J. Appl. Math. & Informatics Vol. **28**(2010), No. 1 - 2, pp. 13 - 30
Website: http://www.kcam.biz

A GENERAL ITERATIVE ALGORITHM FOR A FINITE FAMILY OF NONEXPANSIVE MAPPINGS IN A HILBERT SPACE

SORNSAK THIANWAN

ABSTRACT. Let C be a nonempty closed convex subset of a real Hilbert space H. Consider the following iterative algorithm given by $x_0 \in C$ arbitrarily chosen,

$$x_{n+1} = \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n P_C(I - s_n B)x_n, \forall n \geq 0,$$

where $\gamma > 0$, $B : C \to H$ is a β-inverse-strongly monotone mapping, f is a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$, P_C is a projection of H onto C, A is a strongly positive linear bounded operator on H and W_n is the W-mapping generated by a finite family of nonexpansive mappings $T_1, T_2, \dots T_N$ and $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \dots, \{\lambda_{n,N}\}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. We prove that the sequence $\{x_n\}$ generated by the above iterative algorithm converges strongly to a common fixed point $q \in F := \cap_{i=1}^{N} F(T_i) \cap VI(C, B)$ which solves the variational inequality $\langle (\gamma f - A)q, p - q \rangle \leq 0$ for all $p \in F$. Using this result, we consider the problem of finding a common fixed point of a finite family of nonexpansive mappings and a strictly pseudocontractive mapping and the problem of finding a common element of the set of common fixed points of a finite family of nonexpansive mappings and the set of zeros of an inverse-strongly monotone mapping. The results obtained in this paper extend and improve the several recent results in this area.

AMS Mathematics Subject Classification : 47H10, 47H09, 46B20.
Key word and phrases : Nonexpansive mapping, inverse-strongly monotone mapping, variational inequality, strong convergence, common fixed points.

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and let $B : C \to H$ be a nonlinear map. Let P_C be the projection of H onto

Received June 14, 2009. Received June 22, 2009. Accepted July 16, 2009.

the convex subset C. The classical variational inequality problem, denoted by $VI(C,B)$ is to find $u \in C$ such that

$$\langle Bu, v-u\rangle \geq 0, \tag{1}$$

for all $v \in C$. For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle z-u, u-v\rangle \geq 0, \forall v \in C, \tag{2}$$

if and only if $u = P_C z$. It is known that projection operator P_C is nonexpansive. It is also known that P_C satisfies $\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$, for $x, y \in H$. Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $y \in C$. One can see that the variational inequality problem (1) is equivalent to some fixed point problem. The element $u \in C$ is a solution of the variational inequality (1) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Bu)$, where $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Recall that a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x-y\|$ for all $x, y \in C$. We denote by $F(T) = \{x \in H : Tx = x\}$ the set of fixed points of T. A linear bounded operator A is strongly positive if there is a constant $\bar{\gamma} > 0$ with the property $\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2$ for all $x \in H$. A mapping $f : H \to H$ is said to be a contraction if there exists a coefficient $\alpha (0 < \alpha < 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x-y\|$, for all $x, y \in H$. A mapping $B : C \to H$ is called β-inverse-strongly monotone, if each $x, y \in C$, we have $\langle Bx - By, x-y\rangle \geq \beta\|x-y\|^2$, for a constant $\beta > 0$. It is easy to see that if B is a β-inverse-strongly monotone of C to H, then it is $\frac{1}{\beta}$-Lipschitz continuous.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $u \in Tx$ and $v \in Ty$ imply $\langle x-y, u-v\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,u) \in H \times H$, $\langle x-y, u-v\rangle \geq 0$ for every $(y,v) \in G(T)$ implies $u \in Tx$. Let B be an inverse-strongly monotone mapping of C to H and let $N_C v$ be normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0,\ \forall u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is a maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,B)$; see [10].

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [2 - 6] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{3}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S and b is a given point in H. In [15] (see also [16]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Sx_n, \ n \geq 0,$$

converges strongly to the unique solution of the minimization problem (3) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. In 2006, Marino and Xu [7] introduced a new iterative scheme by the viscosity approximation method which was first introduced by Moudafi [8]:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Sx_n, \ n \geq 0. \tag{4}$$

They proved that the sequence $\{x_n\}$ generated by iterative scheme (4) converges strongly to the unique solution of the variational inequality $\langle (A-\gamma f)x^*, x-x^* \rangle \geq 0, \ x \in C$, which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where C is the fixed point set of a nonexpansive mapping S, h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in a Hilbert space, Iiduka and Takahashi [5] introduced following iterative process: $x_0 \in C$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)SP_C(x_n - \lambda_n Bx_n), \ n \geq 0, \tag{5}$$

where $u \in C$, $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (5) converges strongly to $z \in F(S) \cap VI(C, B)$. In 2007, Chen et al. [2] studied the following iterative process: $x_0 \in C$,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(x_n - \lambda_n Bx_n), \ n \geq 0. \tag{6}$$

They proved that the sequence $\{x_n\}$ generated by (6) converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse strongly-monotone mapping which solves some variational inequality and also obtained a strong convergence theorem by so-called viscosity approximation method [8] for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping and the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of an inverse-strongly monotone mapping.

In this paper, we will consider a finite family of nonexpansive mappings. Let $T_i : C \to C$, where $i = 1, 2, \ldots, N$, be a finite family of nonexpansive mappings. Let $F(T_i)$ denote the fixed point set of T_i, that is, $F(T_i) = \{x \in C : T_i x = x\}$. Finding an optimal point in the intersection $\cap_{i=1}^N F(T_i)$ of the fixed point sets of a family of nonexpansive mappings is a task that occurs frequently in various areas of mathematical sciences and engineering. A simple algorithmic solution to the problem of minimizing a quadratic function over $\cap_{i=1}^N F(T_i)$ is of extreme value in many applications including set theoretic signal estimation; see, e.g., [6, 18]. Atsushiba and Takahashi [1], defined the

$$\begin{aligned} U_{n,0} &= I \\ U_{n,1} &= \lambda_{n,1}T_1U_{n,0} + (1-\lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})I, \\ W_n = U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1} + (1-\lambda_{n,N})I, \end{aligned} \tag{7}$$

where $\{\lambda_{n,1}\},\{\lambda_{n,2}\},\dots,\{\lambda_{n,N}\} \in (0,1]$. Such a mapping W_n is called the W-mapping generated by $T_1,T_2,\dots,T_N$ and $\{\lambda_{n,1}\},\{\lambda_{n,2}\},\dots,\{\lambda_{n,N}\}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. In [17], Yao introduced the iterative scheme

$$x_{n+1} = \alpha_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)W_nx_n, \ \forall n \geq 0, \tag{8}$$

where γ,β are two positive real numbers such that $\beta < 1$, $f : H \to H$ is a contraction with a coefficient α $(0 < \alpha < 1)$, A is a strongly positive linear bounded operator on H and W_n are the self-mappings of H defined by (7). Under suitable hypotheses on the sequences $\{\lambda_{n,i}\}_{i=1}^N$ and α_n, he prove that the sequence $\{x_n\}$ generated by (8) converges strongly to the unique solution of the variational inequality $\langle(\gamma f - A)q, p-q\rangle \leq 0$ for all $p \in \cap_{i=1}^N F(T_i)$. Very recently, Colao et al. [3] defined the mappings

$$\begin{aligned} U_0 &= I \\ U_1 &= \lambda_1T_1U_0 + (1-\lambda_1)I, \\ U_2 &= \lambda_2T_2U_1 + (1-\lambda_2)I, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1}T_{N-1}U_{N-2} + (1-\lambda_{N-1})I, \\ W = U_N &= \lambda_NT_NU_{N-1} + (1-\lambda_N)I, \end{aligned} \tag{9}$$

where λ_1, λ_2, $\dots$, λ_N are real numbers such that $0 < \lambda_i \leq 1$ for every $i = 1,2,\dots,N$. Such a mapping W is called the W-mapping generated by T_1, T_2, $\dots$, T_N and λ_1, λ_2, $\dots$, λ_N. Moreover, in [3], Lemma 2.8, it is shown that

$$\lim_{n\to\infty} \|W_nx - Wx\| = 0 \text{ for all } x \in C,$$

where W_n and W are the W-mappings defined by (7) and (9), respectively. It is an important tool for the proof of the main results in this paper. The concept of W-mappings was introduced in [12, 13].

Motivated by the recent works, we introduce a general iterative process as follows: $x_0 \in C$,

$$\begin{aligned} x_{n+1} &= \alpha_n \gamma f(W_n x_n) + \beta_n x_n \\ &\quad + ((1-\beta_n)I - \alpha_n A)W_n P_C(I - s_n B)x_n, \ \ \forall n \geq 0, \end{aligned} \tag{10}$$

where $\gamma > 0$, $B : C \to H$ is a β-inverse-strongly monotone mapping, f is a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$, P_C is a projection of H onto C, A is a strongly positive linear bounded operator on H and W_n is defined by (7).

We prove that the sequence $\{x_n\}$ generated by the iterative algorithm (10) converges strongly to a common element of the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of the variational inequality for the inverse-strongly monotone mapping B in a real Hilbert space, which solves another variation inequality $\langle(\gamma f - A)q, p - q\rangle \leq 0$ for all $p \in F$, where $F := \cap_{i=1}^N F(T_i) \cap VI(C, B)$ and is also the optimality condition for the minimization problem $\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). Moreover, using this results, we obtain a strong convergence theorem for finding a common fixed point of a finite family of nonexpansive mappings and a strictly pseudocontractive mapping. Further, we consider the problem of finding a common element of the set of common fixed points of a finite family of nonexpansive mappings and the set of zeros of an inverse-strongly monotone mapping.

Now, we recall some well known concepts and results.

It is well known that for all $x, y \in H$ and $\lambda \in [0, 1]$ there holds

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Recall that a space X is said to satisfy *Opial's condition* [9] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ implying that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$.

A mapping $S : C \to C$ is called strictly pseudocontractive if there exists k with $0 \leq k < 1$ such that $\|Sx - Sy\|^2 \leq \|x-y\|^2 + k\|(I-S)x - (I-S)y\|^2$ for every $x, y \in C$. If $k = 0$, then S is nonexpansive. Put $B = I - S$, where $S : C \to C$ is a strictly pseudocontractive mapping with k. Then B is $\frac{1-k}{2}$ -inverse-strongly monotone. Actually, we have, for all $x, y \in C$, $\|(I - B)x - (I - B)y\|^2 \leq \|x - y\|^2 + k\|Bx - By\|^2$. On the other hand, since H is a real Hilbert space, we have $\|(I-B)x - (I-B)y\|^2 = \|x-y\|^2 + \|Bx - By\|^2 - 2\langle x - y, Bx - By\rangle$. Hence we have $\langle x - y, Bx - By\rangle \geq \frac{1-k}{2}\|Bx - By\|^2$.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1 ([14, 15]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n,$$

where γ_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\displaystyle\sum_{n=1}^{\infty} \gamma_n = \infty;$

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 ([7]). *Assume A is a strong positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.*

Lemma 3 ([11]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 4 ([1]). *Let C be a nonempty closed convex set of a strictly convex Banach space. Let $T_1, T_2, \ldots, T_N$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^N Fix(T_i) \ne \emptyset$, and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots, N-1$ and $0 < \lambda_N \le 1$. Let W be the W-mapping of C into itself generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then $Fix(W) = \cap_{i=1}^N Fix(T_i)$.*

Lemma 5 ([3]). *Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ be sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i (i = 1, \ldots, N)$. Moreover for every $n \in \mathbb{N}$, let W and W_n be the W-mappings generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $T_1, T_2, \ldots, T_N$ and $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \ldots, \{\lambda_{n,N}\}$, respectively. Then for every $x \in C$, it follows that $\lim_{n\to\infty} \|W_n x - Wx\| = 0$.*

The following lemma is well known.

Lemma 6. *In a real Hilbert space H, there holds the following inequality $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$, for all $x, y \in H$.*

2. Main results

Firstly, we prove a strong convergence theorem for a finite family of nonexpansive mappings and inverse-strongly monotone mappings. The following lemma which will be used in the proofs for the main results.

Lemma 7. *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$, $0 < \beta < 1$ and $0 < \alpha \le (1-\beta)\|A\|^{-1}$. Then $\|(1-\beta)I - \alpha A\| \le 1 - \beta - \alpha\bar{\gamma}$.*

Proof. Since A is a strongly positive linear bounded operator on a Hilbert space H. Using the standard result in functional analysis for linear bounded self-adjoint operator on H, we have $\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2$ and $\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}$. Now for $x \in H$ with $\|x\| = 1$, we have

$$\langle((1-\beta)I - \alpha A)x, x\rangle = 1 - \beta - \alpha\langle Ax, x\rangle \geq 1 - \beta - \alpha\|A\| \geq 0$$

(i.e., $(1-\beta)I - \alpha A$ is positive). It follows that

$$\begin{aligned}\|(1-\beta)I - \alpha A\| &= \sup\{|\langle((1-\beta)I - \alpha A)x, x\rangle| : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta - \alpha\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1 - \beta - \alpha\bar{\gamma}.\end{aligned}$$

Theorem 1. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, $B : C \to H$ a β-inverse-strongly monotone mapping, W_n and W are the W-mappings defined by (7) and (9), respectively and $T_1, T_2, \ldots, T_N$ a finite family of nonexpansive mappings of C into H such that the common fixed points set $F := \cap_{i=1}^{N} F(T_i) \cap VI(C, B) \neq \emptyset$. Let f be a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\lambda_{n,i} \to \lambda_i$ $(i = 1, 2, \ldots, N)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0,1)$ and $\{s_n\} \subset [0, 2\beta)$ such that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{s_n\}$ are chosen so that $s_n \in [a, b]$ for some a, b with $0 < a < b < 2\beta$ satisfying the following conditions:*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(*ii*) $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(*iv*) $\sum_{n=0}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$ *for all* $i = 1, 2, \ldots, N$.

Then the sequence $\{x_n\}$ defined by (10) converges strongly to $q \in F$, which solves the following variational inequality:

$$\langle(\gamma f - A)q, p - q\rangle \leq 0 \quad \textit{for all } p \in F.$$

Proof. Since $\alpha_n \to 0$ as $n \to \infty$ by the condition (i), we may assume, without loss of generality, that $0 < \alpha_n \leq (1-\beta_n)\|A\|^{-1}$ for all n. We also have $0 < \alpha_n \leq \|A\|^{-1}$ for all n. By using Lemma 2 and Lemma 7, we have $\|I - \alpha_n A\| \leq 1 - \alpha_n\bar{\gamma}$ and $\|(1-\beta_n)I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n\bar{\gamma}$, respectively. Next, we will assume that $\|I - A\| \leq 1 - \bar{\gamma}$. First, we show $I - s_n B$ is nonexpansive. Indeed, since B is β-inverse-strongly monotone mapping and $\{s_n\} \subset (0, 2\beta)$, we have

$$\|(I - s_n B)x - (I - s_n B)y\|^2 = \|(x - y) - s_n(Bx - By)\|^2$$

$$
\begin{aligned}
&\leq \|x-y\|^2 - 2s_n\langle x-y, Bx-By\rangle \\
&\quad + s_n^2\|Bx-By\|^2 \\
&\leq \|x-y\|^2 - 2s_n\beta\|Bx-By\|^2 + s_n^2\|Bx-By\|^2 \\
&= \|x-y\|^2 + s_n(s_n-2\beta)\|Bx-By\|^2 \\
&\leq \|x-y\|^2,
\end{aligned}
$$

which implies the mapping $I - s_nB$ is nonexpansive. Next we prove that $\{x_n\}$ is bounded. Indeed, pick $x^* \in F$. Putting $y_n = P_C(I - s_nB)x_n$ for all $n \geq 0$, we have

$$
\begin{aligned}
\|y_n - x^*\| &= \|P_C(I-s_nB)x_n - x^*\| \\
&\leq \|(I-s_nB)x_n - (I-s_nB)x^*\| \\
&\leq \|x_n - x^*\|.
\end{aligned} \tag{11}
$$

Using (10) and (11), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\| &= \|\alpha_n\gamma f(W_nx_n) + \beta_nx_n + ((1-\beta_n)I - \alpha_nA)W_ny_n - x^*\| \\
&= \|\alpha_n(\gamma f(W_nx_n) - Ax^*) + \beta_n(x_n - x^*) \\
&\quad + ((1-\beta_n)I - \alpha_nA)(W_ny_n - x^*)\| \\
&\leq (1-\beta_n-\alpha_n\bar{\gamma})\|W_ny_n - x^*\| \\
&\quad + \beta_n\|x_n - x^*\| + \alpha_n\|\gamma f(W_nx_n) - Ax^*\| \\
&\leq (1-\beta_n-\alpha_n\bar{\gamma})\|y_n - x^*\| + \beta_n\|x_n - x^*\| \\
&\quad + \alpha_n\|\gamma f(W_nx_n) - \gamma f(x^*) + \gamma f(x^*) - Ax^*\| \\
&\leq (1-\beta_n-\alpha_n\bar{\gamma})\|x_n - x^*\| + \beta_n\|x_n - x^*\| \\
&\quad + \alpha_n\gamma\|f(W_nx_n) - f(x^*)\| + \alpha_n\|\gamma f(x^*) - Ax^*\| \\
&\leq (1-\alpha_n\bar{\gamma})\|x_n - x^*\| + \alpha_n\gamma\alpha\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Ax^*\| \\
&= (1-(\bar{\gamma}-\gamma\alpha)\alpha_n)\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Ax^*\|,
\end{aligned}
$$

which gives that

$$
\|x_n - x^*\| \quad \leq \max\{\|x_0 - x^*\|, \tfrac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma}-\gamma\alpha}\}, \quad n \geq 0.
$$

Hence $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{W_ny_n\}$, $\{Bx_n\}$ and $\{f(W_nx_n)\}$. Since $I - s_nB$ is nonexpansive and $y_n = P_C(I - s_nB)x_n$, we also have

$$
\begin{aligned}
\|y_{n+1} - y_n\| &= \|P_C(I-s_{n+1}B)x_{n+1} - P_C(I-s_nB)x_n\| \\
&\leq \|(I-s_{n+1}B)x_{n+1} - (I-s_nB)x_n\| \\
&= \|(I-s_{n+1}B)x_{n+1} - (I-s_{n+1}B)x_n + (s_n - s_{n+1})Bx_n\| \\
&\leq \|x_{n+1} - x_n\| + |s_n - s_{n+1}|\|Bx_n\| \\
&\leq \|x_{n+1} - x_n\| + M_1|s_n - s_{n+1}|,
\end{aligned} \tag{12}
$$

where $M_1 = \sup_{n\geq 1}\|Bx_n\|$. Define $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$, $n \geq 0$. Then we have

$$y_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}, \quad n \geq 0.$$

It follows that

$$\begin{aligned}
y_{n+1} - y_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}\gamma f(W_{n+1}x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)W_{n+1}y_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma f(W_n x_n) + ((1-\beta_n)I - \alpha_n A)W_n y_n}{1-\beta_n} \\
&= (\frac{\alpha_{n+1}}{1-\beta_{n+1}})\gamma f(W_{n+1}x_{n+1}) - (\frac{\alpha_n}{1-\beta_n})\gamma f(W_n x_n) \\
&\quad + W_{n+1}y_{n+1} - W_n y_n + (\frac{\alpha_n}{1-\beta_n})AW_n y_n - (\frac{\alpha_{n+1}}{1-\beta_{n+1}})AW_{n+1}y_{n+1} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(W_{n+1}x_{n+1}) - AW_{n+1}y_{n+1}) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(AW_n y_n - \gamma f(W_n x_n)) \\
&\quad + W_{n+1}y_{n+1} - W_{n+1}y_n + W_{n+1}y_n - W_n y_n.
\end{aligned}$$

Observe that from (12), we obtain

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(W_{n+1}x_{n+1}) - AW_{n+1}y_{n+1}) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(AW_n y_n - \gamma f(W_n x_n)) \\
&\quad + W_{n+1}y_{n+1} - W_{n+1}y_n + W_{n+1}y_n - W_n y_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(W_{n+1}x_{n+1})\| + \|AW_{n+1}y_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\|AW_n y_n\| + \|\gamma f(W_n x_n)\|) \\
&\quad + \|W_{n+1}y_{n+1} - W_{n+1}y_n\| + \|W_{n+1}y_n - W_n y_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(W_{n+1}x_{n+1})\| + \|AW_{n+1}y_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\|AW_n y_n\| + \|\gamma f(W_n x_n)\|) \\
&\quad + \|y_{n+1} - y_n\| + \|W_{n+1}y_n - W_n y_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(W_{n+1}x_{n+1})\| + \|AW_{n+1}y_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\|AW_n y_n\| + \|\gamma f(W_n x_n)\|) \\
&\quad + \|x_{n+1} - x_n\| + M_1|s_n - s_{n+1}| + \|W_{n+1}y_n - W_n y_n\|. \qquad (13)
\end{aligned}$$

Next we estimate $\|W_{n+1}y_n - W_n y_n\|$. It follows from the definition of W_n that

$$\begin{aligned}\|W_{n+1}y_n - W_n y_n\| &= \|\lambda_{n+1,N}T_N U_{n+1,N-1}y_n + (1-\lambda_{n+1,N})y_n \\ &\quad - \lambda_{n,N}T_N U_{n,N-1}y_n - (1-\lambda_{n,N})y_n\| \\ &\le |\lambda_{n+1,N} - \lambda_{n,N}|\|y_n\| + \|\lambda_{n+1,N}T_N U_{n+1,N-1}y_n \\ &\quad - \lambda_{n,N}T_N U_{n,N-1}y_n\| \\ &\le |\lambda_{n+1,N} - \lambda_{n,N}|\|y_n\| + \|\lambda_{n+1,N}(T_N U_{n+1,N-1}y_n \\ &\quad - T_N U_{n,N-1}y_n)\| + |\lambda_{n+1,N} - \lambda_{n,N}|\|T_N U_{n,N-1}y_n\| \\ &\le M_2|\lambda_{n+1,N} - \lambda_{n,N}| \\ &\quad + \lambda_{n+1,N}\|U_{n+1,N-1}y_n - U_{n,N-1}y_n\|, \end{aligned} \tag{14}$$

where M_2 is an appropriate constant such that

$$M_2 \ge \max\{\sup_{n\ge1}\|y_n\|,\ \sup_{n\ge1}\|T_N U_{n,N-1}y_n\|\}.$$

Next, we consider

$$\begin{aligned}\|U_{n+1,N-1}y_n - U_{n,N-1}y_n\| &= \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n \\ &\quad + (1-\lambda_{n+1,N-1})y_n \\ &\quad - \lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n - (1-\lambda_{n,N-1})y_n\| \\ &\le |\lambda_{n+1,N-1} - \lambda_{n,N-1}|\|y_n\| \\ &\quad + \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n \\ &\quad - \lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n\| \\ &\le |\lambda_{n+1,N-1} - \lambda_{n,N-1}|\|y_n\| \\ &\quad + \|\lambda_{n+1,N-1}(T_{N-1}U_{n+1,N-2}y_n \\ &\quad - T_{N-1}U_{n,N-2}y_n)\| \\ &\quad + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|\|T_{N-1}U_{n,N-2}y_n\| \\ &\le M_3|\lambda_{n+1,N-1} - \lambda_{n,N-1}| \\ &\quad + \lambda_{n+1,N-1}\|U_{n+1,N-2}y_n - U_{n,N-2}y_n\|, \end{aligned}$$

where M_3 is an appropriate constant such that

$$M_3 \ge \max\{\sup_{n\ge1}\|y_n\|,\ \sup_{n\ge1}\|T_{N-1}U_{n,N-2}y_n\|\}.$$

In a similar way, we obtain

$$\|U_{n+1,N-1}y_n - U_{n,N-1}y_n\| \le M_4\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|, \tag{15}$$

where M_4 is an appropriate constant such that

$$M_4 \ge \max\{\sup_{n\ge1}\|y_n\|,\ \sup_{n\ge1}\{\|T_i U_{n,i-1}y_n\| : i = 1,2,\ldots,N\}.$$

Substitute (15) into (14) yields that

$$\begin{aligned}\|W_{n+1}y_n - W_ny_n\| &\le M_2|\lambda_{n+1,N} - \lambda_{n,N}| + \lambda_{n+1,N}M_4\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\ &\le M_5\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|,\end{aligned} \tag{16}$$

where M_5 is an appropriate constant such that $M_5 \ge \max\{M_2, M_4\}$. Substitute (16) into (13) yields that

$$\begin{aligned}\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \le{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(W_{n+1}x_{n+1})\| + \|AW_{n+1}y_{n+1}\|) \\ &+ \frac{\alpha_n}{1-\beta_n}(\|AW_ny_n\| + \|\gamma f(W_nx_n)\|) \\ &+ M_1|s_n - s_{n+1}| + M_5\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|,\end{aligned}$$

which implies that (noting that $(i) - (iv)$)

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence by Lemma 3, we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{17}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|y_n - x_n\| = 0. \tag{18}$$

Note that

$$\begin{aligned}\|x_n - W_ny_n\| &= \|x_n - x_{n+1} + x_{n+1} - W_ny_n\| \\ &\le \|x_{n+1} - x_n\| + \|x_{n+1} - W_ny_n\| \\ &\le \|x_{n+1} - x_n\| + \|\alpha_n\gamma f(W_nx_n) + \beta_nx_n \\ &\quad + ((1-\beta_n)I - \alpha_nA)W_ny_n - W_ny_n\| \\ &= \|x_{n+1} - x_n\| + \|\alpha_n(\gamma f(W_nx_n) - AW_ny_n) \\ &\quad + ((1-\beta_n)I - \alpha_nA)(W_ny_n - W_ny_n) + \beta_n(x_n - W_ny_n)\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(W_nx_n) - AW_ny_n\| + \beta_n\|x_n - W_ny_n\|.\end{aligned}$$

This implies

$$(1-\beta_n)\|x_n - W_ny_n\| \le \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(W_nx_n) - AW_ny_n\|.$$

From condition (i), (iii) and (18), we have $\|x_n - W_ny_n\| \to 0$ as $n \to \infty$. This together with (17), we obtain

$$\|y_n - W_ny_n\| \le \|y_n - x_n\| + \|x_n - W_ny_n\| \to 0 \ \text{(as } n \to \infty). \tag{19}$$

Observe that $P_F(\gamma f + (I - A))$ is a contraction. Indeed, for all $x, y \in H$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, we have

$$\begin{aligned}
\|P_F(\gamma f + (I - A))x - P_F(\gamma f + (I - A))y\| \\
&\leq \|(\gamma f + (I - A))x - (\gamma f + (I - A))y\| \\
&\leq \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\
&\leq \gamma\alpha\|x - y\| + (1 - \bar{\gamma})\|x - y\| \\
&= (\gamma\alpha + 1 - \bar{\gamma})\|x - y\| \\
&\leq \|x - y\|.
\end{aligned}$$

Banach's Contraction Mapping Principle guarantees that $P_F(\gamma f + (I - A))$ has a unique fixed point, say $q \in H$. That is, $q = P_F(\gamma f + (I - A))q$, by (2) we obtain that $\langle(\gamma f - A)q, p - q\rangle \leq 0$ for all $p \in F$. Next, we show that $\limsup_{n\to\infty}\langle(\gamma f - A)q, y_n - q\rangle \leq 0$. To see this, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)q, y_n - q\rangle = \limsup_{n\to\infty}\langle(\gamma f - A)q, y_{n_i} - q\rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to p. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup p$. Next, we show that $p \in F$. First, we prove $p \in F(W)$. To see this, we observe that we may assume (by passing to a further subsequence if necessary) $\lambda_{n_i,k} \to \lambda_k \in (0,1)$ $(k = 1, 2, \ldots, N)$. Let W be the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then by Lemma 5, we have, for every $x \in C$,

$$W_{n_i}x \to Wx. \tag{20}$$

Moreover, from Lemma 4 it follows that $F(W) = \cap_{i=1}^N F(T_i)$. Suppose for contradiction that $p \notin F(W)$. Then $p \neq Wp$. Since Hilbert space are Opial's spaces, from (19) and (20), we have

$$\begin{aligned}
\liminf_{i\to\infty}\|y_{n_i} - p\| &< \liminf_{i\to\infty}\|y_{n_i} - Wp\| \\
&= \liminf_{i\to\infty}\|y_{n_i} - W_{n_i}y_{n_i} + W_{n_i}y_{n_i} - W_{n_i}p + W_{n_i}p - Wp\| \\
&\leq \liminf_{i\to\infty}\|W_{n_i}y_{n_i} - W_{n_i}p\| \\
&\leq \liminf_{i\to\infty}\|y_{n_i} - p\|,
\end{aligned}$$

which derives a contradiction. Thus, we have $p \in F(W)$. It follows from $F(W) = \cap_{i=1}^N F(T_i)$ that $p \in \cap_{i=1}^N F(T_i)$. Next, we prove $p \in VI(C, B)$. Put

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \neq C, \end{cases}$$

where $N_C v$ is normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}$, which yields that T is maximal monotone. Let $(v, w) \in G(T)$. Since $w - Bv \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, w - Bv\rangle \geq 0$. On the other hand, from $y_n = P_C(I - s_nB)x_n$ and (2), we have $\langle v - y_n, y_n - (I - s_nB)x_n\rangle \geq 0$ and hence $\langle v - y_n, \frac{y_n - x_n}{s_n} + Bx_n\rangle \geq 0$. It follows that

$$\begin{aligned}
\langle v - y_{n_i}, w\rangle &\geq \langle v - y_{n_i}, Bv\rangle \\
&\geq \langle v - y_{n_i}, Bv\rangle - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{s_{n_i}} + Bx_{n_i}\rangle \\
&= \langle v - y_{n_i}, Bv - Bx_{n_i} - \frac{y_{n_i} - x_{n_i}}{s_{n_i}}\rangle \\
&= \langle v - y_{n_i}, Bv - By_{n_i}\rangle \\
&\quad + \langle v - y_{n_i}, By_{n_i} - Bx_{n_i}\rangle - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{s_{n_i}}\rangle \\
&\geq \langle v - y_{n_i}, By_{n_i} - Bx_{n_i}\rangle - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{s_{n_i}}\rangle,
\end{aligned}$$

which implies that $\langle v - p, w\rangle \geq 0$. We have $p \in T^{-1}0$ and hence $p \in VI(C, B)$. That is, $p \in F$. It follows from the variational inequality $\langle(\gamma f - A)q, p - q\rangle \leq 0$ for all $p \in F$ that

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\gamma f - A)q, y_n - q\rangle &= \limsup_{n\to\infty}\langle(\gamma f - A)q, y_{n_i} - q\rangle \\
&= \langle(\gamma f - A)q, p - q\rangle \leq 0.
\end{aligned} \tag{21}$$

Using (19) and (21), we have

$$\limsup_{n\to\infty}\langle(\gamma f - A)q, W_ny_n - q\rangle \leq 0. \tag{22}$$

Moreover, from (17) and (21), we have

$$\limsup_{n\to\infty}\langle(\gamma f - A)q, x_n - q\rangle \leq 0. \tag{23}$$

Finally, we prove that $x_n \to q$. Using (10), (11) and Lemma 6, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n\gamma f(W_nx_n) + \beta_nx_n + ((1 - \beta_n)I - \alpha_nA)W_ny_n - q\|^2 \\
&= \|((1 - \beta_n)I - \alpha_nA)(W_ny_n - q) \\
&\quad + \beta_n(x_n - q) + \alpha_n(\gamma f(W_nx_n) - Aq)\|^2 \\
&\leq \|((1 - \beta_n)I - \alpha_nA)(W_ny_n - q) + \beta_n(x_n - q)\|^2 \\
&\quad + \alpha_n^2\|\gamma f(W_nx_n) - Aq\|^2 + 2\beta_n\alpha_n\langle x_n - q, \gamma f(W_nx_n) - Aq\rangle \\
&\quad + 2\alpha_n\langle((1 - \beta_n)I - \alpha_nA)(W_ny_n - q), \gamma f(W_nx_n) - Aq\rangle \\
&\leq ((1 - \beta_n - \alpha_n\bar{\gamma})\|W_ny_n - q\| \\
&\quad + \beta_n\|x_n - q\|)^2 + \alpha_n^2\|\gamma f(W_nx_n) - Aq\|^2 \\
&\quad + 2\beta_n\gamma\alpha_n\langle x_n - q, f(W_nx_n) - f(q)\rangle \\
&\quad + 2\beta_n\alpha_n\langle x_n - q, \gamma f(q) - Aq\rangle \\
&\quad + 2(1 - \beta_n)\gamma\alpha_n\langle W_ny_n - q, f(W_nx_n) - f(q)\rangle \\
&\quad + 2(1 - \beta_n)\alpha_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad - 2\alpha_n^2\langle A(W_ny_n - q), \gamma f(W_nx_n) - Aq\rangle \\
&\leq (1-\beta_n-\alpha_n\bar{\gamma})^2\|W_ny_n - q\|^2 + \beta_n^2\|x_n - q\|^2 \\
&\quad + 2(1-\beta_n-\alpha_n\bar{\gamma})\beta_n\|W_ny_n - q\|\|x_n - q\| \\
&\quad + \alpha_n^2\|\gamma f(W_nx_n) - Aq\|^2 \\
&\quad + 2\beta_n\gamma\alpha_n\langle x_n - q, f(W_nx_n) - f(q)\rangle \\
&\quad + 2\beta_n\alpha_n\langle x_n - q, \gamma f(q) - Aq\rangle \\
&\quad + 2(1-\beta_n)\gamma\alpha_n\langle W_ny_n - q, f(W_nx_n) - f(q)\rangle \\
&\quad + 2(1-\beta_n)\alpha_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\alpha_n^2\langle A(W_ny_n - q), \gamma f(W_nx_n) - Aq\rangle \\
&\leq ((1-\beta_n-\alpha_n\bar{\gamma})^2 + \beta_n^2 + 2(1-\beta_n-\alpha_n\bar{\gamma})\beta_n)\|x_n - q\|^2 \\
&\quad + \alpha_n^2\|\gamma f(W_nx_n) - Aq\|^2 \\
&\quad + 2\beta_n\gamma\alpha_n\langle x_n - q, f(W_nx_n) - f(q)\rangle \\
&\quad + 2\beta_n\alpha_n\langle x_n - q, \gamma f(q) - Aq\rangle \\
&\quad + 2(1-\beta_n)\gamma\alpha_n\langle W_ny_n - q, f(W_nx_n) - f(q)\rangle \\
&\quad + 2(1-\beta_n)\alpha_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\alpha_n^2\langle A(W_ny_n - q), \gamma f(W_nx_n) - Aq\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\leq ((1-\beta_n-\alpha_n\bar{\gamma})^2 + \beta_n^2 + 2(1-\beta_n-\alpha_n\bar{\gamma})\beta_n + 2\gamma\alpha_n\alpha)\|x_n - q\|^2 \\
&\quad + \alpha_n^2\|\gamma f(W_nx_n) - Aq\|^2 + 2\beta_n\alpha_n\langle x_n - q, \gamma f(q) - Aq\rangle \\
&\quad + 2(1-\beta_n)\alpha_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\alpha_n^2\langle A(W_ny_n - q), \gamma f(W_nx_n) - Aq\rangle \\
&= (1 - 2\alpha_n\bar{\gamma} + \alpha_n^2\bar{\gamma}^2 + 2\alpha\gamma\alpha_n)\|x_n - q\|^2 \\
&\quad + \alpha_n^2\|\gamma f(W_nx_n) - Aq\|^2 + 2\beta_n\alpha_n\langle x_n - q, \gamma f(q) - Aq\rangle \\
&\quad + 2(1-\beta_n)\alpha_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\alpha_n^2\langle A(W_ny_n - q), \gamma f(W_nx_n) - Aq\rangle \\
&\leq (1 - 2(\bar{\gamma} - \alpha\gamma)\alpha_n)\|x_n - q\|^2 + \alpha_n^2\bar{\gamma}^2\|x_n - q\|^2 \\
&\quad + \alpha_n^2\|\gamma f(W_nx_n) - Aq\|^2 + 2\beta_n\alpha_n\langle x_n - q, \gamma f(q) - Aq\rangle \\
&\quad + 2(1-\beta_n)\alpha_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle \\
&\quad + 2\alpha_n^2\|A(W_ny_n - q)\|.\|\gamma f(W_nx_n) - Aq\| \\
&= (1 - 2(\bar{\gamma} - \alpha\gamma)\alpha_n)\|x_n - q\|^2 + \alpha_n^2(\bar{\gamma}^2\|x_n - q\|^2 \\
&\quad + \|\gamma f(W_nx_n) - Aq\|^2 + 2\|A(W_ny_n - q)\|.\|\gamma f(W_nx_n) - Aq\|) \\
&\quad + 2\beta_n\alpha_n\langle x_n - q, \gamma f(q) - Aq\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(1-\beta_n)\alpha_n\langle W_ny_n - q, \gamma f(q) - Aq\rangle \\
&= (1-2(\bar{\gamma}-\alpha\gamma)\alpha_n)\|x_n - q\|^2 + \alpha_n(\alpha_n(\bar{\gamma}^2\|x_n - q\|^2 \\
&+ \|\gamma f(W_nx_n) - Aq\|^2 + 2\|A(W_ny_n - q)\|.\|\gamma f(W_nx_n) - Aq\|) \\
&+ 2\beta_n\langle x_n - q, \gamma f(q) - Aq\rangle + 2(1-\beta_n)\langle W_ny_n - q, \gamma f(q) - Aq\rangle).
\end{aligned}
$$

Since $\{x_n\}$, $\{f(W_nx_n)\}$ and $\{W_ny_n\}$ are bounded, we can take a constant $\eta > 0$ such that

$$\eta \geq \bar{\gamma}^2\|x_n - q\|^2 + \|\gamma f(W_nx_n) - Aq\|^2 + 2\|A(W_ny_n - q)\|.\|\gamma f(W_nx_n) - Aq\|)$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - q\|^2 \leq (1-2(\bar{\gamma}-2\gamma)\alpha_n)\|x_n - q\|^2 + \alpha_n\zeta_n, \tag{24}$$

where $\zeta_n = 2\beta_n\langle x_n - q, \gamma f(q) - Aq\rangle + 2(1-\beta_n)\langle W_ny_n - q, \gamma f(q) - Aq\rangle + \alpha_n\eta$. Using (i), (22) and (23), we have $\limsup_{n\to\infty}\zeta_n \leq 0$. By applying Lemma 1 to (24), it conclusion that $x_n \to q$ as $n \to \infty$. The proof is completed.

Finally, we prove two theorems in a Hilbert space by using Theorem 1. We apply the iterative scheme (10) for finding a common fixed point of a finite family of nonexpansive mappings and a k-strictly pseudocontractive mapping and also apply Theorem 1 for finding a common element of the set of common fixed points of a finite family of nonexpansive mappings and the set of zeros of an inverse-strongly monotone mapping.

Theorem 2. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, $T_1, T_2, \ldots, T_N$ a finite family of nonexpansive mappings of C into H, W_n and W are the W-mappings defined by (7) and (9), respectively and S a k-strictly pseudocontractive mapping of C into itself with β such that the common fixed points set $F := \cap_{i=1}^N F(T_i) \cap F(S) \neq \emptyset$. Let f be a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\lambda_{n,i} \to \lambda_i$ $(i = 1, 2, \ldots N)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0,1)$ and $\{s_n\} \subset [0, 1-k)$ such that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{s_n\}$ are chosen so that $s_n \in [a, b]$ for some a, b with $0 < a < b < 1-k$ satisfying the following conditions:*

(*i*) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(*ii*) $\sum_{n=1}^{\infty}|s_{n+1} - s_n| < \infty$;

(*iii*) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(*iv*) $\sum_{n=0}^{\infty}|\lambda_{n,i} - \lambda_{n-1,i}| < \infty$ *for all* $i = 1, 2, \ldots, N$.

Then the sequence $\{x_n\}$ generated by $x_0 \in C$ and

$$x_{n+1} = \alpha_n\gamma f(W_nx_n) + \beta_nx_n + ((1-\beta_n)I - \alpha_nA)W_n((1-s_n)x_n - s_nSx_n), \quad n \geq 0,$$

converges strongly to $q \in F$, which solves the following variational inequality:

$$\langle(\gamma f - A)q, p - q\rangle \leq 0 \ \text{for all } p \in F.$$

Proof. Taking $B = I - S$. Then B is $\frac{1-k}{2}$ -inverse strongly monotone. We have $P_C(I - s_nB)x_n = (1 - s_n)x_n + s_nSx_n$ and $F(S) = VI(C, B)$. So by Theorem 1, we obtain the desired result.

Theorem 3. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, $T_1, T_2, \ldots, T_N$ a finite family of nonexpansive mappings of C into H, W_n and W are the W-mappings defined by (7) and (9), respectively and B a β-inverse-strongly monotone mapping of H into itself such that the common fixed points set $F := \cap_{i=1}^N F(T_i) \cap B^{-1}0 \neq \emptyset$. Let f be a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\lambda_{n,i} \to \lambda_i$ $(i = 1, 2, \ldots N)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0, 1)$ and $\{s_n\} \subset [0, 2\beta)$ such that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{s_n\}$ are chosen so that $s_n \in [a, b]$ for some a, b with $0 < a < b < 2\beta$ satisfying the following conditions:*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(*ii*) $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(*iv*) $\sum_{n=0}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$ *for all* $i = 1, 2, \ldots, N$.

Then the sequence $\{x_n\}$ generated by $x_0 \in C$ and

$$x_{n+1} = \alpha_n\gamma f(W_nx_n) + \beta_nx_n + ((1 - \beta_n)I - \alpha_nA)W_n(I - s_nB)x_n, \ \ n \geq 0,$$

converges strongly to $q \in F$, which solves the following variational inequality:

$$\langle(\gamma f - A)q, p - q\rangle \leq 0 \ \text{for all } p \in F.$$

Proof. Taking $P_H = I$. We have $B^{-1}0 = VI(C, B)$. So by Theorem 1, we obtain the desired result.

Remark 1. If we take $\gamma = 1$, $W_n = S$, a single nonexpansive mapping and $A = I$, the identity mapping, then the iterative scheme (10) reduces to the following scheme:

$$x_{n+1} = \alpha_nf(Sx_n) + \beta_nx_n + (1 - \beta_n - \alpha_n)SP_C(I - s_nB)x_n, \qquad (25)$$

which is a modification of the scheme (6) in Proposition 3.1 defined by Chen, Zhang and Fan [2], and also apply Theorem 1, we obtain strong convergence of the sequence $\{x_n\}$ generated by (25) under the sufficient conditions of Theorem 1 without the hypothesis $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

3. Acknowledgments

The author would like to thank the Thailand Research Fund, The Commission on Higher Education, and Naresuan Phayao University, Phayao, Thailand for financial support during the preparation of this paper.

References

1. S. Atsushiba and W. Takahashi, *Strong convergence theorems for a finite family of non-expansive mappings and applications*, Indian J. Math., **41** (1999), 435–53.
2. J.M. Chen, L.J. Zhang, and T.G. Fan, *Viscosity approximation methods for nonexpansive mappings and monotone mappings*, J. Math. Anal. Appl., **334(2)** (2007), 1450–1461.
3. V. Colao, G. Marino, and H.K. Xu, *An iterative method for finding common solutions of equilibrium and fixed point problems*, J. Math. Anal. Appl., **344(1)** (2008), 340–352.
4. F. Deutsch and I. Yamada, *Minimizing certain convex functions over the intersection of the fixed point set of nonexpansive mappings*, Numer. Funct. Anal. Optim., **19** (1998), 33–56.
5. H. Iiduka and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal., **61(3)** (2005), 341–350.
6. A.N. Iusem and A.R. De Pierro, *On the convergence of Hans method for convex programming with quadratic objective*, Math. Program, Ser. B, **52** (1991), 265–284.
7. G. Marino and H.K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **318(1)** (2006), 43–52.
8. A. Moudafi, *Viscosity approximation methods for fixed points problems*, J. Math. Anal Appl., **241(1)** (2000), 46–55.
9. Z. Opial, *Weak convergence of successive approximations for nonexpansive mappins*, Bull. Amer. Math. Soc., **73** (1967), 591–97.
10. R.T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., **149** (1970), 75–88.
11. T. Suzuki, *Strong convergence of Krasnoselskii and Manns type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl., **305(1)** (2005), 227-239.
12. W. Takahashi, *Weak and strong convergence theorems for families of nonexpansive mappings and their applications*, Ann. Univ. Mariae Curie-Sklodowska, **51** (1997), 277–292 .
13. W. Takahashi and K. Shimoji, *Convergence theorems for nonexpansive mappings and feasibility problems*, Math. Comput. Modelling, **32(11-13)** (2000), 1463–1471.
14. H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66** (2002), 240–256 .
15. H.K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl., **116(3)** (2003), 659–678.
16. I. Yamada, *The hybrid steepest descent method for the variational inequality problem of the intersection of fixed point sets of nonexpansive mappings, in: D. Butnariu, Y. Censor and S. Reich (Eds.), Inherently Parallel Algorithm for Feasibility and Optimization*, Elsevier, 2001, 473–504.
17. Y. Yao, *A general iterative method for a finite family of nonexpansive mappings*, Nonlinear Anal. **66(12)** (2007), 2676-2687.
18. D.C. Youla, *Mathematical theory of image restoration by the method of convex projections, in: H. Stark (Ed.), Image Recovery: Theory and Applications*, Academic Press, Florida, 1987, 29–77.

Sornsak Thianwan received his B.Sc. (Mathematics) from Naresuan University, Phitsanuloke, Thailand, M.Sc. (Mathematics) and Ph.D. (Mathematics) at Chiang Mai University, Chiang Mai, Thailand under the Thailand Research Fund (RGJ Project) and the Royal Golden Jubilee Grant PHD/0160/2004. He works as a lecturer at the Department of Mathematics at Naresuan Phayao University since 2003. Recently, he received the 2008 Thesis Award from the National Research Council of Thailand. In 2007, he was granted for a 2-month research fellow scholarship working with Professor Wataru Takahashi at Tokyo Institute of Technology, Japan. His research interests focus on Fixed Point Theory and Applications and Geometric Property of Banach Spaces.

School of Science and Technology, Naresuan Phayao University, Phayao, 56000, Thailand
e-mail: sornsakt@nu.ac.th