|f-f^I\|_\infty+N^{-2}\|f\|_\infty)\|\lambda^i\|_1$$
$$\leq CN^{-2}.$$

It is enough to estimate

$$K^*(x_i)=\sum_{k=1}^{N}\bar{h}_k c_k u_k\lambda_k^i-\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}(cu)^I\lambda^i dx-(\sum_{k=1}^{N}\bar{h}_k f_k\lambda_k^i-\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}f^I\lambda^i dx)$$
$$=\langle(cu)^I,\lambda^i\rangle_h-\langle f^I,\lambda^i\rangle_h,$$

where $\langle g,\lambda^i\rangle_h=\sum_{k=1}^{N}g_{h,k}\lambda_k^i-\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}g\lambda^i dx$. Direct computation by Simpson rule in the above integral yields

$$\langle g,\lambda^i\rangle_h=\frac{1}{6}\sum_{k=1}^{N-1}(h_k(g_k^- - g_{k-1}^+)-h_{k+1}(g_{k+1}^- - g_k^+))\lambda_k^i.$$

In order to estimate the above term, we need to use the decomposition of the solution $u=v+w$. Then we have

$$K^*(x_i)=\langle(cv)^I,\lambda^i\rangle_h+\langle(cw)^I,\lambda^i\rangle_h-\langle f^I,\lambda^i\rangle_h. \tag{4.9}$$

Now, the first term of the above equation will be

$$|\langle(cv)^I,\lambda^i\rangle_h| \leq C\|c\|\sum_{k=1}^{N-1}|h_k(v_k^- - v_{k-1}^+)-h_{k+1}(v_{k+1}^- - v_k^+)|\lambda_k^i$$
$$\leq C\sum_{k=1}^{N-1}|h_k^2 v'(\xi_k)-h_{k+1}^2 v'(\xi_{k+1})|\lambda_k^i, \quad \xi_k\in[x_{k-1},x_k]$$
$$\leq CN^{-2}\|v''\|_\infty\sum_{k=1}^{N-1}(\xi_{k+1}-\xi_k)\lambda_k^i$$
$$\leq CN^{-2}\|\lambda^i\|_\infty,$$

by using Theorem 1.0.2, $h_i\leq CN^{-1}, i=1(1)N-1$ and $\|\lambda^i\|_\infty\leq C$. Finally, we get

$$|<(cv)^I,\lambda^i>_h|\leq CN^{-2}. \tag{4.10}$$

Secondly, we evaluate the third term of the equation (4.9)

$$|\langle f^I,\lambda^i\rangle_h|\leq CN^{-2}\|f''\|_\infty\|\lambda^i\|_\infty,$$

since $h_i \leq CN^{-1}$, $\|\lambda^i\|_\infty \leq C$ and $\|f''\|_\infty \leq C$. Finally, we get

$$|\langle f^I, \lambda^i\rangle_h| \leq CN^{-2}. \tag{4.11}$$

To estimate the second term, we separately analyze on smooth and layer region. Let us denote the coefficient in $< (cw)^I, \lambda^i >_h$ corresponding to λ_k^i by m_k. Depending on the values of index k, we consider different cases. In general, $g_k^{\pm}$ denotes right-limit and left-limit of a function g at a mesh point x_k.

Case 1: When $1 \leq k \leq \frac{N}{4} - 1$ or $\frac{N}{2} + 1 \leq k \leq \frac{3N}{4} - 1$. That is, $[x_{k-1}, x_{k+1}] \subset \Omega_S$. The coefficient m_k can be estimated by

$$\begin{aligned} |m_k| &= |h_k(c_k^- w_k^- - c_{k-1}^+ w_{k-1}^+) - h_{k+1}(c_{k+1}^- w_{k+1}^- - c_k^+ w_k^+)| \\ &\leq C\bar{h}_k \|w\|_{L_\infty[x_{k-1}, x_{k+1}]} \\ &\leq C\bar{h}_k [\max_{x\in\Omega^-\cap\Omega_S} |e^{-\frac{\beta}{\varepsilon}(d-x)}| + \max_{x\in\Omega^+\cap\Omega_S} |e^{-\frac{\beta}{\varepsilon}(1-x)}|], \text{ from 1.0.2 and (4.6)}. \end{aligned}$$

$$| \, m_k \, | \leq C\bar{h}_k N^{-\tau_0}. \tag{4.12}$$

Case 2: When $\frac{N}{4} + 1 \leq k \leq \frac{N}{2} - 1$ or $\frac{3N}{4} + 1 \leq k \leq N$. That is, the subinterval $[x_{k-1}, x_{k+1}] \subset \Omega_0$. The layer part will be calculated by estimating m_k. We have

$$\begin{aligned} m_k &= h_k(c_k^- w_k^- - c_{k-1}^+ w_{k-1}^+) - h_{k+1}(c_{k+1}^- w_{k+1}^- - c_k^+ w_k^+) \\ &= c_k(h_k(-w_{k+1} + 2w_k - w_{k-1}) + (h_k - h_{k+1})(w_{k+1} - w_k)) + h_k(c_k - c_{k-1}) \\ &\quad (w_{k-1} - w_k) + h_{k+1}(c_{k+1} - c_k)(w_k - w_{k+1}) + w_k(-h_{k+1} \\ &\quad c_{k+1} + (h_k + h_{k+1})c_k - h_k c_{k-1}). \end{aligned}$$

Using the Taylor's expansion for each of the terms in the previous expression yields

$$\begin{aligned} h_k c_k(-w_{k+1} + 2w_k - w_{k-1}) &= h_k(h_k - h_{k+1})c_k w_k' - \frac{h_k^3}{2} c_k w''(\theta_k) \\ &\quad - \frac{h_k h_{k+1}^2}{2} c_k w''(\theta_{k+1}), \\ (h_k - h_{k+1})c_k(w_{k+1} - w_k) &= h_{k+1}(h_k - h_{k+1})c_k w'(\xi_{k+1}), \\ h_k(c_k - c_{k-1})(w_{k-1} - w_k) &= -h_k^3 c'(\rho_k) w'(\xi_k), \\ h_{k+1}(c_{k+1} - c_k)(w_k - w_{k+1}) &= -h_{k+1}^3 c'(\rho_{k+1}) w'(\xi_{k+1}), \\ w_k(-h_{k+1}c_{k+1} + (h_k + h_{k+1})c_k - h_k c_{k-1}) &= (h_k^2 - h_{k+1}^2)c_k' w_k - \frac{1}{2}(h_k^3 c''(\eta_k) \\ &\quad + h_{k+1}^3 c''(\eta_{k+1}))w_k, \end{aligned}$$

where $\theta_k, \xi_k, \rho_k, \eta_k \in [x_{k-1}, x_k]$.

To derive an estimate for $|m_k|$, we need the following lemma.

Lemma 4.2.1. *For the points $x_{k-1}, x_k, x_{k+1} \in \Omega_0$, $x_k \neq d = x_{\frac{N}{2}}$ of the mesh with $\tau_0 \geq 2$ the following holds*

$$\begin{aligned} |\,(h_k - h_{k+1})(w_{k+1} - w_k)\,| &\leq Ch_{k+1}N^{-2}, \\ |\,(h_k - h_{k+1})w_k^{'}\,| &\leq CN^{-2}. \end{aligned}$$

Proof. Let $x_{k-1},\ x_k,\ x_{k+1} \in \bar{\Omega}_0 \cap \Omega^-$ and $x_k \neq d = x_{\frac{N}{2}}$

$$|\ h_k - h_{k+1}\ | = \tau_0 \frac{\varepsilon}{\beta} N^{-1}\ |\ \phi^{'}(\rho_k) - \phi^{'}(\rho_{k+1})\ |$$

for $\rho_k,\ \rho_{k+1} \in (t_{k-1}, t_{k+1})$. Also $|w_{k+1} - w_k| \leq h_{k+1}|w_1^{'}(\alpha_{k+1})|$, $\alpha_{k+1} \in (x_k, x_{k+1})$

$$\begin{aligned} |\,(h_k - h_{k+1})(w_{k+1} - w_k)\,| &\leq C\varepsilon h_{k+1}N^{-2}\ |\ \phi_1^{''}(\psi_k)\ ||\ w_1^{'}(\alpha_{k+1})\ | \\ &\leq Ch_{k+1}N^{-2}(\psi_1(t_{k+1}))^{-2} e^{-\frac{\beta}{\varepsilon}(d-\alpha_{k+1})}. \end{aligned}$$

Using the fact that $\max|\psi_1^{'}| = C$ and $e^{-\frac{\beta}{\varepsilon}(d-\alpha_{k+1})} \leq \psi_1(t_k)^2 + N^{-\tau_0}$, we have

$$|\ (h_k - h_{k+1})(w_{k+1} - w_k)\ | \leq Ch_{k+1}N^{-2},$$

since $\tau_0 \geq 2$. When $[x_{k-1}, x_{k+1}] \subset [d, d + \sigma_2]$, the above estimate is also true for these intervals. From the previous analysis, we get

$$\begin{aligned} h_k c_k(-w_{k+1} + 2w_k - w_{k-1}) &\leq Ch_kN^{-2} + Ch_kN^{-2}\max|\ \psi_1^{'}\ |, \\ (h_k - h_{k+1})c_k(w_{k+1} - w_k) &\leq Ch_{k+1}N^{-2}. \end{aligned}$$

Applying the above Lemma 4.2.1 to each of the terms in m_k of Case 2, we have

$$|m_k| \leq C\bar{h}_kN^{-2}\max|\psi'|^2. \tag{4.13}$$

Now it remains to prove the estimates at the transition points.

Case 3: When $x_k, k \in \{\frac{N}{4}, \frac{3N}{4}\}$ and $i \neq \frac{N}{2}$. At these points $w_k, w_{k\pm1}$ are bounded by $CN^{-\tau_0}$. Then, using the expression for $|m_k|$ given in Case 2,

$$|\ m_k\ | \leq C\bar{h}_kN^{-\tau_0}. \tag{4.14}$$

Case 4: When $k = \frac{N}{2}$. That is, $x_k = d$

$$\begin{aligned} m_k &= h_k(c_k^- w_k^- - c_{k-1}^+ w_{k-1}^+) - h_{k+1}(c_{k+1}^- w_{k+1}^- - c_k^+ w_k^+) \\ |m_k| &\leq h_k|(c_k^+ - c_{k+1})w_k^+ + (c_k^- - c_{k-1})w_k^-| + h_k|c_{k+1}(w_k^+ - w_{k+1}) \\ &\quad + a_{k-1}(w_k^- - w_{k-1})| \\ &\leq Ch_kh_{k+1}|w_k^+| + Ch_k^2|w_k^-| + Ch_k(h_k(c_{k-1} - c_k^-)\bar{w}_k' - \frac{1}{2}h_{k+1}^2 c_k^- \bar{w}''(\vartheta_k) \\ &\quad + \frac{1}{2}h_k^2 c_{k-1}\bar{w}''(\vartheta_i) + R), \quad \vartheta_i \in [x_{i-1}, x_i]. \end{aligned}$$

We use the asymptotic expansion of the layer components $w = \bar{w} + R$, that can be derived using the technique from [14]. It can be concluded that the leading part $\bar{w}'$ of w' is continuous at $x = d$, enabling us to use Taylor's expansions for estimating $w_k^+ - w_{k+1}$ and $w_k^- - w_{k-1}$. Since R contains lower order terms, we have

$$|\ m_k\ | \leq C\bar{h}_k\varepsilon N^{-1} + C\bar{h}_k\varepsilon N^{-2}\max|\psi'|^2 + C\bar{h}_kN^{-2}\max|\psi'|^2, \tag{4.15}$$

and we use the estimate of $\max|\psi'|$ in the above result to obtain

$$| m_k | \leq \begin{cases} C\bar{h}_k(\varepsilon + N^{-1})N^{-1}\ln^2 N, & \text{for Shishkin mesh,} \\ C\bar{h}_k(\varepsilon + N^{-1})N^{-1}, & \text{for Bakhalov-Shishkin mesh.} \end{cases}$$

Collecting estimates (4.12)–(4.15) from the previously analyzed cases and using $\varepsilon \leq CN^{-1}$, we have

$$\begin{aligned} |< (cw)^I, \lambda^i >_h| &\leq \frac{1}{6}\sum_{k=1}^{N-1} |m_k|\lambda_k^i \\ &\leq C(N^{-\tau_0} + N^{-2}\max|\psi'|)\sum_{k=1}^{N-1} \bar{h}_k\lambda_k^i \\ &\leq CN^{-2}\max|\psi'|\|\lambda^i\|_{L^1(\Omega)} \\ &\leq CN^{-2}\max|\psi'|, \end{aligned}$$

since $\tau_0 \geq 2$ and $\|\lambda^i\|_{L^1(\Omega)} \leq C$. From equations (4.9)–(4.10) and the above estimates, we have $|K^*(x_i)| \leq C\varepsilon N^{-1} + N^{-2}\max|\psi^{'}|^2$. Then from (4.8), the consistency error will be

$$|K(x_i)| \leq \varepsilon N^{-1} + N^{-2}\max|\psi^{'}|^2, \tag{4.16}$$

and if $\varepsilon \leq CN^{-1}$ and from equations (3.2), (4.7) and (4.16), we get

$$|u(x_i) - u_h(x_i)| \leq CN^{-2}\max|\psi^{'}|^2. \tag{4.17}$$

On the whole domain $\bar{\Omega}$ we have

$$\|u - u_h\| = \max_{x\in\bar{\Omega}}|u(x) - u_h(x)| \leq \|u - u^I\|_\infty + \max_{0\leq i\leq N}|u_h(x_i) - u(x_i)|. \tag{4.18}$$

Theorem 4.2.2. *Let u and u_h be the solutions of the BVP (1.1)-(1.2) and (2.3) respectively and the maximum norm of the error satisfies with $\tau_0 \geq 2$*

$$\|u - u_h\| \leq \begin{cases} CN^{-2}\ln^2 N, & \textit{for Shishkin mesh} \\ CN^{-2}, & \textit{for Bakhvalov-Shishkin mesh.} \end{cases}$$

Proof. The result follows from (4.18), Corollary (3.2.2) and (4.17).

Remark 4.2.3. In case of a convection-diffusion problem (1.1)–(1.2) with a point source $\bar{\delta}(x - d)$ instead of discontinuous source term f, we can prove the same result as in Theorem 4.2.2 with the condition $-\varepsilon[u^{'}](d) = 1, \quad d \in \Omega$.

5.Scaled first derivative estimate

For $x \in [x_{i-1}, x_i]$, from (3.2), the first derivative of the error representation can be derived as

$$(u - u_h)^{'}(x) = (u - u^I)^{'}(x) + \frac{1}{h_i}B_h(u^I - u, \lambda^i - \lambda^{i-1}). \tag{5.1}$$

Using the same procedure adapted in the proof of Theorem 3.2.1, the first derivative of the interpolation error can be obtained as

$$\|(u-u^I)'\|_\infty \le CN^{-1}\varepsilon^{-1}\max|\psi'|.$$

The second term of (5.1) can be estimated as

$$|\frac{1}{h_i}B_h(u^I-u,\lambda^i-\lambda^{i-1})| \le \frac{1}{h_i}(C\|u^I-u\|_\infty+\varepsilon N^{-1}+\varepsilon N^{1-\tau_0}).$$

If additionally the mesh generating function φ_i have the property $\min|\varphi_i'| \ge C$, on $x_i \in \Omega_0$, then $h_i \ge C\varepsilon N^{-1}$ and if $x_i \in \Omega_S$, we have $h_i \ge CN^{-1}$. Therefore the above term will be

$$|\frac{1}{h_i}B_h(u^I-u,\lambda^i-\lambda^{i-1})| \le \begin{cases} C\varepsilon^{-1}N^{-1}+C\varepsilon^{-1}N\|u^I-u\|_\infty, & x\in\Omega_0 \\ CN^{-1}+CN\|u^I-u\|_\infty, & x\in\Omega_S. \end{cases}$$

Corollary 5.0.4. *The first derivative estimate of the error between the exact and finite element solution of the BVP (1.1)-(1.2), for $\tau_0 \ge 2$ is given by*

$$\varepsilon\|(u-u_h)'\| \le \begin{cases} CN^{-1}\ln N, & \textit{Shishkin mesh} \\ CN^{-1}, & \textit{Bakhvalov-Shishkin mesh} \end{cases}$$

5.1. Superconvergence results. Using the technique from [9], we prove the first order derivative estimates of superconvergence results for the midpoints of the meshes. First, we derive the following result

Lemma 5.1.1. *Let the assumptions of Lemma 4.0.3 hold true. Then we have, for $\tau_0 \ge 2$*

$$|\varepsilon(u'-u_h',v_h')| \le CN^{-2}(\max|\psi'|)^2\|v_h'\|_1, \quad \forall v_h \in V_h.$$

Proof. Using the definition of B and B_h, we have

$$\varepsilon(u'-u_h',v_h') = \begin{cases} ((u_h-u)bv_h)(1)+(u-u_h,b'v_h)+(u-u_h,bv_h') \\ +\sum_{k=1}^N\int_{x_{k-1}}^{x_k}\varepsilon\delta_k u''bv_h'dx+\sum_{k=1}^N\int_{x_{k-1}}^{x_k}\delta_k(u_h-u)'b^2v_h'dx \\ +\sum_{k=1}^N\int_{x_{k-1}}^{x_k}\delta_k(u_h-u)bcv_h'dx. \end{cases} \tag{5.2}$$

To estimate the above equation, we analyze seperately each term of the RHS. Since b is sufficiently smooth function, we get

$$|(u_h-u)(1)b(1)v_h(1)| \le C\|u-u_h\|_\infty\|v_h'\|_1.$$

and we have same bounds for $|(u-u_h,b'v_h)|$ and $|(u-u_h,bv_h')|$. Following the procedure adapted in the proof of the Lemma 7.2 of [11], we can prove the following results

$$
\begin{aligned}
|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} \varepsilon\delta_k u^{''} b v_h^{'} dx| &\leq C\varepsilon(N^{-1}+N^{-\tau_0})\|v_h^{'}\|_1,\\
|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} \delta_k (u_h-u)^{'} b^2 v_h^{'} dx| &\leq C[(N^{-1}\max|\psi^{'}|)^2\|v_h^{'}\|_1\\
&\quad +N^{-1}\|u_h-u\|_\infty\|v_h^{'}\|_1],\\
|\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} \delta_k (u_h-u) bc v_h^{'} dx| &\leq CN^{-1}\|u_h-u\|_\infty\|v_h^{'}\|_1.
\end{aligned}
$$

By substituting all the above results in (5.2), finally we have,

$$|\varepsilon(u^{'}-u_h^{'},v_h^{'})| \leq CN^{-2}(\max|\psi^{'}|)^2\|v_h^{'}\|_1.$$

Using the above Lemma, we can prove the following result [9]:
On shishkin meshes, $\tau_0 \geq 2$

$$\varepsilon\|(u^I-u_h)^{'}\|_\infty \leq CN^{-2}(\max|\psi^{'}|)^2.$$

Theorem 5.1.2. *On Shishkin mesh and $\tau_0 \geq 2$, the midpoints are defined as $x_{i-\frac{1}{2}} := \frac{x_{i-1}+x_i}{2}$ and at these points the error between scaled derivative of the finite element approximation and its exact solution is*

$$\varepsilon|(u-u_h)^{'}(x_{i-\frac{1}{2}})| \leq CN^{-2}(\max|\psi^{'}|)^2, \quad x_{i-\frac{1}{2}} \in \bar{\Omega}_\varepsilon^N.$$

Proof. Since the superconvergent points $x_{i-\frac{1}{2}}$ of $(u-u_h)^{'}$ are same as that of $(u-u^I)^{'}$, therefore we shall analyze $(u-u^I)^{'}$ instead of $(u-u_h)^{'}$. We follow the same procedure adapted in [11] and arrive the required result.

6. Numerical experiments

In this section we experimentally verify our theoretical results proved in the previous section.

Example 6.0.3. *Consider the BVP*

$$-\varepsilon u''(x)+(1+x)u'(x)+x^2u(x) = 2x+\bar{\delta}_{0.5}, \quad x \in \Omega^- \cup \Omega^+, \tag{6.1}$$

$$u(0)=0, \quad \varepsilon u(1)=0.5, \tag{6.2}$$

For our tests, we take $\varepsilon = 2^{-18}$, which is sufficiently small to bring out the singularly perturbed nature of the problem. We measure the accuracy in the maximum norm and the rates of convergence r^N and $r^{1,N}$ are computed using the following formula:

$$r^N = log_2(\frac{E^N}{E^{2N}}), \quad r^{1,N} = log_2(\frac{E^{1,N}}{E^{1,2N}}),$$

where

$$E^N = \{ \max_{x_i \in \bar{\Omega}_\varepsilon^N} |(u_h)^N(x_i) - (u_h^I)^{4096}(x_i)| \},$$
$$E^{1,N} = \{ \max_{x_{i-\frac{1}{2}} \in \bar{\Omega}_\varepsilon^N} |(u_h^{'})^N(x_i) - ((u_h^I)^{'})^{4096}(x_{i-\frac{1}{2}})| \}$$

and u_h^I denotes the piecewise linear interpolant of u_h. In Table 1, we present values of $E^N, E^{1,N}, r^N, r^{1,N}$ for the solution of the BVP (6.1)-(6.2) for Shishkin and Bakhavlov-Shishkin meshes respectively.

From the table it is obvious that the method presented in this paper works better than the standard upwind difference scheme on Shishkin mesh. Some extent the numerical results support the theoretical results.

TABLE 1. Values of $E^N, E^{1,N}$ and $r^N, r^{1,N}$ for the solution of the BVP (6.1)–(6.2).

N	Shishkin mesh				Bakhavlov-Shishkin mesh			
	E^N	r^N	$E^{1,N}$	$r^{1,N}$	E^N	r^N	$E^{1,N}$	$r^{1,N}$
32	3.0660e-2	1.1254	3.1324e-4	1.6256	2.8931e-2	1.0403	2.8838e-5	1.9302
64	1.4054e-2	1.0358	1.0151e-4	1.9232	1.4067e-2	1.0368	7.5671e-6	1.9804
128	6.8548e-3	1.0537	2.6765e-5	2.1231	6.8563e-3	1.0540	1.9176e-6	2.0170
256	3.3021e-3	1.1028	6.1439e-6	2.4266	3.3022e-3	1.1027	4.7380e-7	2.0626
512	1.5375e-3	1.2240	1.1428e-6	1.1600	1.5376e-3	1.2241	1.1342e-7	2.1493
1024	6.5821e-4	-	5.1142e-7	-	6.5821e-4	-	2.5568e-8	-

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